Angle Trisection and Morley's Theorem

By Jean-Marie Laborde, Grenoble, France & Danny Gremillion, Dallas, TX

Mathematicians have tried for many years to devise a construction for trisecting an angle. All have failed, as predicted and proved by Pierre Laurent Wantzel in 1837. Trisection of an angle is not achievable with compass and ruler. While addition, subtraction, multiplication, division and the square root of a number can be found using compass and ruler construction, it turns out there is no way to realize the cube root function which can be shown to be equivalent to a trisection construction.[1]

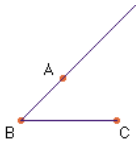
Cabri Geometry II™ is a modern geometry tool which makes trisection of an angle possible, not from a classical Euclidean construction, but through a synthetic construction. Even so, there are limits of numerical precision. But the result is sufficient to allow us to investigate a theorem involving trisection known as Morley's Theorem. The process is explained below and can be performed on the TI-92 or Cabri Geometry II (now available for Macintosh® and MS-DOS™).

Creating a macro for angle trisection

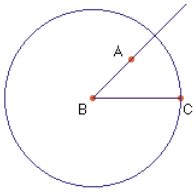
Begin with three points *A*, *B*, and *C* positioned as shown. You can add a label immediately after creating the point by typing the appropriate letter.



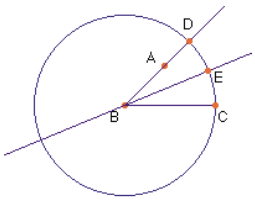
Construct the ray $\overrightarrow{BA}$ and the segment *BC*.



Construct a circle with center point *B* and radius *C*. (The circle must be attached to *C*.)



Create point *D* at the intersection of the circle and ray $\overrightarrow{BC}$. Then, create the perpendicular bisector between *D* and *C*. This bisector bisects $\angle ABC$. Create point *E* at the intersection of the perpendicular bisector and the circle.

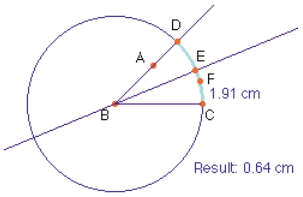


Construct arc $\overset{\frown}{DEC}$ and measure it using the Distance and Length tool. (The arc is shown here in thick outline for emphasis.)

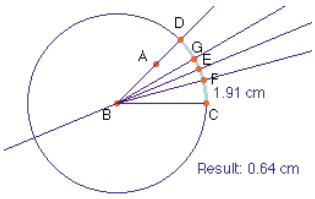
Open the Calculator tool, select the arc length measurement, and divide it by three. Copy the result to the drawing window. (This is automatically done for you on the TI-92) This result is the basis of your trisection.



Transfer the trisection measurement to the circle using the Measurement Transfer tool. Select the result first, then select the circle, then select *C*. A trisection point that is one-third the length of the arc is created. Label this point *F*.

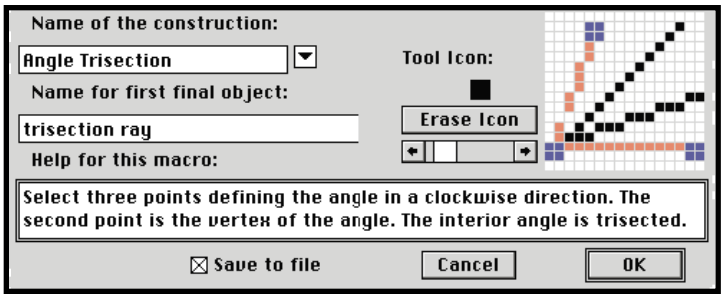


Use Reflection to create the final trisection point. Reflect *F* with respect to the perpendicular bisector and label this point *G*. Now, construct rays $\overrightarrow{BF}$ and $\overrightarrow{BG}$. (Of course, you could use Measurement Transfer to create *G*, but variety is the spice of life and sometimes instructional.)



Construct a macro for trisecting an angle. Order is important since the interior and exterior angle trisections are different. We will choose to trisect the interior angle when three points which define the angle are selected in clockwise order. (See screen sample below.)

- Select Initial Objects in the Macro toolbox and select, in order, points *C*, *B*, and *A*.
- Select Final Objects in the Macro toolbox and select the two rays which trisect the angle. Hide the points *F* and *G* so that they will not appear when your macro is executed.
- Select Define Macro in the Macro toolbox and complete the dialog with descriptive information. Be sure to save your macro so that it is easily accessible in the future.



(see Angle Trisection on page 14)

TI-92 Use In SIMMS

(cont'd from page 6)

- Students who are not normally mathematically inclined will try to gain the understanding.
- The link between real world problems and academic problems will be solidified.
- More high school students will take such courses as physics, chemistry, trig, and calculus.
- TI-92s will take the place of computers in the classroom.
- The TI-92 is, in many ways, more user-friendly than the 82. It will lead to even greater emphasis (and understanding) of concepts through less emphasis on computations.
- We junior high teachers get very little time to experiment with computer programs, so the size and compactness of the 92 will enable me to expand my horizons. In my district, I am having considerable difficulty in justifying calculators in the classroom. A demonstration of 92 use from middle school through college level would be impressive.

Preservice Teachers

During the fall, a class of Modeling with Technology was taught at The University of Montana. Twelve prospective secondary teachers in the class had the TI-92s for only one week of work, during which they were asked to compare their use with the software that SIMMS recommends for schools. They were to consider limits, graphing, geometry problems, symbolic manipulators and statistics.

- The symbol manipulator is much easier to use than anything that was available to us. (Students had access to and instruction for Mathematica, while Maple and MathCad were also available. One student thought the 92 was far easier to use than Derive.)
- The spreadsheet does not allow us to do all that we would like to do. I need row logic sometimes.
- The Cabri-like geometry is good. I could do virtually all of the problems that we had explored on the computer package.
- It is a good tool. Teachers will have to re-think all traditional programs in mathematics if the tool is used. Why should anyone waste time talking about 90% of algebra if this tool was available?

The comments reveal that sentiments of preservice teachers and inservice teachers are more shared than might be expected. Some people expect more of a tool than can be delivered. It was very troublesome to hear preservice teachers express that technology should not be used in the classroom. Such statements might be expected (though not necessarily accepted) from experienced, rather than inexperienced teachers.

Selected Bibliography

Bead, J., et al. *Integrated Mathematics: Definitions, Issues, and Implications*. Helena, MT: Montana Council of Teachers of Mathematics, 1990.

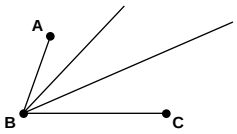
Montana Council of Teachers of Mathematics. *SIMMS Curriculum, Levels 1-4*. Needham Heights, MA: Simon & Schuster Custom Publishing Company, 1995.

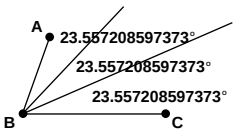
National Council of Teachers of Mathematics. *Curriculum and Evaluation Standards for School Mathematics*. Reston, VA: NCTM, 1989.

Angle Trisection

(*continued from page 7)*

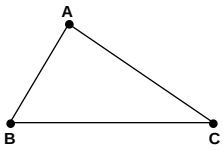
Apply the macro to see how it works. Try selecting three points in both clockwise and counter-clockwise order. Measure the angles and extend the precision of the results to determine the accuracy of the construction.



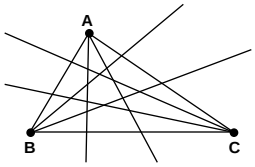


An interesting theorem involving trisected angles is Morley's Theorem. This theorem was discovered by F. Morley in 1899 and proved in 1914. It states that the three points of intersection of adjacent trisectors of the angles of any triangle form an equilateral triangle. Let's give it a try.

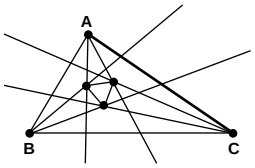
Construct a triangle.



Trisect the angles of your triangle using your trisection macro.



Using the Triangle tool, connect the intersections of the trisection rays at adjacent angles.



You may wish to measure the sides of the triangle to verify that they are indeed equal or use the Equidistant tool.

There are many ways in which to investigate further. Take off in any direction you desire. The discovery is worth the trip. To get you started, observe what happens when you connect the intersections of the external angle trisectors.

Note: This article was written using Cabri Geometry II for DOS (released February 1996) and Microsoft Word. All illustrations were copied and pasted from Cabri II.[1]

[1] *The Trisectors, U. Dudley, MAA, 1987, pg. XV*
Cabri Geometry II™ is a trademark of the Universitè Joseph Fourier.
Macintosh is a trademark of Apple.
MS-DOS is a trademark of Microsoft.