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RANDOM VERSUS PERIODIC CELLS IN HOMOGENIZATION

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A common assumption in the homogenization method is to assume that the inclusions are periodically distributed in the material. However, this assumption might often not be in accordance with the situation in real-world composites, where the inclusions in most cases are randomly distributed. We will in this paper compare the effect of the geometries and the distributions of the inclusions on the effective conductivity of some composites.

We start by giving some basic results from the theory of homogenization. Let Ω be a open bounded subset of $\mathbf{R}^n$, Y the unit cube in $\mathbf{R}^n$ and $\langle \cdot, \cdot \rangle$ the Euclidean inner product. Let us define $A^h(x) = A(x/\varepsilon_h)$, where the conductivity matrix $A(x) = (a_{ij}(x))$ satisfies $a_{ij} = a_{ji}$, $a_{ij} \in L^\infty(\Omega)$ and is Y -periodic for every $i, j = 1, \dots, n$. Consider the weakly formulated stationary heat conduction problem

$$\begin{cases} \int_{\Omega} \langle A^h(x) \nabla u_h, \nabla \phi \rangle dx = \int_{\Omega} f \phi dx & \text{on } \Omega, \\ u_h \in H_0^{1,2}(\Omega), \end{cases}$$

where $f \in L^2(\Omega)$ is the internal heat source and (ε_h) is a sequence of positive real numbers converging to zero. The main theorem in the homogenization theory then states that

$$u_h \rightarrow u_0 \text{ weakly in } H_0^{1,2}(\Omega)$$

and

$$A^h(x) \nabla u_h \rightarrow B \nabla u_0 \text{ weakly in } (L^2(\Omega))^n,$$

where u_0 is the unique solution of the homogenized problem

$$\begin{cases} \int_{\Omega} \langle B \nabla u_0, \nabla \phi \rangle dx = \int_{\Omega} f \phi dx & \text{on } \Omega, \\ u_0 \in H_0^{1,2}(\Omega). \end{cases}$$

The constant matrix B is given by the expression

$$B e_k = \frac{1}{|Y|} \int_Y A(x) (e_k + \nabla w_k) dx,$$

where w_k is the unique solution of the cell problem

$$\begin{cases} \int_Y \langle A(x) (e_k + \nabla w_k), \nabla \phi \rangle dx = 0 & \text{on } Y, \\ w_k \in H_{\square}^{1,2}(Y). \end{cases}$$

Here e_k denotes the k :th unit vector and $H_{\square}^{1,2}(Y)$ is the subset of $H^{1,2}(Y)$ of Y -periodic functions u with mean value zero. A proof of this theorem can be found in e.g. [3].

For a given number of inclusions in one periodic cell, we generate a series of different realizations where the inclusions are randomly dispersed. On an overall basis, each realization yields a periodic material, but as the number of inclusions in each periodic cell goes to infinity, the material will become totally random. By taking the mean values and standard deviations of the effective conductivities for at least 200 different realizations, we get an approximation of the effective conductivity for the random material.

Each quasi-random realization are generated by a Monte Carlo procedure of Metropolis type, see e.g. [4]. This is done by first placing every inclusion in a rectangular or hexagonal array. Each disk is then given a small tentative displacement in a random direction. The move is accepted or rejected whether or not the move will cause the disk to overlap a neighboring disk. One iteration step consists of trying to move each disk once. The length of the displacement is chosen such that the ratio of acceptance is 30-50%. This process is repeated several times until an equilibrium state is obtained, see Figure 1.

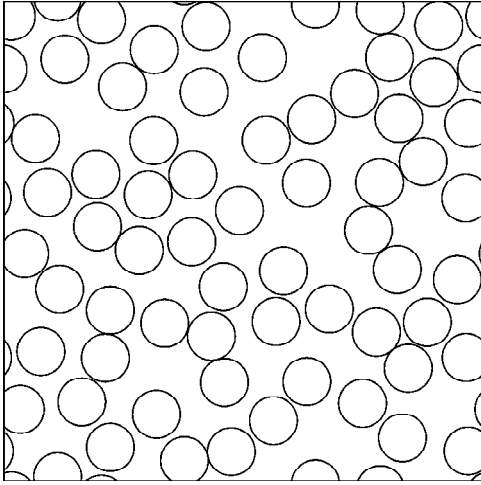


Figure 1: One realization of a random cell with 64 inclusions of volume fraction 0.5.

In the first example, we consider a quasi-random composite with 4, 16, 64 and 144 inclusions in each periodic cell. We assume that $a_1 = 1$, $a_2 = 10$ with a corresponding volume fraction of $\nu_2 = 0.5$. The first column shows the number of inclusions in each periodic cell, the second column shows the mean value and the standard deviation of the effective conductivity and the third column shows the mean value and the standard deviation of b_{12}/b_1 , a measure of the anisotropy of the composite.

Table 1:

Incl.	$\bar{b} \pm St.D.$	Anisotropy
4	2.476 ± 0.093	-0.001 ± 0.019
16	2.543 ± 0.076	0.001 ± 0.017
64	2.546 ± 0.036	0.000 ± 0.009
144	2.546 ± 0.023	-0.001 ± 0.005

For this specific composite, we draw the conclusion that 64 inclusions are enough. We note that the standard deviations of the effective conductivity and the measure of anisotropy decrease more or less like the square root of the number of inclusions, which is in agreement with what we expect from a Monte Carlo process.

In the second example, we compare random distribution with hexagonal and rectangular distributions of the inclusions for different volume fractions. We present the results in Table 2 and Figure 2, where they are plotted together with the lower Hashin-Shtrikman bound. We note that hexago-

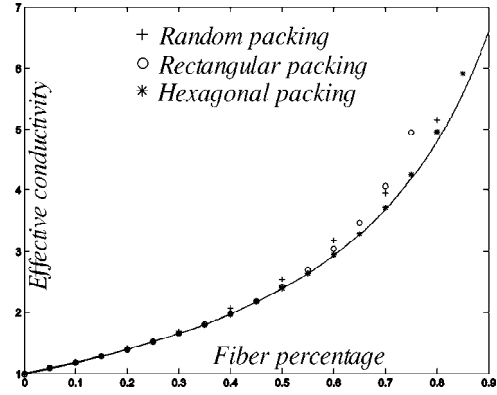


Figure 2: The results of Table 2.

nal distribution gives the lowest effective conductivity, near the lower Hashin-Shtrikman bound, followed by rectangular distribution (except for high volume fractions, due to percolation effects) whereas random distribution gives the highest effective conductivity. We refer to [1] and [2] for more examples of this type.

Table 2:

ν_2	Hex.	Rect.	Rand. $\bar{b} \pm St.D.$
0.1	1.1782	1.1782	1.182 ± 0.006
0.2	1.3913	1.3915	1.410 ± 0.018
0.3	1.6506	1.6520	1.697 ± 0.034
0.4	1.9733	1.9806	2.068 ± 0.054
0.5	2.3864	2.4155	2.543 ± 0.076
0.6	2.9375	3.0372	3.170 ± 0.098
0.7	3.7190	4.0617	3.945 ± 0.086
0.8	4.9488	—	5.153 ± 0.062

References

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