

An evaluation of different models for prediction of elastic properties of woven composites

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Received 6 August 1999; accepted 30 September 1999

Abstract

A critical analysis of two simple and convenient analytical models for calculation of elastic properties of woven fabric composites is performed. Predictions of these models are compared with results obtained using the method of reiterated homogenization and with experimental data for plain weave glass fiber and carbon fiber polyester composites. Three different scales are identified in the analysis. The first scale predictions, which are the tow properties (obtained by applying Hashin's concentric cylinder model, the Halpin–Tsai expressions or mathematical homogenization technique), are the most critical because they form the input information for woven composite modeling. It appears that the uncertainty in this information causes larger differences in predictions than the deviations between models of different degree of accuracy. This fact sets limits on the required accuracy of the models. Model comparisons reveal that the woven composite model based on isostrain assumption in the composite plane and isostress assumption for out-of-plane components is in very good agreement with both experimental data and the reiterated homogenization method, whereas the modified mosaic parallel model fails to describe composites with large interlaced regions. © 2000 Elsevier Science Ltd. All rights reserved.

Keywords: B. Elasticity; Reiterated homogenization

1. Introduction

Although woven composites have no evident advantages in the elastic properties versus laminated composites, their relative usage have increased over the recent years. This popularity comes from lower production costs (resin transfer molding), higher fracture toughness and better control over the thermomechanical properties. These benefits are related to the current textile technologies, which allows production of fabrics with accurate, repeatable unit cells (RUC). The elastic properties modeling can be considered as the first step in the design of a new material.

By using the term *elastic properties* in the context of woven composites, we are mainly thinking about the average elastic response of a large material region to load. We are virtually replacing a part of the heterogeneous material with a homogeneous one. Hence, in order to obtain the elastic properties of woven composites, we need to formulate a homogenization problem over some representative volume element (RVE) that is large enough to provide the effective properties. This homogenization problem may

contain several scales, and then the final homogenization must be performed over all scales. Let us assume some volume of the composite to be representative with respect to its elastic properties, i.e. let this volume be the RVE, see Fig. 1(a). This is the first scale. Usually, the RVE contains several woven fabric layers stacked together in the thickness direction (z-direction) with some number of RUCs in each layer, and in general, the RUCs are randomly shifted with respect to each other. As we will see below, this causes a problem—the boundary conditions for each RUC are unknown. The next scale (Fig. 1(b)) is the RUC scale. The constituents of the RUC are tows with different orientations, resin rich regions and possibly voids. The last scale is the homogenization over fibers, matrix and voids on the tow level (Fig. 1(c)). Thus we can recognize three different scales in the homogenization problem.

Existing homogenization models can be divided into three different categories—semi-analytic engineering models, models based on finite element method (FEM) calculations and the relatively new mathematical homogenization method (MH). MH has been developed in the studies of partial differential equations, see e.g. Refs. [1–4]. MH has been applied to woven composites in Refs. [5,6], and in particular, in the present paper.

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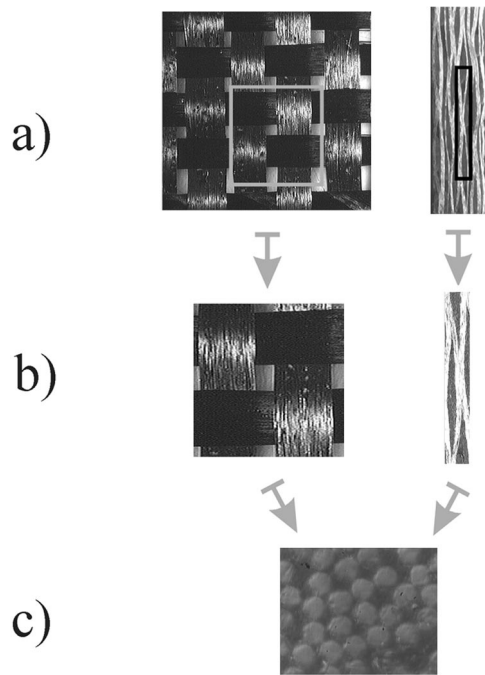


Fig. 1. Different scales of woven composites: (a) representative volume element; (b) repeatable unit cell of fabric; and (c) tow structure.

The semi-analytical models can be divided into two groups, namely the classical laminate theory (CLT) type of approach and the energy approach. The basic ideas of the energy approach models are described in Ref. [7].

A huge spectrum of studies contains the CLT type of approach. As one of the earliest here can be noted the fiber inclination model by Chou et al. [8,9]. This model divides the volume element into a number of inclined unidirectional laminates. The mosaic model proposed by Ishikawa and Chou [10,11] divides the unit cell into blocks of cross-ply laminates. The fiber continuity and undulation is taken into account by the fiber undulation model [12], and further, the bridging model [13] proposed by the same authors, combines the fiber undulation and mosaic models. Karayaka and Kurath [14] have proposed a homogenization model where they assemble flat matrix pockets as well as in-plane and interlaced flat tow regions in the 3D representative volume element. Naik and Ganesh [15] have proposed a refined version of Ishikawa and Chou's model by using a refined geometry of the RUC. The work [7] contains a more complete list of the semi-analytical models.

In general, these models contain certain analytic expressions which give relationships between the properties of the constituents (tows and matrix) and the final elastic properties of the woven composites with known morphologies of the waviness. Most of the models predict the in-plane thermo-elastic properties, but some of them can also predict the out-of-plane properties [14,15].

Practically, all semi-analytic models work on the RUC scale. The randomly shifted fabric layers can be taken into account in the homogenization procedure as it is done in

Karayaka and Kurath's model [14]. Naik and Ganesh [15] also considered a possible shift of the layers with respect to each other in their papers.

Since semi-analytic models do not cover the tow scale, experimental data for unidirectional composite laminates [16] or different engineering model expressions could be used here. Expressions like Halpin–Tsai [16] or Hashin's cylindrical model [17] are also useful to obtain the tow elastic properties.

The main advantage of semi-analytic models is their convenience of use, allowing fast evaluation of a new materials. The possibility to incorporate those models in FEM software is of great practical interest for structural design. Some of these models also allow to model damage and fracture processes [14,15].

With their simplicity, semi-analytic models have several disadvantages. This is because these models are not based on the exact solution of the governing equations, hence the obtained material properties correspond to certain assumptions. Other problems are related to the tow properties that are the input data for all semi-analytic models, therefore the final results depend on the tow level model.

Nowadays, FEM is a common tool to obtain the stress–strain state in a constrained elastic or inelastic solid. High computer performance in combination with easy-to-use commercial FEM software supports this application, and at least theoretically, it is easy to apply FEM to solid constructions with almost all levels of complexity. Many authors [18–20] use FEM in order to obtain the elastic and fracture properties on the RUC scale. The disadvantages of the FEM approach for the current problem are also well known. It is not easy to use FEM on the RVE scale—shifted layers imply complex boundary conditions and several calculations must be performed in order to obtain the elastic properties in the case of randomly shifted layers. The development time for a geometrical model of waviness is high, and it is almost impossible to create models for n -harness satin with $n \geq 5$ by using standard FEM software. The majority of all FEM models are 3D. The number of nodes necessary to get acceptable accuracy in the 3D case may be extremely high, and sometimes it is out of the range allowed by the software (or hardware). The tow properties for FEM calculations require a separate model. Since tow properties as well as the geometrical details of the RUC are only known approximately, the obtained high accuracy corresponds to the model but not to the real case. Finally, the output of the finite element model is the stress–strain field. Since we are interested in the average material properties some homogenization over the RUC is necessary. This homogenization sometimes introduces new assumptions, and the advantages of the exact solution of the stress–strain problem are lost.

The mathematical homogenization (MH) approach does not contain some of the problems discussed before. The basic idea in the MH method is that one introduces a small parameter that describes the fineness of the microstructure.

In this sense, we have a sequence of problems, one for each choice of the parameter. By letting the small parameter tending to zero, we obtain a limit problem which is used to define the effective properties. One advantage with this general approach is that the microstructure does not have to be periodic. For example, we can have almost periodicity or a realization of a stationary random field. Moreover, it is possible to apply the method to nonlinear problems. In this paper, we use the so-called reiterated homogenization (RH), described in Ref. [5] and also below in Section 4, to study a linear elastic problem with periodic microstructure. Unfortunately, the complex geometry of the woven composites RUC requires the use of numerical methods [6], therefore, the work consumption in the modeling of a new material is high compared to semi-analytical methods. In this paper, results obtained by RH are used as an exact reference in order to estimate the accuracy of simple engineering models.

The objective of this study is to develop the homogenization method for stiffness matrix computation of woven composites, and to perform a critical evaluation of the existing semi-analytical models based on experimental data. We apply several semi-analytic models and the developed RH method in order to predict the elastic properties of the tows and also of the whole 3D stiffness matrix of the woven composite.

2. Materials and testing procedures

2.1. Materials

Two different composite materials are used in the experimental studies. The E-glass fiber woven composite is the first material. The corresponding fabric is rowing plain weave cloth Ahlström R3-290 with a fabric area density of 290 g/m^2 . The second material is a TENAX HTA pitch based carbon fiber woven composite with fabric code Brochier Injectex GB200. The area density of the fabric is 205 g/m^2 .

Every fifth fill tow is twisted, therefore some local thickness inhomogeneity is introduced. In any other physical characteristics (tow density, tow type, number of fibers per tow), the fill and warp directions are equal. Although such a woven pattern differs from plain weave, the authors have not been able to find any experimentally observable difference in the elastic properties between the warp and fill directions. Thus from now on, the pattern is assumed to be plain weave.

The high-modulus unsaturated polyester resin Neste S910-4138 is used as matrix for both the glass and carbon composites. The woven laminates were manufactured at the Swedish Institute of Composites (SICOMP). The carbon plates contain eight fabric layers and the glass plates contain nine layers of fabric. For the matrix filling in the reinforcement, vacuum bag molding was used. According to the resin manufacturer's recommendations, the composite plate was

after the filling cured inside the bag during 30 min at 30°C temperature.

The complete post-curing, which continued for 4 h at 80°C (as prescribed by the resin manufacturer), was only done for the carbon fiber material. The composite specimens with glass fiber yarns only had 30 min of post-curing at 80°C , and hence the curing process was not completed. To make the modeling correct, the elastic properties of polyester samples with the same thermal history as the woven fabric composites were measured. The corresponding values of Young's moduli then differed, being 3.00 and 3.43 GPa for partially and fully cured resins (called matrix A and matrix B in the following), respectively. This difference is taken into account in the further calculations.

The thicknesses of the glass fabric and carbon fabric composite plates after curing were about 1.75 and 2.20 mm, respectively.

2.2. Testing procedures

Because of the simple test procedure and the possibility to measure two elastic constants at once, the tensile test method is chosen as the method to identify the elastic properties of the samples. The tests were carried out on a 25 kN universal servohydraulic Instron model 1272 test machine at a crosshead speed of 2 mm/min.

The specimens were cut out from a composite plate in the 45 and 0° directions with respect to the weave pattern. After polishing the specimens edges, tabs made of TE 630 glass-epoxy woven material were glued. The distance between the tabs (gauge length) was 150 mm and the specimens width were about 25 mm. Specimens were made according to ASTM standard D3878 (related to high-modulus fibers and their composites). Since the ASTM standards do not contain direct recommendations for the testing of woven composites, the standards described in Ref. [21] were used to decide specimen dimensions and loading rate.

The standard Epsilon 3542-050M-025-ST extensometer with a gauge length of 50 mm was used for the strain measurements in the load direction. The strains perpendicular to the load direction were measured with 5 mm long strain gages. Although the size of the characteristic woven RUC ($4 \times 4 \text{ mm}$) can be covered by this gage, the gage may appear too short. However, several tests performed using both the extensometer and the gage oriented in the same direction showed that the measured strain values from those different sources differed only negligible (about 1%). Neither did the standard deviations of the properties obtained from different specimens show any difference between the gage measurements and the extensometer measurements.

To insure a linear-elastic response of the material, a maximal elongation of 0.2% axial strain was allowed. Several loading-unloading steps were performed for each specimen. The in-plane moduli and Poisson's ratios were

Table 1
Characteristics of the plain weave composites RUCs

Property	E-glass/Matrix A	HTA/Matrix B
Size of the RUC λ (mm)	4.4	4.17
Thickness of the RUC h (mm)	0.2	0.28
Width of the tows L (mm)	2.0	1.37
Volume fraction of the tows	0.91	0.66
Fiber volume fraction in the tows V_f^Y	0.66	0.62
Overall fiber volume fraction V_f^C	0.60	0.41
Interlaced angle ^a θ (°)	30.0	11.3

^a Related to the other items in the table through Eq. (23).

obtained through the average elastic values on the unloading part of the loading sequence.

The size of the RUC required for the modeling was obtained by counting yarns over one meter of non-impregnated woven fabric. The tow width for the final composite was obtained from optical microscopy with a digital camera. The NIH-IMAGE software was used for both—digital image capturing and the following tow size measurements. The input values for the modeling were obtained by averaging over about 10 images of a single tow. Table 1 contains the measured characteristics of the RUC.

In order to obtain the elastic modulus of the resin, several unsaturated polyester samples with different extent of curing were tested in a three-point bending test (ASTM standard D790M for flexural properties of unreinforced and reinforced plastics). Poisson's ratio of the resin was measured in a tensile test, as described in ASTM D638-96.

3. Semi-analytical models of stiffness reduction

Since the goal of this paper is not to try all existing semi-analytical models, only two are described. A modified mosaic parallel model (MMPM) is chosen because it is simple. The Karayaka and Kurath model (KKM) is more complex and more accurate.

3.1. A modified mosaic parallel model

Semi-analytical models have different levels of complexity. The benefits of simple models are their ready-to-use, but the disadvantages are their low accuracy. As an example of a model which may be the simplest possible for the considered problem, we can mention the mosaic model proposed by Ishikawa and Chou [10,11]. This model greatly simplifies the fabric geometry by ignoring fiber continuity and undulation. The sequence of assumptions for the unit cell is schematically shown in Fig. 2.

It follows from the figure that the composite can be treated as an assemblage of pieces of unidirectional cross-ply laminates. The real RUC of the fabric is three-dimensional (3D) (Fig. 2(b)), but the mosaic model simplifies it by breaking up the plate into two types of plies. The plies are considered

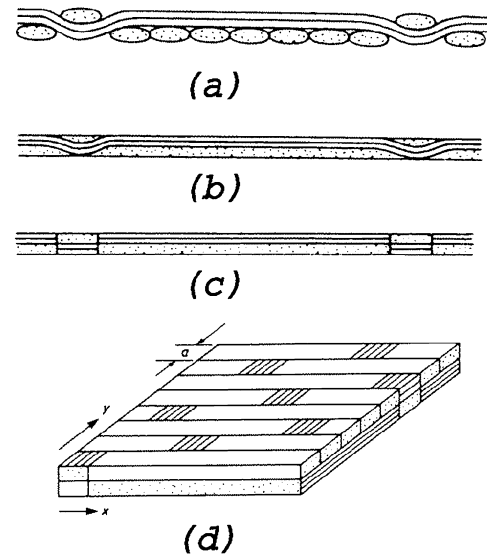


Fig. 2. Idealization of the mosaic model: (a) cross-sectional view of a woven fabric; (b) row of transverse oriented bundles "replaced" by transverse ply; (c) interlaced regions "replaced" by two plies (0 and 90°) linked in parallel; and (d) mosaic model of repeating unit cell for 8 harness satin.

to be linked in parallel (as in a cross-ply laminate) or serially, which corresponds directly to the isostrain and isostress assumptions, respectively. Knowing the properties of the constituents, the stiffness of each such a ply can be calculated by using micro-mechanics. The isostrain and isostress assumptions give upper and lower bounds for the modulus of the construction. As it is shown in the original work [11], the difference between the upper and lower bounds is high and comparable to the absolute value of the modulus for plain weave fabrics. Thus the mosaic model is not valid for plain weave fabric based composites in its initial form. However, there exists a modification of the mosaic model suggested by Naik and Ganesh (called the modified mosaic parallel model MMPM). Based on experimental data and predictions of their other models, they claim that instead of the yarn fiber volume fraction V_f^Y , the experimentally determined fiber volume fraction of the overall composite V_f^C , should be used to calculate the properties of each unidirectional ply. Then the MMPM accuracy is improved if the isostrain conditions for the in-plane and isostress conditions for the out-of-plane properties are used.

3.2. Karayaka and Kurath model

Karayaka and Kurath [14] have proposed an approach based on simple homogenization for modeling the elastic properties and failure behavior of a textile composite. Their approach leads to the determination of 3D stresses in the interlacing region of the composite. The main features of the elastic analysis are described below. The representative volume element of the woven composite is homogenized as a combination of unidirectional and interlaced regions. The 3D unit cell is considered to be assembled from idealized

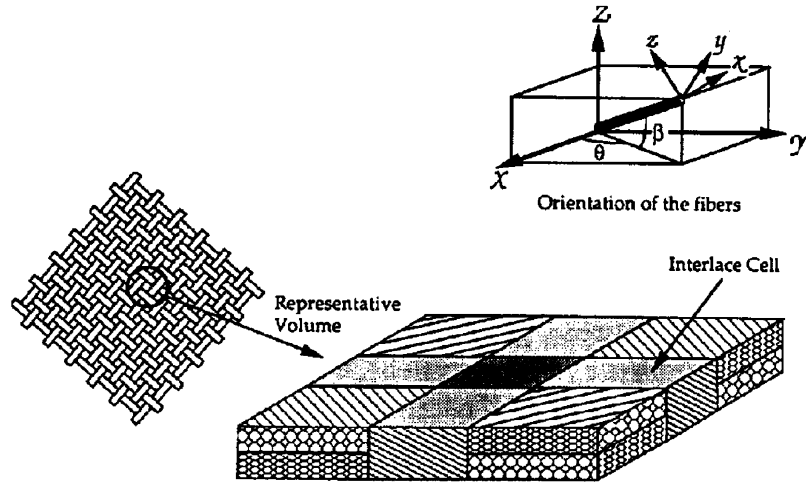


Fig. 3. Homogenization unit cell by Karayaka and Kurath.

sub-cells (Fig. 3). Two coordinate systems are chosen. The global coordinates X and Y are along the longitudinal and transverse directions of the specimen, respectively. The Z coordinate is along the thickness direction. The local coordinate system is defined by the triad x, y, z , with x along the local fiber direction. For resin pockets in between yarns, the local and global systems are equal. In the local coordinate system, the linear elastic constitutive behavior of each sub-region is given by

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} C_{11}^l & C_{12}^l & C_{13}^l & 0 & 0 & 0 \\ C_{21}^l & C_{22}^l & C_{23}^l & 0 & 0 & 0 \\ C_{31}^l & C_{32}^l & C_{33}^l & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44}^l & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55}^l & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66}^l \end{bmatrix} \begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix}, \quad (1)$$

where the superscript l denotes the stiffness tensor in the local basis. The transformation of constitutive equations for each sub-cell is done using the usual tensor transformation relations. In general this leads to a fully populated stiffness matrix defined in the global axes

$$\begin{Bmatrix} \sigma_{XX} \\ \sigma_{YY} \\ \sigma_{ZZ} \\ \tau_{YZ} \\ \tau_{XZ} \\ \tau_{XY} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_{XX} \\ \epsilon_{YY} \\ \epsilon_{ZZ} \\ \gamma_{YZ} \\ \gamma_{XZ} \\ \gamma_{XY} \end{Bmatrix}. \quad (2)$$

Eq. (2) is then decomposed into the in-plane components

$$\begin{Bmatrix} \sigma_{XX} \\ \sigma_{YY} \\ \tau_{XY} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{16} \\ C_{21} & C_{22} & C_{26} \\ C_{61} & C_{62} & C_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_{XX} \\ \epsilon_{YY} \\ \gamma_{XY} \end{Bmatrix} + \begin{bmatrix} C_{13} & C_{14} & C_{15} \\ C_{23} & C_{24} & C_{25} \\ C_{63} & C_{64} & C_{65} \end{bmatrix} \begin{Bmatrix} \epsilon_{ZZ} \\ \gamma_{YZ} \\ \gamma_{XZ} \end{Bmatrix} \quad (3)$$

and out-of-plane components

$$\begin{Bmatrix} \sigma_{ZZ} \\ \tau_{YZ} \\ \tau_{XZ} \end{Bmatrix} = \begin{bmatrix} C_{31} & C_{32} & C_{36} \\ C_{41} & C_{42} & C_{46} \\ C_{51} & C_{52} & C_{56} \end{bmatrix} \begin{Bmatrix} \epsilon_{XX} \\ \epsilon_{YY} \\ \gamma_{XY} \end{Bmatrix} + \begin{bmatrix} C_{33} & C_{34} & C_{35} \\ C_{43} & C_{44} & C_{45} \\ C_{53} & C_{54} & C_{55} \end{bmatrix} \begin{Bmatrix} \epsilon_{ZZ} \\ \gamma_{YZ} \\ \gamma_{XZ} \end{Bmatrix}. \quad (4)$$

Eqs. (3) and (4) can be written in a shorter form as follows

$$\begin{cases} \sigma_{in} = C_1 \epsilon_{in} + C_2 \epsilon_{out}, \\ \sigma_{out} = C_3 \epsilon_{in} + C_4 \epsilon_{out}, \end{cases} \quad (5)$$

where the subscripts 'in' and 'out' denote the in-plane and out-of-plane components of the corresponding stresses and strains.

The effective stresses and strains are defined as the volumetric average of the actual stresses and strains over every sub-cell, that is, uniform stress and strain are assumed in each sub-cell. The following boundary conditions are used at the interface of the sub-cells:

$$\begin{cases} \bar{\epsilon}_{in} = \epsilon_{in}^k, & k = 1, \dots, N, \\ \bar{\sigma}_{out} = \sigma_{out}^k, & k = 1, \dots, N, \end{cases} \quad (6)$$

where N is the number of sub-cells in the representative volume and the average stresses and strains are $\bar{\sigma}$ and $\bar{\epsilon}$, respectively. By rearranging Eq. (5), averaging the result over all sub-cells and using Eq. (6), the effective in-plane stresses and out-of-plane strains are obtained as

$$\begin{cases} \bar{\sigma}_{\text{in}} = C_1^* \bar{\epsilon}_{\text{in}} + C_2^* \bar{\sigma}_{\text{out}}, \\ \bar{\epsilon}_{\text{out}} = C_3^* \bar{\sigma}_{\text{out}} - C_4^* \bar{\epsilon}_{\text{in}}, \end{cases} \quad (7)$$

where

$$C_1^* = \sum_{k=1}^N f_k (C_1 - C_2 C_4^{-1} C_3)_k, \quad (8)$$

$$C_2^* = \sum_{k=1}^N f_k (C_2 C_4^{-1})_k, \quad (9)$$

$$C_3^* = \sum_{k=1}^N f_k (C_4^{-1})_k, \quad (10)$$

$$C_4^* = \sum_{k=1}^N f_k (C_4^{-1} C_3)_k, \quad (11)$$

and f_k is the volume fraction of the k th sub-cell in the representative unit cell.

By solving the system of Eq. (7), we obtain the relation between the average stresses and average strains as

$$\begin{aligned} \begin{Bmatrix} \bar{\sigma}_{\text{in}} \\ \bar{\sigma}_{\text{out}} \end{Bmatrix} &= \begin{bmatrix} C_1^* + C_2^* C_3^{*-1} C_4^* & C_2^* C_3^{*-1} \\ C_3^{*-1} C_4^* & C_3^{*-1} \end{bmatrix} \begin{Bmatrix} \bar{\epsilon}_{\text{in}} \\ \bar{\epsilon}_{\text{out}} \end{Bmatrix} \\ &= Q \begin{Bmatrix} \bar{\epsilon}_{\text{in}} \\ \bar{\epsilon}_{\text{out}} \end{Bmatrix}. \end{aligned} \quad (12)$$

Since the matrix Q in Eq. (12) is the stiffness matrix of the homogenized composite, the in-plane elastic constants are

$$\begin{aligned} E_X &= 1/S_{11}, & E_Y &= 1/S_{22}, & G_{XY} &= 1/S_{66}, \\ \nu_{XY} &= -S_{12}/S_{11}, \end{aligned} \quad (13)$$

and out-of-plane elastic constants are

$$\begin{aligned} E_Z &= 1/S_{33}, & G_{YZ} &= 1/S_{44}, & G_{XZ} &= 1/S_{55}, \\ \nu_{YZ} &= -S_{23}/S_{22}, & \nu_{XZ} &= -S_{13}/S_{11}, \end{aligned} \quad (14)$$

where $S = Q^{-1}$. From the symmetry of our plain-weave pattern and the assumption of transverse isotropy of the tows follows that $E_X = E_Y$, $G_{XY} = G_{XZ}$ and $\nu_{XZ} = \nu_{YZ}$.

Both the methods (MMPM and KKM) described above are implemented in ANSI C source code and tested in the Linux OS. These codes are freely obtainable by contacting the authors. The results of the modeling for different woven composites are presented in the discussion part.

4. Reiterated homogenization

The idea of reiterated homogenization can for instance be found in the book by Bensoussan et al. [1]. In order to derive the governing equations, we consider a linear elastic body which occupies a region Ω in $\mathbf{R}^3$ and introduce a Cartesian coordinate system (x_i) . Moreover, we define $\sigma = (\sigma_{ij})$, $\mathbf{f} = (f_i)$, $\mathbf{t} = (t_i)$, $\mathbf{u} = (u_i)$ and $\mathbf{n} = (n_i)$, as the stress tensor, the internal force field, the surface force field, the displacement field and the outer unit normal to the boundary $\partial\Omega$ of Ω , respectively. The equilibrium equations for the body are

$$\begin{cases} \nabla \cdot \sigma + \mathbf{f} = \mathbf{0} \text{ in } \Omega, \\ \mathbf{u} = \mathbf{0} \text{ on } \Gamma_1 \text{ and } \sigma \mathbf{n} = \mathbf{t} \text{ on } \Gamma_2, \end{cases}$$

where $\Gamma_1 \cap \Gamma_2 = \emptyset$ and $\Gamma_1 \cup \Gamma_2 = \partial\Omega$. Hooke's law states that

$$\sigma = C e, \quad (15)$$

where the product between the fourth-order tensor $C = (C_{ijkl})$ and the second-order tensor $e = (e_{kl})$ should be interpreted as

$$(Ce)_{ij} = C_{ijkl} e_{kl}$$

from now on. Here $e = (e_{kl})$ is the strain tensor defined as

$$e(\mathbf{u}) = \frac{1}{2} [\nabla \mathbf{u} + (\nabla \mathbf{u})^T]$$

and C_{ijkl} is the tensor of elastic moduli (or stiffness tensor) with the symmetries $C_{ijkl} = C_{ijlk} = C_{jikl} = C_{klij}$. Using these equations, the equilibrium equations become

$$\begin{cases} -\nabla \cdot (Ce(\mathbf{u})) = \mathbf{f} \text{ in } \Omega, \\ \mathbf{u} = \mathbf{0} \text{ on } \Gamma_1 \text{ and } Ce(\mathbf{u})\mathbf{n} = \mathbf{t} \text{ on } \Gamma_2. \end{cases}$$

Let us now assume that the body consists of two or more different linear elastic materials which are periodically distributed in the sense that we can define unit cells which are periodically repeated. In this case, we assume that we have two different scales of periodicity. Furthermore, assume that $C(\mathbf{y}, \mathbf{z})$ is Y - Z -periodic, i.e. this means that $C(\mathbf{y}_1, \mathbf{z}_1) = C(\mathbf{y}_2, \mathbf{z}_2)$ whenever $(\mathbf{y}_1, \mathbf{z}_1)$ and $(\mathbf{y}_2, \mathbf{z}_2)$ occupy the same positions in their corresponding cells. Thus we introduce the local variables $\mathbf{y} = \mathbf{x}/\epsilon$ and $\mathbf{z} = \mathbf{x}/\epsilon^2$ and put $C^\epsilon = C(\mathbf{x}/\epsilon, \mathbf{x}/\epsilon^2) = C(\mathbf{y}, \mathbf{z})$.

We now consider the following class of problems, one problem for each choice of ϵ ,

$$\begin{cases} -\nabla \cdot (C^\epsilon e(\mathbf{u}^\epsilon)) = \mathbf{g} \text{ in } \Omega, \\ \mathbf{u}^\epsilon = \mathbf{0} \text{ on } \Gamma_1 \text{ and } C^\epsilon e(\mathbf{u}^\epsilon)\mathbf{n} = \mathbf{t} \text{ on } \Gamma_2. \end{cases}$$

We want to approximate the solutions $\mathbf{u}^\epsilon$ by means of a function $\mathbf{u}$ which solves the problem corresponding to a homogenized material,

$$\begin{cases} -\nabla \cdot (\bar{C} e(\mathbf{u})) = \mathbf{f} \text{ in } \Omega, \\ \mathbf{u} = \mathbf{0} \text{ on } \Gamma_1 \text{ and } \bar{C} e(\mathbf{u})\mathbf{n} = \mathbf{t} \text{ on } \Gamma_2. \end{cases} \quad (16)$$

where $\bar{C}$ is a constant tensor whose response is close (in

Table 2
Measured in-plane properties of the plain weave composite

Property	E-glass/matrix A	HTA/Matrix B
Young's modulus (GPa)	26.3 ± 2	46.5 ± 2
Poisson's ratio	0.14 ± 0.01	0.05 ± 0.009
Shear modulus ^a (GPa)	4.8 ± 0.3	4.2 ± 0.3
Shear modulus ^b (GPa)	4.5 ± 0.3	4.0 ± 0.3

^a Modulus $E_X^{(45)}$ of 45° specimen was used (Eq. (21)).

^b Modulus $E_X^{(45)}$ and Poisson's ratio $\nu_{XY}^{(45)}$ was used (Eq. (22)).

some context) to the response of the actual heterogeneous tensor C . To find this tensor, we assume a multiple scales expansion of $\mathbf{u}^\epsilon(\mathbf{x})$ of the form

$$\mathbf{u}^\epsilon(\mathbf{x}) = \mathbf{u}^{(0)}(\mathbf{x}, \mathbf{y}, \mathbf{z}) + \epsilon \mathbf{u}^{(1)}(\mathbf{x}, \mathbf{y}, \mathbf{z}) + \epsilon^2 \mathbf{u}^{(2)}(\mathbf{x}, \mathbf{y}, \mathbf{z}) + \dots,$$

where $\mathbf{u}^{(n)}$, $n = 0, 1, 2, \dots$, are Y -periodic in the coordinate $\mathbf{y}$ and Z -periodic in the coordinate $\mathbf{z}$. Let us denote $e_a(\mathbf{u}) = 1/2[\nabla_a \mathbf{u} + (\nabla_a \mathbf{u})^T]$ for $a = x, y, z$. Then reiterated homogenization gives the homogenized tensor

$$\bar{C} = \frac{1}{|Y|} \int_Y \hat{C}(\mathbf{y}) + \hat{C}(\mathbf{y}) e_y(W_\eta) d\mathbf{y}, \quad (17)$$

where $W_\eta = (W_\eta^{rs})$ are the Y -periodic weak solutions of the Y cell problem

$$\nabla_y \cdot (\hat{C}(\mathbf{y}) e_y(\eta + W_\eta)) = \mathbf{0}, \quad (18)$$

with the boundary conditions $\eta_i^{rs} = \delta_{ir} y_s$. Here, $\hat{C}(\mathbf{y})$ is the Z cell homogenized tensor given by

$$\hat{C}(\mathbf{y}) = \frac{1}{|Z|} \int_Z C(\mathbf{y}, \mathbf{z}) + C(\mathbf{y}, \mathbf{z}) e_z(V_\zeta) d\mathbf{z}, \quad (19)$$

where $V_\zeta = (V_\zeta^{rs})$ are the Z -periodic weak solutions of the Z cell problem

$$\nabla_z \cdot (C(\mathbf{y}, \mathbf{z}) e_z(\zeta + V_\zeta)) = \mathbf{0}, \quad (20)$$

with the boundary conditions $\zeta_i^{rs} = \delta_{ir} z_s$. We also obtain the following expression for the vector $\mathbf{g}$:

$$\mathbf{g} = \frac{1}{|Y||Z|} \int_Y \int_Z \mathbf{f} d\mathbf{y} d\mathbf{z}.$$

See Appendix A for a simple derivation of the above

Table 3
Fiber properties

Property	E-glass/Matrix A	HTA/Matrix B
Axial Young's modulus (GPa)	72.4	238
Transverse Young's modulus (GPa)	72.4	40 ^a
In plane shear modulus (GPa)	29.8	22 ^b
Major Poisson's ratio	0.22	0.20
Transverse Poisson's ratio	0.22	0.25 ^b
Density (g/cm ³)	2.54	1.77

^a Transverse properties for PAN based carbon fibers.

^b Back calculation from ply properties of tested GB200/RTM6 plies.

formulae. In short, the overall homogenization can be computed by the reiterated homogenization procedure:

1. solve the Z cell problem (Eq. (20)) numerically;
2. compute the Z homogenized tensor (Eq. (19));
3. solve the Y cell problem (Eq. (18)) numerically;
4. compute the overall homogenized tensor by Eq. (17);
5. insert the homogenized tensor into the homogenized Eq. (16) and solve the problem.

5. Results and discussions

5.1. Experimental results

The results of the uniaxial tensile tests of woven composites are summarized in Table 2. The values of Young's modulus and Poisson's ratio are measured directly on specimens with zero degree orientation (0,90) with respect to the weave pattern. The shear modulus G_{XY} is calculated from the elastic properties obtained in tests on (0,90) and (± 45) specimens. Since both the modulus and Poisson's ratio are measured for all specimens, there are two separate ways of how to calculate the shear moduli based on stiffness tensor transformations. The expressions of G_{XY} are (see Ref. [16])

$$G_{XY} = m^2 n^2 \left[\frac{1}{E_X^{(45)}} - \frac{m^4}{E_X} - \frac{n^4}{E_Y} + \frac{2\nu_{XY}}{E_X} \right]^{-1} \quad (21)$$

and

$$G_{XY} = m^2 n^2 \left[\frac{\nu_{XY}^{(45)}}{E_X^{(45)}} + \frac{m^4}{E_X} + \frac{n^4}{E_Y} - (m^4 + n^4) + \frac{2\nu_{XY}}{E_X} \right]^{-1}, \quad (22)$$

where $E_X^{(45)}$ and $\nu_{XY}^{(45)}$ are the elastic properties of the (± 45) specimens, $m^2 = \cos^2 \theta = 1/2$ and $n^2 = \sin^2 \theta = 1/2$. From Table 2, shear modulus values obtained with different data reduction procedures differ by 8%. This difference may correspond to the overall accuracy of the measurements. A possible source of this may be the natural deviations in specimen properties. Also, the loading curve nonlinearities of the (± 45) specimens (softening) and the HTA carbon/Matrix B (0,90) specimens (fiber stiffening) are open for discussion.

5.2. Input data for modeling

The necessary properties of the fibers are collected in Table 3. Only the longitudinal modulus E_L and the fiber density were given by the producer of the carbon fibers (the fiber density is used to obtain the volume fractions V_f^Y and V_f^C). All other properties come from different sources, and the accuracies of these data are unknown.

The properties of the other constituents are more accurate. The glass fibers are made of standard E-glass, see Table 3. The matrix test results are summarized in Table 4.

Elastic properties of the unidirectional composites are the

Table 4
Measured polyester NesteS910-4138 matrix properties

Property	E-glass/Matrix A	HTA/Matrix B
Postcuring time (h)	0.5	4
Young's modulus (GPa)	3.0 ± 0.05	3.4 ± 0.04
Poisson's ratio	0.38 ± 0.02	0.36 ± 0.02

input data for both semi-analytical models (tow properties in KKM and effective layer properties in MMPM). We have used two different analytical models and mathematical homogenization theory for the predictions of them. In all models, the composite is assumed to be macroscopically transversely isotropic, therefore

$$G_{TT} = \frac{E_T}{2(1 + \nu_{TT})}$$

with L being the fiber direction and T being the transverse direction. The Halpin–Tsai expressions with the parameters suggested in Ref. [16] and Hashin's model [17] (random concentric cylinder cell, lower bound) are chosen because they are more accurate compared to the constant stress models and can be used for anisotropic fibers. Halpin–Tsai's model can only predict transverse and shear in-plane moduli. The rule of mixtures is used for the rest of the elastic properties. This approach will be denoted “Halpin–Tsai”. Hexagonal (in the plane of isotropy) unit cells were used for mathematical homogenization, because they are the simplest cells which fulfill the requirement of transverse isotropy. Tables 5 and 6 contains tow properties for both glass and carbon fibers. From the table, all micro-mechanics models predict almost the same longitudinal modulus E_L and major Poisson's ratio ν_{LT} . The situation with the rest of the elastic properties is more complex, because predictions of different models for transverse modulus E_T and shear modulus G_{LT} differs a lot, with the maximum error being about 30% of the absolute value. Halpin–Tsai's model have a tendency to give higher E_T and G_{LT} compared to homogenization theory, while Hashin's model give lower values (since it is a lower bound). However, in the glass-fiber case Hashin's model gives an extremely high transverse Poisson's ratio ν_{TT} . Still, the predictions of Hashin's model seem closer to the MH results than Halpin–Tsai's model does. Do the exact results of mathematical homogenization correspond to the

Table 5
Tow properties; results of different models

E-glass/Matrix A, $V_f^Y = 0.66$			
Property	MH	Hashin	Halpin–Tsai
Axial Young's modulus E_L (GPa)	48.6	48.6	48.6
Transverse Young's modulus E_T (GPa)	13.6	11.0	15.6
Major Poisson's ratio ν_{LT}	0.27	0.27	0.27
Transverse Poisson's ratio ν_{TT}	0.25	0.26	0.27
Shear modulus G_{LT} (GPa)	4.41	4.53	6.25

Table 6
Tow properties; results of different models

HTA carbon/Matrix B, $V_f^Y = 0.62$			
Property	MH	Hashin	Halpin–Tsai
Axial Young's modulus E_L (GPa)	149	149	149
Transverse Young's modulus E_T (GPa)	11.5	10.0	13.1
Major Poisson's ratio ν_{LT}	0.25	0.25	0.26
Transverse Poisson's ratio ν_{TT}	0.42	0.54	0.30
Shear modulus G_{LT} (GPa)	4.05	4.38	5.94

reality? We do not know, because of the inaccuracy of the constituent properties.

5.3. Results of modeling

We used the semi-analytical models as described in Section 3 and the RH method with RUC dimensions from Table 1 in order to obtain the elastic properties of the woven composites E-glass/Matrix A and HTA carbon/Matrix B. In order to emulate random lay-up of composites in the Z (thickness) direction, we used a RVE for RH that consisted of two layers of the plain weave, shifted with respect to each other by $DX = \lambda/4$, $DY = \lambda/4$ in the XY plane. The results of all discussed models are presented in Tables 7–10. Tables 7 and 8 contain predictions for the in-plane properties of E-glass/Matrix A and HTA carbon/Matrix B, respectively. Let us focus on the results from the more exact RH model and evaluate the model's agreement with the experiments, see Table 2. The comparison concludes that in the E-glass/Matrix A composite case, the agreement for the in-plane modulus E_X and Poisson's ratio ν_{XY} is good (relative deviation $\leq 3\%$). For the HTA carbon/Matrix B composite, the E_X value has a deviation of about 2%, which is good. One possible reason for this disagreement may be wrong assumptions about the crimp of yarn in the unit cell—to obtain a flat RUC we assumed that the value of the fiber inclination angle θ is related to the other RUC

Table 7
Predicted in-plane properties of the E-glass/Matrix A woven composite

Property	Tow model	Homogenization		
		RH	KKM	MMPM ^a
Young's modulus $E_X = E_Y$ (GPa)	MH	26.5	27.4	28.0
	Hashin		26.2	27.0
	Halpin–Tsai		28.4	30.0
Poisson's ratio ν_{XY}	MH	0.12	0.12	0.11
	Hashin		0.11	0.10
	Halpin–Tsai		0.14	0.13
Shear modulus G_{XY} (GPa)	MH	3.9	4.1	3.6
	Hashin		3.7	3.8
	Halpin–Tsai		5.8	5.3

^a Fiber volume fraction V_f^C for each ply is 0.60.

Table 8
Predicted in-plane properties of the HTA carbon/Matrix B woven composite

Property	Tow model	Homogenization		
		RH	KKM	MMPM ^a
Young's modulus $E_X = E_Y$ (GPa)	MH	45.5	44.3	53.7
	Hashin		43.9	53.0
	Halpin–Tsai		45.2	53.9
Poisson's ratio ν_{XY}	MH	0.052	0.047	0.039
	Hashin		0.044	0.036
	Halpin–Tsai		0.052	0.045
Shear modulus G_{XY} (GPa)	MH	3.0	3.1	2.5
	Hashin		3.3	2.7
	Halpin–Tsai		4.3	3.4

^a Fiber volume fraction V_f^C for each ply is 0.41.

characteristics as (see Fig. 4)

$$\sin \theta = \frac{h}{\lambda - 2L} \quad (23)$$

where λ , L and h come from Table 1. Any other θ value leads to a Z-twisted RUC. To obtain correct values of θ , more micro-structural investigations must be performed.

The values of the shear moduli are quite inaccurate for both glass and carbon composites. In both cases, the theories predict lower shear moduli, which is not surprising, since the predicted shear moduli of the tows (Table 3) have quite low values (except the predictions of the Halpin–Tsai model, which we assume to be more inaccurate). For glass fibers, the tow shear modulus from the MH method is even lower than the overall measured shear modulus of the composite. The sources of this inaccuracy are not known.

The out-of-plane properties were not measured, therefore a comparison with experiments is not available. Comparisons between different models (Tables 7 and 8 for the in-plane properties) show that the simple MMPM (with MH or Hashin ply properties) slightly overestimate E_X and E_Y for the E-glass/Matrix A composite, but it fails to predict the moduli E_X , E_Y of the carbon HTA carbon/Matrix B

Table 9
Predicted out-of-plane properties of the E-glass/Matrix A woven composite

Property	Tow model	Homogenization		
		RH	KKM	MMPM ^a
Young's modulus E_Z (GPa)	MH	12.4	12.7	12.5
	Hashin		11.9	12.0
	Halpin–Tsai		13.6	13.5
Major Poisson's ratio $\nu_{XZ} = \nu_{YZ}$	MH	0.39	0.39	0.40
	Hashin		0.49	0.49
	Halpin–Tsai		0.31	0.29
Shear modulus $G_{XZ} = G_{YZ}$ (GPa)	MH	3.8	4.3	3.9
	Hashin		3.7	2.9
	Halpin–Tsai		5.5	5.2

^a Fiber volume fraction V_f^C for each ply is 0.60.

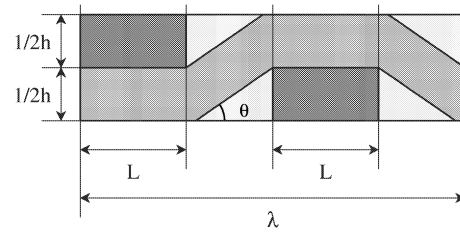


Fig. 4. Dimensions of idealized RUC.

composite. This is related to the geometry of the different RUCs, see Table 1 and Fig. 4. The glass fiber fabric contains large flat regions with short interlaced parts, which corresponds to a high volume fraction of the tows. Because the amount of the flat tow regions (Fig. 4) is so high (83% of the XY area in the RUC), the elastic properties of the woven composite are almost the same as for a cross-ply laminate, and thus the MMPM model works fine. The studied carbon fabric has a smaller amount of flat tow regions (only 43% of the XY area in the RUC), and hence regions with fiber inclination are more important. The cross-ply model with straight fibers only (MMPM) does in this case not work. Even if the out-of-plane properties for the MMPM assumptions are extremely rough, MMPM still gives acceptable (as compared to the RH and KKM models) out-of-plane elastic constants (Tables 9 and 10) for both the HTA carbon/Matrix B as well as for the E-glass/Matrix A woven composites.

The KKM (MH tows) and RH model gives quite close in-plane and out-of-plane values of the elastic properties (Tables 7–10). At least these values are practically always much closer to each other than to the experimental data. This gives some conclusions about the necessary complexity of the model—on the RUC scale, there is no point in using models that are more complex such as KKM. The major problem is to obtain the right tow elastic properties and to find a good approximation of the RUC geometry, because these parameters affect a larger part of the final model's error.

Table 10
Predicted out-of-plane properties of the HTA carbon/Matrix B woven composite

Property	Tow model	Homogenization		
		RH	KKM	MMPM ^a
Young's modulus E_Z (GPa)	MH	9.2	8.7	9.1
	Hashin		8.5	9.1
	Halpin–Tsai		8.9	8.8
Major Poisson's ratio $\nu_{XZ} = \nu_{YZ}$	MH	0.43	0.47	0.44
	Hashin		0.51	0.49
	Halpin–Tsai		0.41	0.34
Shear modulus $G_{XZ} = G_{YZ}$ (GPa)	MH	2.7	2.8	2.5
	Hashin		2.6	2.1
	Halpin–Tsai		3.1	3.1

^a Fiber volume fraction V_f^C for each ply is 0.41.

The results of parametric studies of the influence of the tow inclination angle θ on the in-plane modulus E_X by using KKM are shown in Fig. 5 (the calculations show that the variations of θ from 0 to 35° give negligible effect on the values of all other elastic constants). The two curves correspond to the E-glass/Matrix A and HTA carbon/Matrix B woven composites. As follows from previous discussions, the drop in the modulus value with an increase in θ for the HTA carbon/Matrix B composite is much higher than for the E-glass/Matrix A composite. The big dots on the curves correspond to the θ values from Table 1 and the square marks correspond to the measured E_X values (Tables 7 and 8). The difference in θ between these two markers for HTA carbon/Matrix B are only 3°, thus it is fully possible that a wrong choice of the interlaced angle θ causes an observable disagreement between the predicted and the measured values. Even if it requires a very high accuracy of the measurement, it is possible to measure the real value of θ to obtain a verification of the analysis above.

6. Conclusions

1. The applicability of approximate analytical models for elastic properties determination of woven composites is analyzed comparing with results obtained using the reiterated homogenization method and comparing with test data. Calculations are performed on several scales: one on the fiber scale to obtain tow properties and one on the RUC scale to determine the macroscopic elastic properties of the woven composite.
2. Tow properties are obtained using the homogenization method as well as Hashin's concentric cylinder model and the Halpin–Tsai expressions.
3. The MMPM by Naik and Ganesh, KKM by Karayaka and Kurath and RH are used to obtain macro-properties.
4. Two plain weave fabric reinforced polyester matrix composites (glass fiber and carbon fiber) are used in order to create experimental data for evaluation of the models.

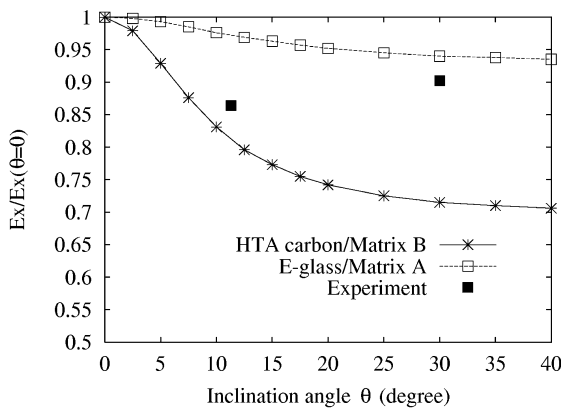


Fig. 5. Parametric studies of the influence of the inclination angle on the normalized in-plane modulus E_X .

5. Analysis show that the most critical is the tow elastic properties determination. Both Hashin model and the Halpin–Tsai expressions give rather different results than the homogenization method. The exact architecture of the reinforcement (the orientation angle in the interlaced region) may have significant effect on the predictions, and the introduced inaccuracy in this step may explain the deviations with the test results.
6. Starting with the same tow properties, the MMPM model may in some cases (carbon fiber composite) lead to rather erroneous results even for in-plane properties. Karayaka and Kurath's model gives results that are accurate enough for practical applications.

Acknowledgements

We want to thank Professor Lars-Erik Persson, Department of Mathematics, Luleå University of Technology, Sweden, for generous advises and fruitful discussions concerning questions connected to this paper. Moreover, we thank the Swedish Research Council for Engineering Sciences (grant TFR 96-422) for financial support.

Appendix A. Derivation of reiterated homogenization expressions

We recall the model problem

$$\begin{cases} -\nabla \cdot (C^\epsilon e(\mathbf{u}^\epsilon)) = \mathbf{f} & \text{in } \Omega, \\ \mathbf{u}^\epsilon = \mathbf{0} & \text{on } \Gamma_1 \text{ and } C^\epsilon e(\mathbf{u}^\epsilon) \mathbf{n} = \mathbf{t} & \text{on } \Gamma_2, \end{cases} \quad (\text{A1})$$

and assume a multiple scales expansion of $\mathbf{u}^\epsilon(\mathbf{x})$ on the form

$$\mathbf{u}^\epsilon(\mathbf{x}) = \mathbf{u}^{(0)}(\mathbf{x}, \mathbf{y}, \mathbf{z}) + \epsilon \mathbf{u}^{(1)}(\mathbf{x}, \mathbf{y}, \mathbf{z}) + \epsilon^2 \mathbf{u}^{(2)}(\mathbf{x}, \mathbf{y}, \mathbf{z}) + \dots, \quad (\text{A2})$$

where $\mathbf{u}^{(n)}$, $n = 0, 1, 2, \dots$, Y-periodic in the coordinate $\mathbf{y}$ and Z-periodic in the coordinate $\mathbf{z}$. Moreover, we define $e(\mathbf{u})$ by

$$e(\mathbf{u}) = \frac{1}{2} [\nabla \mathbf{u} + (\nabla \mathbf{u})^T].$$

Then according to the chain rule, we then obtain (recall that $\mathbf{y} = \mathbf{x}/\epsilon$ and $\mathbf{z} = \mathbf{x}/\epsilon^2$)

$$e(\mathbf{u}) = \frac{1}{2} \left[\nabla \mathbf{u}_x + (\nabla \mathbf{u}_x)^T + \frac{1}{\epsilon} (\nabla \mathbf{u}_y + (\nabla \mathbf{u}_y)^T) + \frac{1}{\epsilon^2} (\nabla \mathbf{u}_z + (\nabla \mathbf{u}_z)^T) \right].$$

Now we insert the expression for $\mathbf{u}^\epsilon$ and expand $e(\mathbf{u}^\epsilon)$. Then the strain field $e(\mathbf{u}^\epsilon)$ can be written

$$e(\mathbf{u}^\epsilon) = \epsilon^{-2} e_0 + \epsilon^{-1} e_1 + e_2 + \epsilon e_3 + \dots, \quad (\text{A3})$$

where we have defined $e_\alpha(\mathbf{u}^{(i)})$ by

$$e_\alpha(\mathbf{u}^{(i)}) = \frac{1}{2}[\nabla_\alpha \mathbf{u}^{(i)} + (\nabla_\alpha \mathbf{u}^{(i)})^T], \quad \alpha = x, y, z,$$

and then introduced e_i through

$$\begin{aligned} e_0 &= e_z(\mathbf{u}^{(0)}), & e_1 &= e_y(\mathbf{u}^{(0)}) + e_z(\mathbf{u}^{(1)}), \\ e_i &= e_x(\mathbf{u}^{(i-2)}) + e_y(\mathbf{u}^{(i-1)}) + e_z(\mathbf{u}^{(i)}), & i &= 2, 3, 4, \dots \end{aligned} \quad (\text{A4})$$

Consequently, by Hooke's law (Eq. (15)), the stress field $\sigma(\mathbf{u}^\epsilon)$ can be represented as

$$\sigma(\mathbf{u}^\epsilon) = C^\epsilon(\mathbf{y}, \mathbf{z})e(\mathbf{u}^\epsilon) = \epsilon^{-2}\sigma_0 + \epsilon^{-1}\sigma_1 + \sigma_2 + \epsilon\sigma_3 + \dots, \quad (\text{A5})$$

where

$$\sigma_i = C^\epsilon(\mathbf{y}, \mathbf{z})e_i.$$

Here and from now on, the product between the fourth-order tensor $C = (C_{ijkl})$ and the second-order tensor $e = (e_{kl})$ should be interpreted as

$$(Ce)_{ij} = C_{ijkl}e_{kl},$$

and the divergence of this quantity should be interpreted as

$$(\nabla_\alpha \cdot (Ce))_i = \frac{\partial}{\partial \alpha_j} C_{ijkl}e_{kl}.$$

By putting Eqs. (A3) and (A4) into Eq. (A1) and using

$$\nabla = \nabla_x + \epsilon^{-1}\nabla_y + \epsilon^{-2}\nabla_z,$$

we see that Eq. (A1) can be written as

$$\begin{aligned} &(\epsilon^{-4}A_1 + \epsilon^{-3}A_2 + \epsilon^{-2}A_3 + \epsilon^{-1}A_4 + A_5)(\mathbf{u}^{(0)} + \epsilon\mathbf{u}^{(1)} \\ &+ \epsilon^2\mathbf{u}^{(2)} + \dots) \\ &= \mathbf{f}, \end{aligned}$$

where the operators $A_1, \dots, A_5$ are defined by

$$\begin{aligned} A_1 &= -\nabla_z \cdot (Ce_z), & A_2 &= -\nabla_z \cdot (Ce_y) - \nabla_y \cdot (Ce_z), \\ A_3 &= -\nabla_z \cdot (Ce_x) - \nabla_y \cdot (Ce_y) - \nabla_x \cdot (Ce_z), & (\text{A6}) \\ A_4 &= -\nabla_y \cdot (Ce_x) - \nabla_x \cdot (Ce_y), & A_5 &= -\nabla_x \cdot (Ce_x). \end{aligned}$$

By collecting powers of ϵ , we see that the following five lowest order equations are relevant to solve:

$$\mathbf{0} = A_1\mathbf{u}^{(0)}, \quad (\text{A7})$$

$$\mathbf{0} = A_1\mathbf{u}^{(1)} + A_2\mathbf{u}^{(0)}, \quad (\text{A8})$$

$$\mathbf{0} = A_1\mathbf{u}^{(2)} + A_2\mathbf{u}^{(1)} + A_3\mathbf{u}^{(0)}, \quad (\text{A9})$$

$$\mathbf{0} = A_1\mathbf{u}^{(3)} + A_2\mathbf{u}^{(2)} + A_3\mathbf{u}^{(1)} + A_4\mathbf{u}^{(0)}, \quad (\text{A10})$$

$$\mathbf{f} = A_1\mathbf{u}^{(4)} + A_2\mathbf{u}^{(3)} + A_3\mathbf{u}^{(2)} + A_4\mathbf{u}^{(1)} + A_5\mathbf{u}^{(0)}. \quad (\text{A11})$$

In order to solve these equations, we need the following lemma.

Lemma A1. Let F_i , $i = 1, 2, 3$ be square integrable functions over A and consider the boundary value problem

$$A_1\Phi = \mathbf{F}, \quad \Phi \text{ } A\text{-periodic in } \mathbf{a}.$$

Then the following holds:

1. An A -periodic solution exists if and only if $\frac{1}{|A|} \int_A \mathbf{F} d\mathbf{a} = \mathbf{0}$,
2. If an A -periodic solution exists, then it is unique up to an additive constant (with respect to $\mathbf{a}$) vector $\mathbf{c}$.

Proof. See for instance the book by Persson et al. [4].

Clearly Eq. (A7) has one solution $\mathbf{u}^{(0)} = \mathbf{0}$. Thus by the lemma it follows that

$$\mathbf{u}^{(0)}(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \tilde{\mathbf{u}}_0(\mathbf{x}, \mathbf{y}). \quad (\text{A12})$$

By noting that $\langle \mathbf{A}_2\mathbf{u}^{(0)} \rangle_Z = \mathbf{0}$ (by Z -periodicity), we can use the lemma on Eq. (A8). Thus we see that the solution can be written

$$\mathbf{u}^{(1)}(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \mathbf{V}_\zeta^{rs}(\mathbf{z})(e_y(\tilde{\mathbf{u}}_0(\mathbf{x}, \mathbf{y})))_{rs} + \tilde{\mathbf{u}}_1(\mathbf{x}, \mathbf{y}), \quad (\text{A13})$$

where $V_\zeta(\mathbf{z}) = (\mathbf{V}_\zeta^{rs}(\mathbf{z}))$ is determined by the Z cell problems

$$(A_1\mathbf{V}_\zeta^{rs})_i = \frac{\partial C_{ijrs}}{\partial z_j}, \quad \mathbf{V}_\zeta^{rs}(\mathbf{z}) \text{ are } Z\text{-periodic}. \quad (\text{A14})$$

These cell problems can equivalently be stated as

$$A_1(\zeta + V_\zeta) = \mathbf{0}, \quad V_\zeta \text{ is } Z\text{-periodic}, \quad (\text{A15})$$

with the boundary conditions $\zeta_i^{rs} = \delta_{ir}z_s$. The homogenization theory (see Ref. [4]) now states that we can compute the Z cell homogenized tensor $\hat{C} = \hat{C}(\mathbf{y})$ by

$$\hat{C}(\mathbf{y}) = \frac{1}{|Z|} \int_Z C(\mathbf{y}, \mathbf{z}) + C(\mathbf{y}, \mathbf{z})e_z(V_\zeta) d\mathbf{z}. \quad (\text{A16})$$

Using the lemma on Eq. (A9), we see that there exists a solution if

$$\frac{1}{|Z|} \int_Z (A_2\mathbf{u}^{(1)} + A_3\mathbf{u}^{(0)}) d\mathbf{z} = \mathbf{0}. \quad (\text{A17})$$

Now

$$\begin{aligned} \int_Z A_2\mathbf{u}^{(1)} d\mathbf{z} &= - \int_Z \nabla_z \cdot (Ce_y(\mathbf{u}^{(1)})) + \nabla_y \cdot (Ce_z(\mathbf{u}^{(1)})) d\mathbf{z} \\ &= -\nabla_y \cdot \left[\left(\int_Z Ce_z(\mathbf{V}_\zeta^{rs}) d\mathbf{z} \right) (e_y(\tilde{\mathbf{u}}_0))_{rs} \right] \end{aligned}$$

and

$$\begin{aligned} \int_Z A_3\mathbf{u}^{(0)} d\mathbf{z} &= - \int_Z \nabla_z \cdot (Ce_x(\mathbf{u}^{(0)})) + \nabla_y \cdot (Ce_y(\mathbf{u}^{(0)})) d\mathbf{z} \\ &= -\nabla_y \cdot \left[\left(\int_Z C d\mathbf{z} \right) e_y(\tilde{\mathbf{u}}_0) \right] \end{aligned}$$

by the Z-periodicity of C . Hence the requirement (A17) can be written

$$a_1 \tilde{\mathbf{u}}_0 = \mathbf{0}, \quad (\text{A18})$$

where the operator a_1 is defined by

$$a_1 = -\nabla_y \cdot (\hat{C} e_y). \quad (\text{A19})$$

Since the operator a_1 only depends on $\mathbf{y}$, Eq. (A18) then yields that

$$\mathbf{u}^{(0)}(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \tilde{\mathbf{u}}_0(\mathbf{x}, \mathbf{y}) = \mathbf{u}(\mathbf{x}),$$

and thus

$$\mathbf{u}^{(1)}(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \mathbf{V}_\xi^{rs}(\mathbf{z})(e_y(\mathbf{u}(\mathbf{x})))_{rs} + \tilde{\mathbf{u}}_1(\mathbf{x}, \mathbf{y}) = \tilde{\mathbf{u}}_1(\mathbf{x}, \mathbf{y}).$$

By rewriting Eq. (A9) as

$$\begin{aligned} A_1 \mathbf{u}^{(2)} &= -A_3 \mathbf{u}^{(0)} - A_2 \mathbf{u}^{(1)} \\ &= \nabla_z \cdot (C e_x(\mathbf{u}(\mathbf{x}))) + \nabla_z \cdot (C e_y(\tilde{\mathbf{u}}_1(\mathbf{x}, \mathbf{y}))) \\ &= \nabla_z \cdot [C(e_x(\mathbf{u}(\mathbf{x})) + e_y(\tilde{\mathbf{u}}_1(\mathbf{x}, \mathbf{y})))] \end{aligned} \quad (\text{A20})$$

we see that the lemma applies since

$$\int_Z \nabla_z \cdot [C(e_x(\mathbf{u}(\mathbf{x})) + e_y(\tilde{\mathbf{u}}_1(\mathbf{x}, \mathbf{y})))] d\mathbf{z} = \mathbf{0} \quad (\text{A21})$$

by the Z-periodicity of C_{ijkl} . By using Eq. (A14) in Eq. (A20), it follows by linearity and the lemma that the solution of Eq. (A9) is

$$\mathbf{u}^{(2)}(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \mathbf{V}_\xi^{rs}(\mathbf{z})(e_x(\mathbf{u}) + e_y(\tilde{\mathbf{u}}_1))_{rs} + \tilde{\mathbf{u}}_2(\mathbf{x}, \mathbf{y}). \quad (\text{A22})$$

According to the lemma, Eq. (A10) has a solution if

$$\frac{1}{|Z|} \int_Z A_2 \mathbf{u}^{(2)} + A_3 \mathbf{u}^{(1)} + A_4 \mathbf{u}^{(0)} d\mathbf{z} = \mathbf{0}. \quad (\text{A23})$$

Now

$$\begin{aligned} \int_Z A_2 \mathbf{u}^{(2)} d\mathbf{z} &= - \int_Z \nabla_z \cdot (C e_y(\mathbf{u}^{(2)})) + \nabla_y \cdot (C e_z(\mathbf{u}^{(2)})) d\mathbf{z} \\ &= -\nabla_y \cdot \left[\int_Z C e_z(\mathbf{V}_\xi^{rs})(e_x(\mathbf{u}(\mathbf{x})) + e_y(\tilde{\mathbf{u}}_1(\mathbf{x}, \mathbf{y})))_{rs} d\mathbf{z} \right], \\ \int_Z A_3 \mathbf{u}^{(1)} d\mathbf{z} &= - \int_Z \nabla_z \cdot (C e_x(\mathbf{u}^{(1)})) + \nabla_y \cdot (C e_y(\mathbf{u}^{(1)})) d\mathbf{z} \\ &= -\nabla_y \cdot \left[\int_Z C e_y(\tilde{\mathbf{u}}_1(\mathbf{x}, \mathbf{y})) d\mathbf{z} \right], \end{aligned}$$

and

$$\int_Z A_4 \mathbf{u}^{(0)} d\mathbf{z} = -\nabla_y \cdot \int_Z C e_x(\mathbf{u}(\mathbf{x})) d\mathbf{z}$$

by Z periodicity. Hence Eq. (A23) states that

$$\begin{aligned} &-\nabla_y \cdot \int_Z C e_z(\mathbf{V}_\xi^{rs})(e_x(\mathbf{u}(\mathbf{x})) + e_y(\tilde{\mathbf{u}}_1(\mathbf{x}, \mathbf{y})))_{rs} d\mathbf{z} \\ &-\nabla_y \cdot \left[\int_Z C e_y(\tilde{\mathbf{u}}_1(\mathbf{x}, \mathbf{y})) d\mathbf{z} \right] + -\nabla_y \cdot \int_Z C e_x(\mathbf{u}(\mathbf{x})) d\mathbf{z} \\ &= \mathbf{0}, \end{aligned}$$

or by rearranging

$$\begin{aligned} &-\frac{1}{|Z|} \nabla_y \cdot \left[\left(\int_Z C + C e_z(\mathbf{V}_\xi^{rs}) d\mathbf{z} \right) e_x(\mathbf{u}) \right] \\ &= \frac{1}{|Z|} \nabla_y \cdot \left[\left(\int_Z C + C e_z(\mathbf{V}_\xi^{rs}) d\mathbf{z} \right) e_y(\tilde{\mathbf{u}}_1) \right]. \end{aligned} \quad (\text{A24})$$

Now by putting $\eta_i^{rs} = \delta_{ir} y_s$ we note that

$$\begin{aligned} [e_y(\boldsymbol{\eta}^{rs}[e_x(\mathbf{u}(\mathbf{x}))]_{rs})]_{kl} &= [e_y(\boldsymbol{\eta}^{rs})]_{kl} [e_x(\mathbf{u}(\mathbf{x}))]_{rs} \\ &= \frac{1}{2} \left(\frac{\partial(\delta_{kr} y_s)}{\partial y_l} + \frac{\partial(\delta_{lr} y_s)}{\partial y_k} \right) [e_x(\mathbf{u}(\mathbf{x}))]_{rs} \\ &= \frac{1}{2} (\delta_{kr} \delta_{sl} + \delta_{lr} \delta_{sk}) [e_x(\mathbf{u}(\mathbf{x}))]_{rs} = [e_x(\mathbf{u}(\mathbf{x}))]_{kl}. \end{aligned}$$

Thus by using Eq. (A19), Eq. (A24) can be written

$$a_1(\boldsymbol{\eta}^{rs}[e_x(\mathbf{u}(\mathbf{x}))]_{rs} + \tilde{\mathbf{u}}_1) = \mathbf{0}.$$

We now introduce $W_\eta = (\mathbf{W}_\eta^{rs}(\mathbf{y}))$ by the *Y cell problem*

$$a_1(\boldsymbol{\eta} + W_\eta) = \mathbf{0}, \quad W_\eta \text{ is } Y \text{ periodic}, \quad \eta_i^{rs} = \delta_{ir} y_s. \quad (\text{A25})$$

It then follows by linearity that

$$\tilde{\mathbf{u}}_1 = \mathbf{W}_\eta^{rs} [e_x(\mathbf{u})]_{rs} + \tilde{\tilde{\mathbf{u}}}_1(\mathbf{x}). \quad (\text{A26})$$

Now consider Eq. (A11). A necessary condition for Eq. (A11) to hold is

$$\begin{aligned} &\int \int_{Y \times Z} A_1 \mathbf{u}^{(4)} + A_2 \mathbf{u}^{(3)} + A_3 \mathbf{u}^{(2)} + A_4 \mathbf{u}^{(1)} \\ &+ A_5 \mathbf{u}^{(0)} d\mathbf{y} d\mathbf{z} = \int \int_{Y \times Z} \mathbf{f} d\mathbf{y} d\mathbf{z}. \end{aligned} \quad (\text{A27})$$

However, since the operators A_1 and A_2 only depend on $\mathbf{y}$ and $\mathbf{z}$, it follows by Y and Z periodicity that

$$\int \int_{Y \times Z} A_1 \mathbf{u}^{(4)} d\mathbf{y} d\mathbf{z} = \int \int_{Y \times Z} A_2 \mathbf{u}^{(3)} d\mathbf{y} d\mathbf{z} = \mathbf{0}.$$

Moreover, by using Y and Z periodicity again it follows that

$$\begin{aligned} \int \int_{Y \times Z} A_3 \mathbf{u}^{(2)} \, d\mathbf{y} \, d\mathbf{z} &= -\nabla_x \cdot \int \int_{Y \times Z} C e_z(\mathbf{V}^{rs})(e_x(\mathbf{u}) \\ &\quad + e_y(\tilde{\mathbf{u}}_1))_{rs} \, d\mathbf{y} \, d\mathbf{z}, \\ \int \int_{Y \times Z} A_4 \mathbf{u}^{(1)} \, d\mathbf{y} \, d\mathbf{z} &= -\nabla_x \cdot \int \int_{Y \times Z} C e_y(\tilde{\mathbf{u}}_1) \, d\mathbf{y} \, d\mathbf{z}, \\ \int \int_{Y \times Z} A_5 \mathbf{u}^{(0)} \, d\mathbf{y} \, d\mathbf{z} &= -\nabla_x \cdot \int \int_{Y \times Z} C e_x(\mathbf{u}) \, d\mathbf{y} \, d\mathbf{z}. \end{aligned}$$

Hence the Eq. (A27) becomes

$$\begin{aligned} &-\nabla_x \cdot \int \int_{Y \times Z} C e_z(\mathbf{V}^{rs})(e_x(\mathbf{u}) + e_y(\tilde{\mathbf{u}}_1))_{rs} \, d\mathbf{y} \, d\mathbf{z} \\ &\quad - \nabla_x \cdot \int \int_{Y \times Z} C e_y(\tilde{\mathbf{u}}_1) \, d\mathbf{y} \, d\mathbf{z} - \nabla_x \cdot \int \int_{Y \times Z} C e_x(\mathbf{u}) \, d\mathbf{y} \, d\mathbf{z} \\ &= |Y||Z|\langle \mathbf{f} \rangle_{Y \times Z}. \end{aligned}$$

By inserting the expression (A26) for $\tilde{\mathbf{u}}_1$ we get

$$\begin{aligned} &-\int \int_{Y \times Z} C_{ijkl} + C_{ijmn}[(e_z(\mathbf{V}_\xi^{kl}))_{mn} + (e_z(\mathbf{V}_\xi^{rs}))_{mn}(e_y(\mathbf{W}_\eta^{kl}))_{rs} \\ &\quad + (e_y(\mathbf{W}_\eta^{kl}))_{mn}] \, d\mathbf{y} \, d\mathbf{z} \times \frac{\partial}{\partial x_j}(e_x(\mathbf{u}))_{kl} \\ &= |Y||Z|\langle \mathbf{f}_i \rangle_{Y \times Z}. \end{aligned}$$

This can be written

$$-\bar{C} \nabla_x \cdot e_x(\mathbf{u}) = \langle \mathbf{f} \rangle_{Y \times Z},$$

where

$$\begin{aligned} \bar{C}_{ijkl} &= \langle C_{ijkl} + C_{ijmn}(e_z(\mathbf{V}_\xi^{kl}))_{mn} + (e_z(\mathbf{V}_\xi^{rs}))_{mn}(e_y(\mathbf{W}_\eta^{kl}))_{rs} \\ &\quad + (e_y(\mathbf{W}_\eta^{kl}))_{mn} \rangle_{Y \times Z}. \end{aligned}$$

Thus by comparison with the desired result (Eq. (16)), we see that we have an expression for the *overall homogenized tensor* $\bar{C}$ by Eq. (16) and we can also interpret the right-hand side term $\mathbf{g}$ as

$$\mathbf{g} = \langle \mathbf{f} \rangle_{Y \times Z}.$$

On the other hand, the solution of the Z cell problem (Eq. A14) yields the Z homogenized tensor $\hat{C}$ by Eq. (A16)

$$\hat{C}(\mathbf{y}) = \frac{1}{|Z|} \int_Z C(\mathbf{y}, \mathbf{z}) + C(\mathbf{y}, \mathbf{z}) e_z(\mathbf{V}_\xi) \, d\mathbf{z},$$

or in tensorial formulation

$$\hat{C}_{ijkl} = \frac{1}{|Z|} \int_Z C_{ijkl} + C_{ijmn}(e_z(\mathbf{V}_\xi^{kl}))_{mn} \, d\mathbf{z}.$$

Analogously, we can compute the Y homogenized tensor

$\bar{C}_{ijkl}^Y$ from the Y cell problem by inserting $\hat{C}_{ijkl}$ and get

$$\begin{aligned} \bar{C}_{ijkl}^Y &= \frac{1}{|Y|} \int_Y \hat{C}_{ijkl} + \hat{C}_{ijrs}(e_y(\mathbf{W}_\eta^{kl})) \, d\mathbf{y} \\ &= \frac{1}{|Z||Y|} \int \int_{Y \times Z} [C_{ijkl} + C_{ijmn}(e_z(\mathbf{V}_\xi^{kl}))_{mn} + (e_z(\mathbf{V}_\xi^{rs}))_{mn} \\ &\quad \times (e_y(\mathbf{W}_\eta^{kl}))_{rs} + (e_y(\mathbf{W}_\eta^{kl}))_{mn}] \, d\mathbf{y} \, d\mathbf{z}. \end{aligned} \quad (\text{A28})$$

Thus we see that

$$\bar{C} = \bar{C}^Y,$$

that is, we can compute the homogenized tensor for the overall problem by first solving the Z cell problem and compute the Z homogenized tensor. Then we solve the Y cell problem using this Z homogenized tensor, and finally compute the resulting homogenized tensor $\bar{C}$ by homogenizing over the Y cell.

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