

ICCE/7

SEVENTH ANNUAL INTERNATIONAL CONFERENCE ON
COMPOSITES ENGINEERING

Edited by
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July 2-8, 2000

Denver, Colorado

Sponsored and Organized by
International Community for Composites Engineering and
College of Engineering, University of New Orleans

HEAT CONDUCTIVITY BOUNDS BY HALPIN-TSAI TYPE EQUATIONS AND HOMOGENIZATION

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Let us consider a composite material that occupies a volume $\Omega \subset \mathbf{R}^2$. We assume that the material consists of periodically distributed inclusions of material 1 in the continuous material 2. Moreover, we assume that material 1 occupies a volume fraction of m_1 and has a thermal conductivity of a_1 and that material 2 has a thermal conductivity of a_2 , where $a_2 < a_1$. We will here consider thermal conduction, the elasticity case can be found in [3].

Let us consider the expression

$$\bar{a}(\xi, m_1) = \frac{a_2 \left(\xi + \frac{a_1}{a_2} + m_1 \left(\frac{a_1}{a_2} - 1 \right) \xi \right)}{\xi + \frac{a_1}{a_2} - m_1 \left(\frac{a_1}{a_2} - 1 \right)}.$$

This expression can be rewritten as a Halpin-Tsai type equation, see [2], by

$$\frac{\bar{a}(\xi, m_1)}{a_2} = \frac{1 + \xi \eta m_1}{1 - \eta m_1},$$

where

$$\eta = \frac{\frac{a_1}{a_2} - 1}{\frac{a_1}{a_2} + \xi}$$

and $\xi \geq 0$ is the fitting parameter. The function $\bar{a}(\xi, m_1)$ has the following properties:

1. $\bar{a}(\xi, m_1)$ is an increasing function in m_1 for every fixed ξ and

$$\begin{cases} \bar{a}(\xi, 0) = a_2, \\ \bar{a}(\xi, 1) = a_1. \end{cases}$$

2. $\bar{a}(\xi, m_1)$ is an increasing function in ξ for every fixed m_1 .
3. The lower Reuss-Voigt bound (i.e. the harmonic mean H) is attained at $\xi = 0$, that is,

$$\bar{a}(0, m_1) = H = \frac{1}{\frac{m_1}{a_1} + \frac{1-m_1}{a_2}}.$$

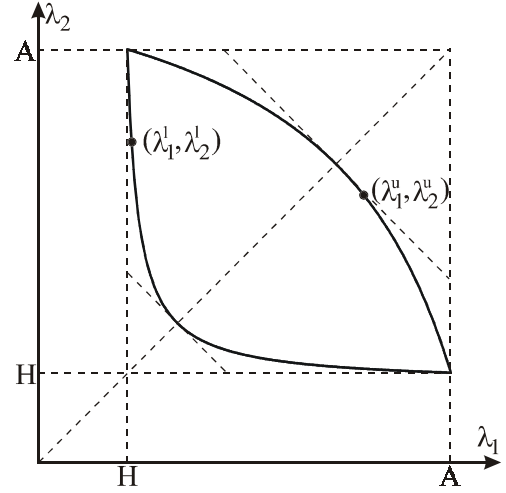


Figure 1: Points on the upper and lower generalized Hashin-Shtrikman bounds.

4. The upper Reuss-Voigt bound (i.e. the arithmetic mean A) is attained in the limit as $\xi \rightarrow \infty$, that is,

$$\lim_{\xi \rightarrow \infty} \bar{a}(\xi, m_1) = A = m_1 a_1 + (1 - m_1) a_2.$$

5. The Hashin-Shtrikman lower bound is attained at $\xi = 1$, that is,

$$\bar{a}(1, m_1) = a_2 + \frac{m_1}{\frac{1}{a_1 - a_2} + \frac{1 - m_1}{2a_2}}.$$

6. The Hashin-Shtrikman upper bound is attained at $\xi = \frac{a_1}{a_2}$, that is,

$$\bar{a}\left(\frac{a_1}{a_2}, m_1\right) = a_1 + \frac{1 - m_1}{\frac{1}{a_2 - a_1} + \frac{m_1}{2a_1}}.$$

7. The generalized Hashin-Shtrikman lower bound given by the eigenvalues $(\lambda_1^l, \lambda_2^l)$, see

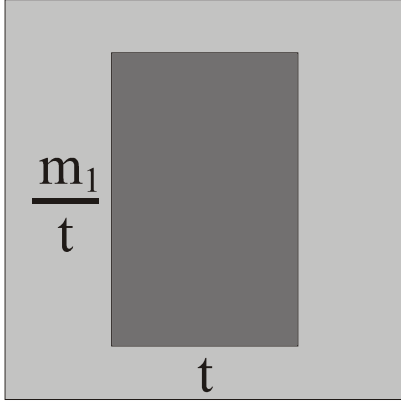


Figure 2: Unit cell with a rectangular inclusion of volume fraction m_1 .

figure 1, is achieved with $\xi = \frac{H(A-\lambda_1^l)}{a_1(\lambda_1^l-H)}$ for $H \leq \lambda_1^l \leq A$, that is,

$$\lambda_2^l(\lambda_1^l) = \bar{a} \left(\frac{H(A-\lambda_1^l)}{a_1(\lambda_1^l-H)}, m_1 \right).$$

8. In particular, the generalized Hashin-Shtrikman lower bound is parametrized by the curve $(\bar{a}(\xi, m_1), \bar{a}(\frac{1}{\xi}, m_1))$ as $0 \leq \xi \leq \infty$.
 9. The generalized Hashin-Shtrikman upper bound given by the eigenvalues $(\lambda_1^u, \lambda_2^u)$, see figure 1, is achieved with $\xi = \frac{Ha_1(A-\lambda_1^u)}{a_2^2(\lambda_1^u-H)}$ for $H \leq \lambda_1^u \leq A$, that is,
- $$\lambda_2^u(\lambda_1^u) = \bar{a} \left(\frac{Ha_1(A-\lambda_1^u)}{a_2^2(\lambda_1^u-H)}, m_1 \right).$$
10. In particular, the generalized Hashin-Shtrikman upper bound is parametrized by the curve $(\bar{a}(\frac{a_1\xi}{a_2}, m_1), \bar{a}(\frac{a_1}{a_2\xi}, m_1))$ as $0 \leq \xi \leq \infty$.
 11. For the special case when we have rectangular inclusions, the generalized Reuss-Voigt lower bound is achieved with $\xi_l = \frac{t-m_1}{1-t}$, where m_1/t^2 is the quotient between the sides of the rectangle, see figure 2. We have

$$\bar{a} \left(\frac{t-m_1}{1-t}, m_1 \right) = \int_0^1 \left(\int_0^1 \frac{1}{a(x,y)} dx \right)^{-1} dy.$$

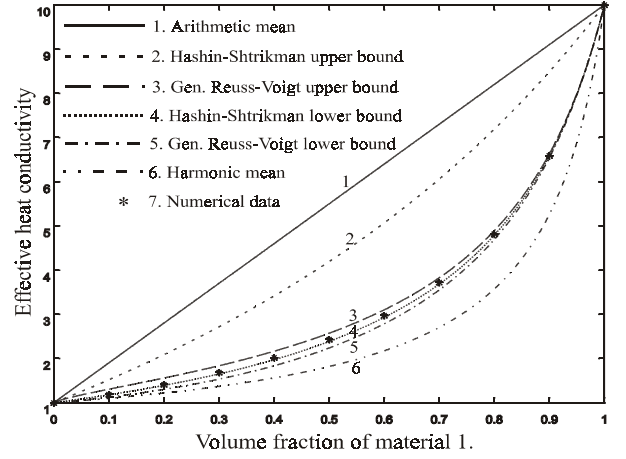


Figure 3: A comparison of different bounds with numerical data from the homogenization of a composite with square inclusions.

12. For the special case when we have rectangular inclusions, the generalized Reuss-Voigt upper bound is achieved with $\xi_u = \frac{t-m_1}{m_1(1-t)}$, where m_1/t^2 is the quotient between the sides of the rectangle, see figure 2. We have

$$\bar{a} \left(\frac{t-m_1}{m_1(1-t)}, m_1 \right) = \frac{1}{\int_0^1 \left(\int_0^1 a(x,y) dy \right)^{-1} dx}.$$

We have square inclusions in the special case when $t = \sqrt{m_1}$, and then $\xi_l = \sqrt{m_1}$ and $\xi_u = \frac{1}{\sqrt{m_1}}$. In this case, the best bounds are the lower Hashin-Shtrikman bound and the generalized Reuss-Voigt upper bound. These bounds are compared with numerical data from the homogenization procedure described in [1], see figure 3.

References

- [1] J. Byström, *The Homogenization Method Applied to the Computation of Effective Thermal Conductivities of Composite Materials*, Master's Thesis 222E, Department of Mathematics, Luleå University of Technology, 1996.
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- [3] P. Wall, A comparison of homogenization, Hashin-Shtrikman bounds and the Halpin Tsai equations, *Appl. Mat.*, 42(4):245-257, 1997.