Høgskolen Stord/Haugesund

The Logistic Equation

Oppgave

Den "Logistiske Ligningen" lyder

$$x_{i+1} = rx_i(1 - x_i)$$

Det er en ligning som beskriver en tallfølge. Du begynner med en verdi x_1, som f.eks. $x_1 = 0.5$. Så beregner du verdien x_2 utifra ligningen og en verdi r som du har valgt.

Har du x_2, så finner du x_3, så x_4, etc. Det er en såkallt rekursiv ligning: du må først beregne alle verdier fram til x_{999} før du kan finne verdien x_{1000}.

I denne ligningen er r en parameter som kan ta verdier mellom 0 og 4.

Denne ligningen ble opprinnelig brukt for å beskrive antall dyr av en spesiel type innenfor et spesielt system. Antallet er beskrevet ikke i form av et heltall som 349, men i form av et tall x som kan ta verdier mellom 0 og 1. Indeksen $_i$ kan tenkes som årstall, uketall, etc.

Antallet har en tendens til å stige når det er mange fra før pga. økt reproduksjon. Samtidig blir det knappere med ressursene; derfor har antallet også en tendens til å synke. Ligningen beskriver denne balansen på enkelst mulig måte.

Har du f.eks. antall 0, så forblir antallet null i all evighet. Har du derimot antall 1 (som er maks) så bryter systemet sammen og du ender opp med null antall i neste skritt.

Din oppgave: Lag et Excel-ark der du beskriver denne ligningen og hva den kan brukes for. Lag deg kolonner med ca. 1000 rader, en rad for hvert år eller uke du ser på. I første kolonne skriver du indeksen $_i$, altså 1, 2, 3, ... fram til 1000. I kolonnen ved siden av skriver du tilhørende x_i-verdier.

Lag en graf av x_i som funksjon av i. Viktig: lag et punkt-diagram, men bare plot prikker, ikke linjer som forbinder prikkene.

Hvorfor? Fordi du skal i dette arket leke med parameteren r. Lag arket slik at man kan sette verdien til r, og hele arket med alle x_i-verdiene blir da oppdatert. Det hadde vært moro å gjøre dette ved hjelp av spin-knapper, der man setter r til verdier mellom 0 og 4, i skritt av 0.01 eller mindre.

Den allerførste verdien, x_1, må du sette selv. Velg f.eks. $x_1 = 0.5$.

Hvis du øker r-verdien langsomt... ser du noe interessant som skjer?

Prøv å dokumentere dine observasjoner i arket ditt.

Bakgrunn

Dette bakgrunnsmateriale trenger du ikke for å løse oppgaven, men kanskje du synes likevel det er interessant. Det er sakset fra boken 'Chaos: Making a New Science' av James Gleick. I denne teksten er det snakk om Robert May, som var blant de første som oppdaget sprengstoffet som ligger i denne ligningen...

"May came to biology though the back door, as it happened. He started as a theoretical physicist in his native Sydney, Australia, the son of a brillant barrister, and he did postdoctoral work in applied mathematics at Harvard. In 1971, he went for a year to the Institute for Advanced Study in Princeton: instead of doing the work he was supposed to be doing, he found himself drifting over to Princeton University to talk to the biologists there.

Even now, biologists tend not to have much mathematics beyond calculus. People who like mathematics and have an aptitude for it tend more toward mathematics or physics than the life sciences. May was an exception. His interests at first tended toward the abstract problems of stability and complexity, mathematical explanations of what enables competitors to coexist. But he soon began to focus on the simplest ecological questions of how single populations behave over time. The inevitably simple models seemed less of a compromise. By the time he joined the Princeton faculty for good – eventually he would become the university's dean for research – he had already spent many hours studying a version of the logistic difference equation [Dette er vår ligning...!], using mathematical analysis and also a primitive hand calculator.

Once, in fact, on a corridor blackboard back in Sydney, he wrote the equation out as a problem for the graduate students. It was starting to annoy him. *"What the Christ happens when r gets bigger than the point of accumulation?* What happened, that is, when a population's rate of growth, its tendency to toward boom and bust, passed a critical point. By trying different values of this nonlinear parameter, May found that he could dramatically change the system's character. Raising the parameter meant rising the degree of nonlinearity, and that changed not just the quantity of the outcome, but also its quality. It affectd not just the final populations at equilibrium, but also whether the population would reach equilibrium at all.

When the parameter was low, May's simple model settled on a steady state. When the parameter was high, the steady state would break apart, and the population would oscillate between two alternating values. When the parameter was very high, the system – the very *same* system – seemed to behave unpredictably. Why? What exactly happened at the boundaries between the different kinds of behavior? May couldn't figure it out (nor could the graduate students).

May carried out a program of intense numerical exploration into the behavior of this simplest of equations. [...] It seemed incredible that its pos-

sibilities for creating order and disorder had not been exhausted long since. But they had not. Indeed, May's program was just the beginning. He investigated hundreds of different values of the paramter, setting the feedback loop in motion and watching to see – and whether – the string of numbers would settle down to a fixed point. He focused more and more closely on the critical boundary between steadines and oscillation. It was as if he had his own fish pond, where he could wield fine mastery over the "boom-and-bustiness" of the fish. Still using the logistic equation, $x_{next} = rx(1-x)$, May increased the parameter r as slowly as he could. If the parameter was 2.7, then the population would be 0.6292. As the parameter rose, the final population rose slightly, too, making a line that rose slightly as it moved from left to the right on the graph.

Suddenly, though, the parameter passed 3, the line broke in two. May's imaginary fish population refused to settle down to a single value, but oscillated between two points in alteranting years. Starting at low number, the population would rise and then fluctuate until it was steadily flipping back and forth. Turning up the know a bit more – raising the parameter a bit more – would split the oscillation again, producing a string of number that settled down to four different values, each returning every fouth year. Now the population rose and fell on a regular four-year schedule. The cycle had doubled again – first from yearly to every two years, and now to four. Once again, the resulting cyclical behavior was stable; different starting values for the population would converge to the same four-year cycle.

[...]

The doublings themselves were bifurcations, and each bifurcation meant that the pattern of repetition was breaking down a step further. [...] These bifurcations would come faster and faster – 4, 8, 16, 32, ... – and suddenly break off. Beyond a certain point, "point of accumulation", periodicity gives way to chaos, fluctuations that never settle down at all. "