

On the effect of stacked fabric layers on the stiffness of a woven composite

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Abstract

Most micromechanical models for stiffness prediction of woven composites assume independence of the Q -matrix on the number of fabric layers in the composite. For example, the moduli of single and 10 layer composites are assumed to be equal in the case when all layers have the same in-plane orientation. Although this statement is likely to be true for isotropic materials or even for unidirectional laminated composites, it may not be valid in some cases of woven composites.

This paper contains experimental and theoretical investigations of plain weave carbon fiber/polyester composites. Specimens with one single and eight layers of fabrics are tested and observable differences of mechanical properties are obtained.

The theoretical part of this article consists of derivation and application of several micromechanical models on these particular composites. The use of those simplified models finally allows us to find the main mechanisms which cause the observed effects.

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1. Introduction

Nowadays, the popularity of woven composites is increasing due to simple processing and acceptable mechanical properties. The regularity and geometrical stability of woven fabric reinforcements allow us to produce complex shapes and large components by using resin transfer molding with low production costs. The same regularity, or to be more precise, the existence of a very accurate repeatable unit cell (RUC) of reinforcement fabric (see Fig. 1 for the plain weave RUC), suggests us to use such a unit cell as a representative volume element (RVE) for calculations of the stiffness matrix of the composite. The stiffness matrix of the composite can be obtained by simulation of different loads on the RUC by the finite element method (FEM), see some examples in Refs. [1–3].

However, this approach is in most cases not totally correct since a real composite usually has several layers of the fabric stacked together in the out-of-plane (Z -) direction. Let us consider such a multi-layered woven composite.

Then we can shift the layers with respect to each other. Fig. 2 shows some possible stacking sequences. The regular stackings (Fig. 2(a) and (b)) are relatively rare. It is not easy, or even impossible, for curved surfaces to obtain and keep a perfect shift between individual fabric layers during processing. Therefore, a random shift of the layers is common as shown in Fig. 2(c). This causes some drawbacks if one considers the micromechanical model. From a micromechanical point of view, the neighboring layers give additional unknown constraints on the upper and lower surfaces of the RUC ($z = \text{const.}$), i.e. the boundary conditions for the RUC are unknown, hence the procedure described earlier is no longer valid. As a consequence, we must choose for our hypothetical FEM model a RVE much larger than the RUC of the fabric, which can easily exceed the limits of the used FEM software/hardware.

On the other hand, the RUC is equal to the RVE for composites consisting of a single fabric layer. Such a RUC has totally unconstrained upper and lower ($z = \text{const.}$) surfaces and it can freely bend in the Z -direction. Thus, one has strong indications that the mechanical responses on load of single- and multi-layered composites may be different.

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Nomenclature

RUC	repeatable unit cell, smallest possible repeatable volume of one single fabric layer	h_{tow}	thickness of the composite tows
RVE	representative volume element, smallest possible volume which can be used to obtain the stiffness matrix of the composite as a homogeneous medium	L	width of the composite tows
RMH	reiterated mathematical homogenization method	l_b	structural length of the beam ($l_b = \lambda/4$)
λ	size of the RUC, wave length of the weave	l_0	curve-length of the beam midplane, see expression (A2)
h_f	thickness of the RUC in the out-of-plane direction	X, Y	in-plane coordinates
		u, v	X- and Z-displacements
		E_l	axial modulus of the composite tow
		E_1	transverse modulus of the composite tow
		E_2	matrix modulus
		D	bending stiffness of a beam $D = \int z^2 E(z) dz$

The objective of this paper is to evaluate the difference between single- and multi-layered woven composites experimentally and compare it to theoretical predictions and to show the main micromechanical mechanisms which cause this difference.

2. Experimental procedure and results

We used Tenax HTA Carbon fiber (Brochier Injectex GB2000)/Unsaturated polyester plain weave woven composite specimens consisting of a single fabric layer and eight fabric layers. For the eight-layer (multi-layer below) composite, the layers have the same in-plane orientation, but they are not specifically ordered with respect to shifting

between each other. Image analysis of pieces of broken specimens show almost random shifting of the fabric layers, which varies from laminate to laminate. For reference, the weave dimensions and other characteristics of the composite are given in Table 1.

Standard size specimens for tensile tests were cut out from the laminate or ply in the direction of the warp yarns (X-direction in Fig. 1). Tensile tests with multiple loading–unloading cycles were applied to several specimens of both kinds. Strains in the load (X-) direction ϵ_{xx} were measured by an extensometer with 50 mm gage length and a strain gage with 5 mm gage length was used for ϵ_{yy} registration.

Typical stress–strain curves are shown in Fig. 3. By comparing Fig. 3(a) for a single-layer composite with

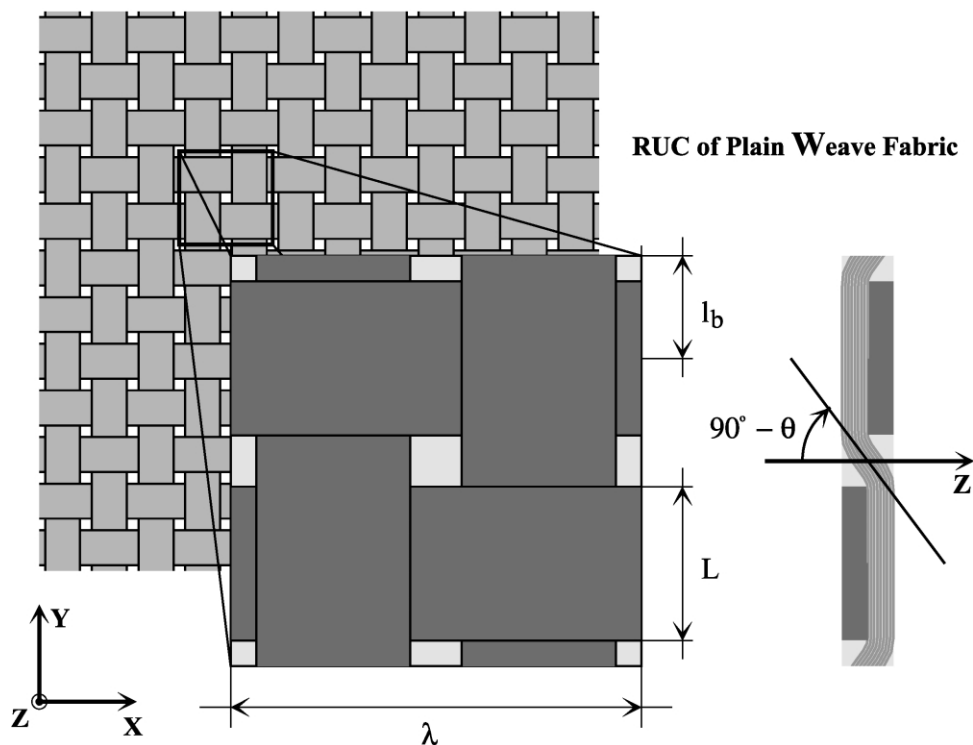


Fig. 1. Schematic drawing of a RUC of plain weave fabric.

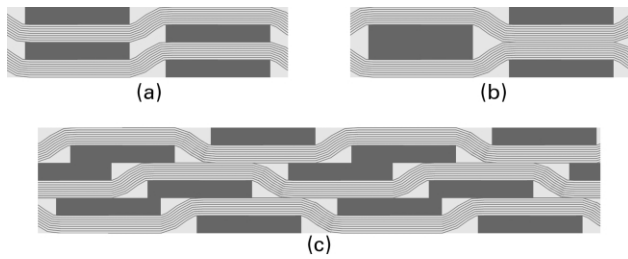


Fig. 2. Stacking order of plain weave fabrics: (a) in-phase, (b) out-of-phase, and (c) random (more realistic).

Table 1
Some characteristics of the material

Wavelength of RUC	λ	4.17 (mm)
Initial thickness of RUC	h_f	0.28 (mm)
Tow width	L	1.37 (mm)
Tow thickness	h_{tow}	0.14 (mm)
Axial modulus of tow	E_l	150 (GPa)
Transverse modulus of tow	E_1	10 (GPa)
Matrix modulus	E_2	4 (GPa)

Fig. 3(b) for multi-layer composites one can conclude that

1. the loading curves for single-layer specimens show much higher non-linearity and residual strains than those for multi-layer specimens,
2. the initial tangent modulus (obtained by the least squares method in the deformation interval 0.2–0.4% on the first unloading path of the load–unload cycle) is higher in the case of multi-layer specimens (31 and 50 GPa for particular specimens, respectively),
3. both loading–unloading curves in Fig. 3(a) and (b) show specimen stiffening (meaning an increase of the tangent modulus).

Thus, the dependency of the number of layers is well observable for a particular reinforcement/matrix.

The influence of the number of stacked layers on the in-plane transverse strain ϵ_{yy} is even higher. This can be

concluded from a comparison between Fig. 4(a) and (b), which contain the ϵ_{yy} dependency on ϵ_{xx} . The values of ϵ_{yy} are collected during the same tests as ϵ_{xx} and σ_{xx} shown in Fig. 3(a) and (b). Not only is high non-linearity and hysteresis during the loading–unloading cycles for single-layer specimen notable, but also a relative difference in Poisson’s ratios of about five times between the two specimens. The small hysteresis loop in the ϵ_{yy} curve in Fig. 4(b) is related to extensometer jumping due to a crack in the tab.

3. Mechanisms

3.1. Multi-layered composites

Let us start our analysis with the multi-layered composites. As mentioned earlier, FEM models based on RVEs with sizes equal to the fabrics RUC having simple periodic boundary conditions are not correct, whereas FEM models describing a large RVE (with a volume of approximately $8^3 = 512$ RUCs each containing at least 1000 3D finite elements) are on the limit of what existing commercial FEM software allow. A more realistic approximate FEM model consisting of just two layers shifted 1/4 of the wavelength in the (X-) and (Y-) directions with respect to each other is analyzed in Ref. [4]. In this approximation, antisymmetric boundary conditions may be used, which reduce the minimal volume of the RVE. The results obtained have good agreement with measurements, see Ref. [4].

In the light of the above discussion, it is even more surprising that several simple semi-analytic models for in-plane properties of composites does work almost as good as FEM models. The early models developed by Ishikawa and Chou [5,6] are mainly based on 2D laminate theory and therefore limited by n -hardness ($n \geq 4$) satin. They are not very good for plain weave ($n = 2$) fabrics. However, the Karayaka and Kurath’s model [7] based on 6×6 Q -matrix representation of each sub-cell have shown to be very accurate for composites of such kind,

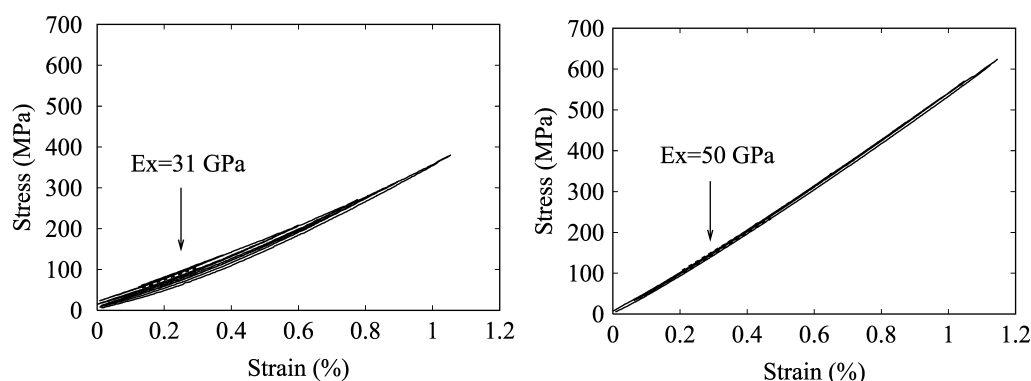


Fig. 3. Stress–strain curves for C/UP woven composites, results of tensile test: (a) composite containing one single fabric layer; (b) composite containing eight fabric layers.

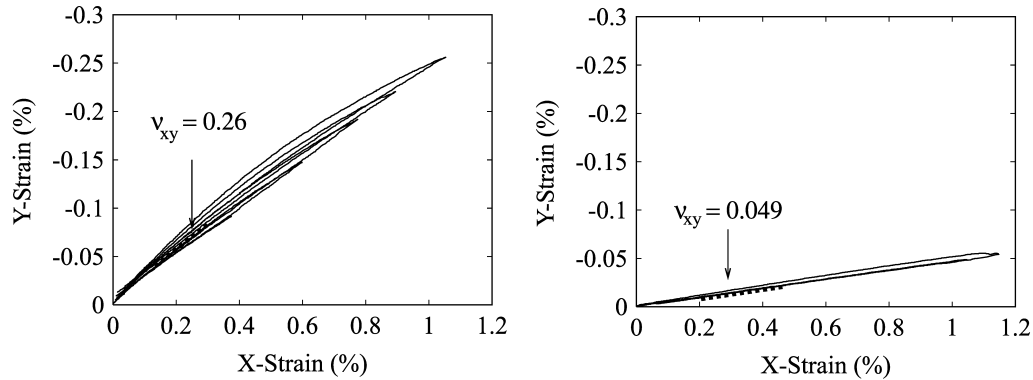


Fig. 4. X-strain (load direction)/Y-strain curves for C/UP woven composites, results of tensile test: (a) composite containing one single fabric layer; (b) composite containing eight fabric layers.

see Ref. [4]. The secret of good performance lies in the averaging procedure; for in-plane characteristics the assumption of isostrains is here used, whereas for out-of-plane characteristics the assumption of isostresses is used, see Fig. 5. Let us look on the isostrain conditions. These conditions basically say that the average in-plane strains in all sub-cells (see Fig. 5) are equal:

$$\varepsilon_{ij}^{(1)} = \varepsilon_{ij}^{(2)} = \varepsilon_{ij}^{(3)} = \varepsilon_{ij}^{(4)} = \varepsilon_{ij}^{(5)} = \varepsilon_{ij}^{(6)} = \varepsilon_{ij}^{(7)}, \quad (3.1)$$

where $i = \{x, y\}$; $j = \{x, y\}$. This is obviously wrong if we consider single-layer composites since the volumes 1, 4, 7 in Fig. 5 usually have much lower E -moduli compared to the E -moduli of the cross-ply (2 + 3) and (5 + 6). Hence, this model does not obey global conservation of integral force over area perpendicular to the load. However, this is fine with multi-layered composites with randomly shifted layers; the average of the force over the YZ -section of several layers reduce the dependency on the section's X -coordinate when the number of randomly shifted layers increases. Moreover, commercial fabrics usually have flat yarns with high aspect ratio L/h_{tow} , which ensure short shear lag zones even for low shear moduli of yarns as in the case of carbon fiber reinforcements. This gives the condition for local in-plane strain conservation.

The existence of neighboring layers gives additional constraints in the Z -direction. This leads to some kind of

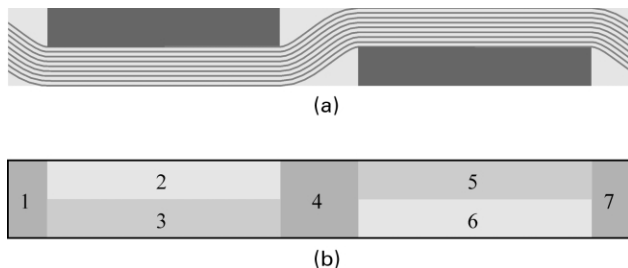


Fig. 5. The isostrain condition: (a) section of plain weave RUC; (b) isostrain conditions state that the average ε_{xx} , ε_{yy} and ε_{xy} in each sub-cell must be equal for all sub-cells (1–7).

similarity between multi-layered composites with equivalent (in the sense of fiber content) laminated cross-ply composites leading to very low values of Poisson's ratio. Like laminates, multi-layered composites have almost linear stress–strain relationships, see Figs. 3(b) and 4(b).

Paper [4] contains a more detailed analysis as well as numeric examples related to the evaluation of different models for particular multi-layered composites.

3.2. An analytic model for single-layer composites

For the ideal (in the meaning of deviations of the shape of RUCs) single-layer composite, the RUC is the same as the RVE. As a consequence, FEM calculations with simple boundary conditions may successfully be used to obtain an exact stiffness matrix.

The statement above does not mean that the simple approximate analytical models are useless. In our case, an analytic model serves as the main tool to study the main mechanisms causing the non-linear behavior observed for single-layer composites.

The model mentioned earlier is a result of the evolution of a model for ceramics composites given in Ref. [8]. The model is easy to modify for complex load conditions and it gives an expression of the tangent modulus in a closed analytic form. The secant modulus can be obtained by integration. At almost the same time a similar model was proposed by Luo and Mitra [9]. The last involves numerical solution of a non-linear system of equations, which from our point of view is a disadvantage. We will use the approach from Ref. [8].

The mechanical response to load for a single ply composite is very similar to that of raw fabric, the only difference being the zero shear modulus of dry tows of fabric (let us not count friction between the fibers, since it is an irreversible process). The matrix gives finite shear and transverse moduli of the tows, which causes them to act as solid beams. If one considers resin inclusions outside the tows, then the moduli of those are almost two orders of

magnitude smaller compared to the tow axial modulus, hence it seems to be meaningful to describe a single-layer composite as a set of four rigid tows. After that, one may have to do necessary (small) corrections for the finite matrix modulus.

Since all derivations are given in Appendix A, let us focus our attention on the resultant expressions only. The prediction of the tangent modulus of the weave in the tow direction is

$$E_x(h_0) = \frac{L\alpha}{4h_{\text{tow}}} [1 - \gamma(h_0)a(h_0)] + E_p \quad (3.2)$$

and the ‘tangent’ Poisson ratio is

$$\nu_{xy} = \gamma(h_0)a(h_0), \quad (3.3)$$

where the internal variable $a(h_0) = a_0 h_0$ is weave geometry dependent and may be assumed to be a linear function of the amplitude h_0 of the weave wave. For sine-shaped inclination of the tows (Fig. 9), we have $a_0 = \pi^2/2\lambda$. The internal variable γ is the ratio between in- and out-of-plane displacements for 1/4 of the RUC. The value of γ corresponds to the minimum of elastic energy stored in the RUC and is a function of $a(h_0)$. Moreover, γ is also load-condition dependent. For a uniaxial load,

$$\gamma(a(h_0)) = a(h_0) \left[a^2(h_0) + 2 \frac{\beta}{\alpha} \right]^{-1}. \quad (3.4)$$

Check Appendix A for the energetic constants α and β (Eqs. (A7) and (A8)) as well as for the derivation of Eqs. (3.2)–(3.4). The formula for modulus correction E_p due to finite modulus of matrix inclusions is given by Eqs. (A21) and (A22).

During the tensile test, the amplitude h_0 of the wave changes and the load-bearing tows becomes straighter, hence from Eqs. (3.2) and (3.4) follows an increase of the tangential modulus E_x .

Let us derive the expression of the secant modulus e_x , defined for the case of monotonic tensile test as $\bar{\sigma}_{xx} = e_x \bar{\epsilon}_{xx}$, where bars denote average over the RUC. The result is given by a rather simple integration of Eqs. (3.2) and (3.4). Actually, from the definition of γ and the definition of Cauchy strain $\epsilon = \Delta/L$, it follows that

$$\frac{dh_0}{d\bar{\epsilon}_{xx}} = -\frac{\lambda}{4} \gamma(h_0), \quad (3.5)$$

which is a simple first order differential equation with respect to h_0 . It may be solved by separation of variables

$$\int_{h_{\text{tow}}/2}^{h_0} \frac{dh}{\gamma(h)} = -\frac{\lambda}{4} \bar{\epsilon}_{xx}. \quad (3.6)$$

The definition of tangent modulus together with Eqs. (3.2) and (3.6) leads to

$$e_x = \frac{\bar{\sigma}_{xx}}{\bar{\epsilon}_{xx}} = \frac{4}{\lambda \bar{\epsilon}_{xx}} \int_0^{h_0(\bar{\epsilon}_{xx})} \frac{E_x(h)}{\gamma(h)} dh = \frac{1}{\bar{\epsilon}_{xx}} \int_0^{\bar{\epsilon}_{xx}} E_x(h_0(\epsilon)) d\epsilon. \quad (3.7)$$

Integration of Eq. (3.7) together with Eq. (3.6) for h_0 give us the secant modulus e_x , which allows us to reconstruct the load curve. Integration of Eqs. (3.6) and (3.7) may be done numerically or analytically. Since the analytic integral (Eq. (3.6)) contains both h_0 and $\ln(h_0)$, a closed form expression of $h_0(\bar{\epsilon}_{xx})$ does not exist, so we have to use numerical integration.

3.3. Verification of the model: comparison with RMH method

For a validation of the model described earlier, the results from reiterated mathematical homogenization (RMH) were used. The main concepts and ideas of RMH are described in Appendix B. Since RMH involves 3D FEM calculations, the weave geometry has been simplified. A cross-section of the simplified weave is drawn in Fig. 6. The main difference, as one may notice, is the shape of the yarn inclination. The yarn used for the 3D RMH model may be assembled from rectangular bars by keeping the overall fiber content constant. This causes higher values of the inclination angle θ compared to the sine midplane representation, see Fig. 9. As a result, the energetic constants α , β and the beam constant a_0 differ from Eqs. (A1)–(A3), (A7), (A8), (A20), and (A21), which seem to be too inaccurate.

For higher accuracy, values of α , β and a_0 are calculated using 2D FEM simulations with boundary conditions (A4) and (A5), see Table 2. The table also contains both FEM and analytic values for the sine-shaped beam, later used for the real composite. From the table, we see that we have lower values of α and higher values of a_0 in the case of the beam from Fig. 6 compared to the beam in Fig. 9. This may be explained by longer midplane length of our simplified beam, since even if it is constrained in the knot points for loading in Mode 1, the beam may still gain energetically by just becoming straighter and not by elongation of its midplane.

The load conditions also differ. In order to use the RMH method most effectively, it is preferable to apply the in-plane load in the X-direction by keeping the RUC constrained in the Y-direction. Thus, Poisson contraction of the whole RUC is not allowed.

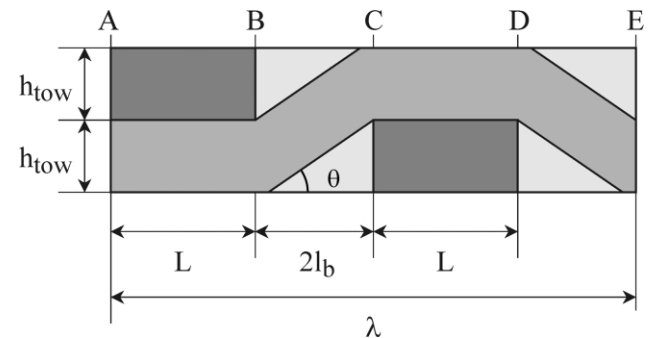


Fig. 6. Section (XZ-plane) of 3D simplified RUC. This kind of RUC is used in the RMH model.

Table 2
Beam characteristics by different models

Number	Model (see text)	$\alpha\lambda/4h_{\text{tow}}$ (GPa)	β/α (–)	$2\alpha_0/h_{\text{tow}}$ (–)
1	Analytic sine	150	0.0076	0.083
2	FEM sine	147	0.0070	0.067
3	FEM simplified	117	0.0097	0.094

Since the new load conditions are biaxial, some changes in the analytic model are required. Equations of biaxial elastic response of the RUC are derived in Appendix A, Section A.3, see Eqs. (A24)–(A26). Those with $\Delta Y = \epsilon_{yy} = 0$ become

$$E_x^c = \frac{\partial \bar{\sigma}_{xx}}{\partial \bar{\epsilon}_{xx}} = \frac{L\alpha}{4h_{\text{tow}}} [1 - \gamma_1(h_0)a_1(h_0)] + E_p \quad (3.8)$$

and

$$E_y^c = \frac{\partial \bar{\sigma}_{yy}}{\partial \bar{\epsilon}_{xx}} = \frac{L\alpha}{4h_{\text{tow}}} \gamma_1(h_0)a_1(h_0), \quad (3.9)$$

where γ_1 is given by

$$\gamma_1(h_0) = a_1(h_0) \left[a_1^2(h_0) + a_2^2(h_0) + 2\frac{\beta}{\alpha} \right]^{-1}. \quad (3.10)$$

The coefficients a_1 and a_2 are still considered to be linear functions of h_0 : $a_1 = a_0 h_0$, $a_2 = a_0(h_{\text{tow}} - h_0)$. One may notice the similarities between Eqs. (3.8) and (3.10) on one hand and Eqs. (3.4) and (3.6) on the other. This leads to the substitutions

$$e_x \rightarrow e_x^c, \quad E_x \rightarrow E_x^c, \quad \gamma(h_0) \rightarrow \gamma_1(h_0), \quad (3.11)$$

in the integrals of Eqs. (3.8) and (3.9) for the secant modulus e_x^c of the constrained RUC and the substitutions

$$e_x \rightarrow e_y^c, \quad E_x \rightarrow E_y^c, \quad \gamma(h_0) \rightarrow \gamma_1(h_0), \quad (3.12)$$

for the transverse secant response e_y^c , where $\bar{\sigma}_{yy} = e_y^c \bar{\epsilon}_{xx}$.

A comparison of the results from the 3D RMH model with the analytic model is made in Fig. 7(a), where the values of e_x^c and e_y^c obtained from the RMH and from the analytic models are plotted versus applied average strain.

Two different sets of beam characteristics α , β , a_0 from Table 2 (No. 1 and 3) are used for the analytic model. From

Fig. 7, it follows that the model with FEM constants (No. 3) agree very well with the RMH 3D model predictions. The remaining differences may be explained with 2D load distribution in between warp and fill tows which is taken into account by the 3D model and missing in the 2D submodel. Additionally, in the 3D case tows must be considered as shells rather as beams, which is how they are considered in the 2D beam model. However, the predictions of the analytic model are still surprisingly accurate.

In Fig. 7(b) predictions of different models of normalized out-of-plane displacements $2\Delta Z/h_{\text{tow}}$ are plotted. The analytic model with the 2D FEM set of beam constants seems to do the work very good again. Therefore, the final conclusion is that the simplified analytic model may be used for the prediction of the non-linear elastic response of the plain weave RUC.

3.4. Comparison with experimental results

The experimentally measured secant modulus e_x as a function of stress together with the prediction of the analytic model is plotted in Fig. 8(a). It follows from the figure that the analytic model gives a correct description of the measured phenomenon. However, the agreement is not as good as with the RMH method. The discrepancy may have several sources. First it may be related to the initial straightening of the specimens. Since single-layer specimens are very thin, it is difficult to load a whole specimen at the same load level. Moreover, the shape of the RUC is only approximately described by the model. Even worse, it is also not uniform, since there are deviations from RUC to RUC. The high deviation between the load and unload curves of different specimens agree with both hypotheses.

Fig. 8(b) shows a plot of the normalized out-of-plane displacements $2\Delta Z/h_{\text{tow}}$ given by the analytic model versus applied strains. From the figure, one may get an impression of the order of tow straightening. The final value of the normalized displacements close to specimen failure is about $0.6h_{\text{tow}}/2$ which means that the load-bearing tows are almost straight. Therefore, the mechanism of the increase of modulus for single-layer plain weave composite is very likely identified; it is prescribed by the tow straightening.

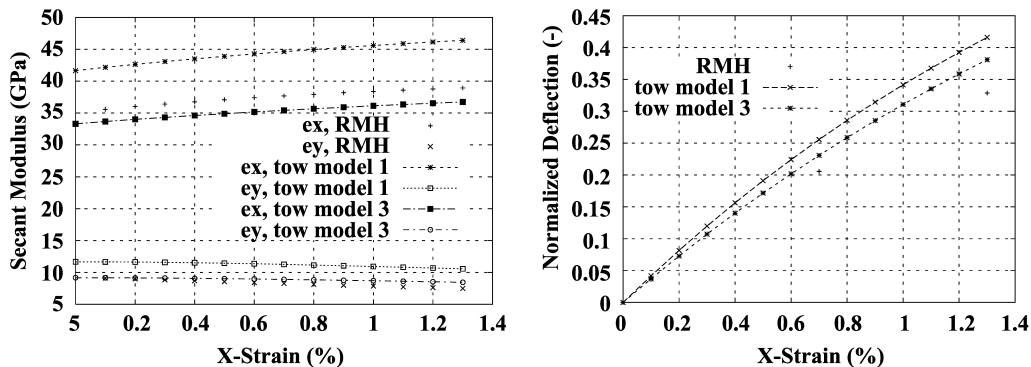


Fig. 7. Comparison of analytic and RMH models: (a) comparison of secant 'constrained' moduli e_x^c and e_y^c ; (b) comparison of normalized displacements $2\Delta Z/h_{\text{tow}}$.

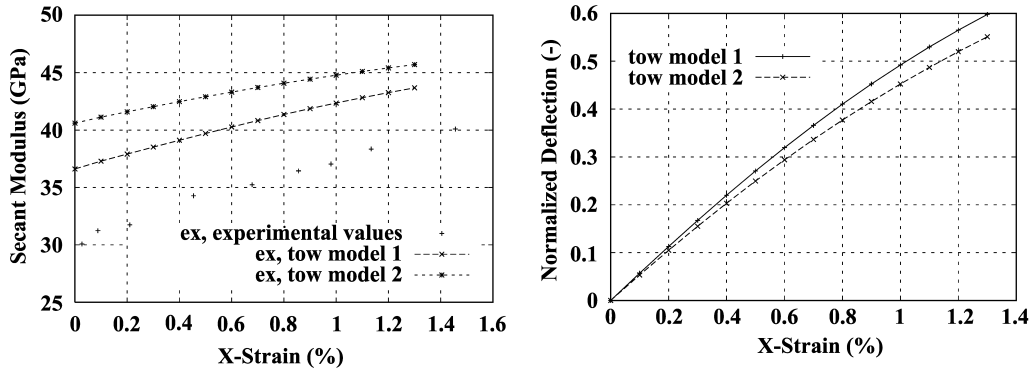


Fig. 8. Predictions of the model compared to experimental results: (a) secant modulus e_x ; (b) normalized displacements $2\Delta Z/h_{\text{tow}}$.

Now what about the hysteresis loops which one may observe in the strain/strain curves in Fig. 4(a)? One explanation of this may be a plasticity phenomenon. Since bending introduce high values of shear strains inside the inclination regions of the tows, the matrix may start to yield, which finally ends with irreversible plastic deformations. Although the limited size of the present paper does not allow to discuss this question in detail we would like to say that our preliminary numerical results indicates this scenario is very likely to be true. The tangent Poisson ratio predicted by the model changes in the interval from 0.25 at low load values to about 0.11 close to failure, which in general agree with experimental measurements.

4. Practical remarks

As seen in the previous two sections, theoretical and experimental arguments exist for considering single- and multi-layered composites not to be equivalent by means of their elastic properties. The properties of multi-layered composites are usually used by engineers, because single-ply composites are very scarcely used. Then isostrain/isostress assumptions are valid. For a single-layer composite, we have to use the RUC approach. What about, let's say, two-layer composites? That kind of lay-up is often used for covering another material, for example, wood constructions. Our preliminary results show that for the used material, a two-layer composite still has almost single-layer mechanical behavior. The exact localization of the boundary between single- and multi-layered woven composite approaches is still not known.

5. Conclusions

- Tensile tests on single- and eight-layer C-fiber/UP woven composites are performed. Experimental results show an observable difference between those two lay-ups.
- The multi-layered composites seem to be more linear and also have a higher initial E -modulus. Like cross-ply laminated composites these composites have a low

Poisson's ratio. Single-layer woven composites are evidently non-linear during the loading, with low tangential modulus at the beginning and high at the end. Moreover, Poisson's ratio for a single-layer woven composite is several times higher than for a multi-layered.

- Semi-analytical micromechanical as well as mathematical homogenization based FEM models are used in order to explain observed differences. The results of the modeling agree with the models and can explain the observed phenomenon.
- Different micromechanical assumptions must be considered in order to explain the observed effects. A single-layer composite has unconstrained deformations in the Z -direction. Multi-layer composites obey isostrain conditions for the in-plane components of the deformations and isostress conditions for the out-of-plane deformations.

Appendix A. A micromechanical model of the reinforced plain weave fabric

A.1. Derivation of the basic model

Our first task is to carefully describe the geometry of the RUC of a plain weave fabric composite, see Fig. 1. As a background one can find all used geometrical quantities as well as the elastic properties of tows and matrix collected in Table 1.

Although the tows of the plain weave RUC merely crosses at 90° angle in the XY -plane, the shapes in the XZ - and YZ -planes are more complex. Moreover, in situ observations of the cross-sections of the composite show high deviations in the inclination angle θ defined in Fig. 1. Of course, multi-layered composites are much worse in that sense.

A common assumption for analytical models, see ref. [10] is a cosine or sine law for the midplane of the warp and fill tows. This leads to the expression for the midplane of the warp tow

$$z = f(x, y) = f(x) = h_0 \sin(2\pi x/\lambda), \quad (A1)$$

$$-\lambda/2 \leq x \leq \lambda/2,$$

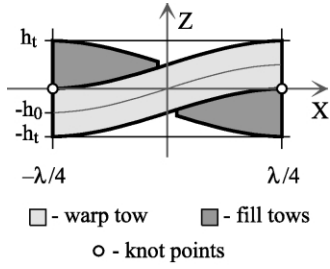


Fig. 9. Section of the RUC consisting of idealized tows. The midplane of each tow is a sine function.

where h_0 is the amplitude of the weave wave. Fig. 9 shows half of the XZ section of the RUC with sine tows.

For the un-twisted, non-deformed RUC, we have $h_0 = 0.5h_{\text{tow}}$ and $\theta = 2\pi h_0/\lambda$. The full thickness of the RUC is equal to $4h_0$ and the length of the midplane is

$$\begin{cases} l_0 = \int_{-\lambda/2}^{\lambda/2} \sqrt{1 + \omega^2 \cos^2\left(\frac{\omega x}{h_0}\right)} dx \\ = 4h_0 \frac{\sqrt{1 + \omega^2}}{\omega} E\left(\frac{\omega}{\sqrt{1 + \omega^2}}\right), \\ \omega = \frac{2\pi h_0}{\lambda}, \end{cases} \quad (\text{A2})$$

where $E(\cdot)$ is the complete elliptic integral of the second kind. But since $4\pi^2 h_0^2/\lambda^2$ usually is very small, we may with acceptable accuracy replace Eq. (A2) by

$$l_0 \approx \lambda + \frac{\pi^2}{\lambda} h_0^2. \quad (\text{A3})$$

The third term of the Taylor series expansion ($-3\pi^4 h^4/4\lambda^3$) in this particular case (see Table 1 for a λ and h_{tow}) is about 500 times smaller than the second term, which validates the usage of only the first two terms.

Due to symmetry, one may consider each tow in the RUC as a set of four beams with equal length $l_b = \lambda/4$. For simplicity, we assume that the cross-section of each beam is rectangular with height h_{tow} and width L . Consider a construction of two beams (like in Fig. 9) and two idealized independent load modes for the construction.

- Mode 1 bending of the beams with boundary conditions, which follow from the RUC symmetry

$$\begin{aligned} \frac{\partial v}{\partial x} \Big|_{x=\pm\lambda/4} &= 0, & v|_{x=\pm\lambda/4, z=\pm h_0} &= \mp \Delta Z, \\ \frac{\partial u}{\partial y} \Big|_{x=\pm\lambda/4} &= 0 \text{ (no rotation)}, & F_x|_{x=\pm\lambda/4} &= 0, \end{aligned} \quad (\text{A4})$$

where u is the X -displacement, v is the deflection of the midplane (Z -direction) and F_x is the integral force acting on the cross-section of the tow.

- Mode 2 extension of the beams in the axial direction by $\pm\Delta X$ with zero deflection in the knot points (see Fig. 9)

$$u|_{x=\pm\lambda/4} = \pm\Delta X, \quad v|_{x=\pm\lambda/4, z=\pm h_0} = 0. \quad (\text{A5})$$

The energy of Mode 1 is

$$\Delta U_1 = c_1 \Delta Z^2 = \frac{L}{2} \beta \Delta Z^2, \quad (\text{A6})$$

where an approximate value of β is given by elementary beam theory (which can be used since h_0 is small with respect to l_b , see Ref. [11]). In this case,

$$\beta = 192D/\lambda^3 = 16E_l \left(\frac{h_{\text{tow}}}{\lambda}\right)^3, \quad D = E_l I, \quad (\text{A7})$$

$$I = \frac{h_{\text{tow}}^3}{12},$$

where E_l is the axial modulus of the beam and D is the bending modulus of the beam. Similarly, for the load energy of Mode 2 one can write

$$\Delta U_2 = c_2 \Delta X^2 \approx \frac{L}{2} \alpha \Delta X^2, \quad \text{where } \alpha = E_l \frac{4h_{\text{tow}}}{\lambda}. \quad (\text{A8})$$

Consider all deformations, which can be described as a superposition of Mode 1 and Mode 2. The elastic energy for a composite deformation may be expressed as positive definite quadratic form with respect to ΔZ and ΔX

$$\Delta U = \bar{c}_1 \Delta Z^2 + \bar{c}_2 \Delta X^2 + \bar{c}_{12} \Delta X \Delta Z. \quad (\text{A9})$$

One may note that $\bar{c}_2 = c_2$ but $\bar{c}_1 \neq c_1$ because the boundary conditions Eq. (A4) allow $\Delta X \neq 0$ for loading in Mode 1. By assuming a conservation of the beam length l_0 in Mode 1, Eq. (A9) may be simplified to

$$\Delta U = c_1 \Delta Z^2 + c_2 (\Delta X - a(h_0) \Delta Z)^2, \quad (\text{A10})$$

where $a(h_0)$ is given by

$$a(h_0) = \frac{1}{4} \frac{\partial l_0}{\partial h_0} \left(\frac{\partial l_0}{\partial \lambda} \right)^{-1} \approx \frac{1}{4} \frac{\partial l_0}{\partial h_0} = \frac{\pi^2}{2\lambda} h_0. \quad (\text{A11})$$

The idealized RUC of plain weave fabric consists of 16 equal beams, four beams per tow. The warp and fill beams constrains each other in the out-of-plane direction. For clarity, let the load be applied in the warp (X -) direction and simulate a standard tensile test with zero value of far field average of all stress tensor components $\bar{\sigma}_{ij}$ except $\bar{\sigma}_{xx}$. In our idealized model, the fill tow lies in the Y -direction and is subjected to Mode 1 deformation only. The warp tows have a mixed load mode. Because both the warp and fill tows have the same deflection ΔZ and each RUC contain two warp tows which carry the load, the energy of the plain weave fabric RUC is

$$\Delta U_{\text{RUC}} = 2[2c_1 \Delta Z^2 + c_2 (\Delta X - a \Delta Z)^2]. \quad (\text{A12})$$

The ratio $\gamma = \Delta Z/\Delta X$ is not an arbitrary number. Indeed, the value of γ gives the deformed state of the fabric and must correspond to the minimum of elastic energy stored in the RUC. Minimization of Eq. (A12) with respect to

ΔZ gives γ as a function of the weave amplitude h_0

$$\gamma(h_0) = a(h_0) \left[a^2(h_0) + 2 \frac{\beta}{\alpha} \right]^{-1}. \quad (\text{A13})$$

From Eq. (A12) and Hooke's law, the E -modulus of the fabric is

$$E_x(h_0) = \frac{L}{4h_{\text{tow}}} [\alpha(1 - \gamma(h_0)a(h_0))^2 + 2\beta\gamma^2], \quad (\text{A14})$$

which may be simplified using Eq. (A13) to

$$E_x(h_0) = \frac{L\alpha}{4h_{\text{tow}}} [1 - \gamma(h_0)a(h_0)]. \quad (\text{A15})$$

Similarly, Poisson's ratio is given by

$$\nu_{xy}(h_0) = a(h_0)\gamma(h_0). \quad (\text{A16})$$

A.2. Improvements of the basic model

Eqs. (A15) and (A16) give predictions for both the E -modulus and Poisson's ratio, which may be measured by a simple tensile test. However, one can find those equations oversimplified. Let us improve the quality of the model.

First of all, the perpendicular tows also participate in forming of the bending modulus D in Eq. (A7). In order to obtain more accurate results, let us replace D in Eq. (A7) by $D^* = k_1 D$, where k_1 is defined by the following expressions

$$k_{1i} = \frac{1}{3D} [E_i(h_{\text{tow}} + h_i)^3 + E_i(h_{\text{tow}} - h_i)^3 + h_i^3(E_i + E_i)] \quad (\text{A17})$$

and h_i is the position of the stress-free surface, that is,

$$h_i = 0.5h_{\text{tow}} \frac{E_i - E_i}{E_i + E_i}. \quad (\text{A18})$$

Index i in Eqs. (A17) and (A18) takes the value 1 for the tow and the value 2 for the matrix. E_1 is assumed to be the tow transverse modulus and E_2 the matrix modulus. The volumetric harmonic average is used to obtain the correction factor k_1

$$k_1 = \lambda \left(\frac{2L}{k_{11}} + \frac{\lambda - 2L}{k_{12}} \right)^{-1}. \quad (\text{A19})$$

Although the transverse modulus of the tow is small, the correction above gives a value of k_1 close to 2, i.e. it almost doubles the value of the flexural energy of the RUC.

The axial modulus of the idealized beam is also affected by the existence of perpendicular tows. However, this effect is much smaller and our estimation gives

$$\alpha^* = \alpha k_2, \quad k_2 \approx 1 + E_1/E_i \approx 1.07. \quad (\text{A20})$$

Finally, we have the correction related to finite modulus of the matrix inclusions. For this reason, one may imagine two additional slices of material in both sides of the load-bearing tows. The approximate modulus of each slice from rules of

mixture and geometrical considerations is

$$E_{\text{slice}} = \frac{\lambda}{4} \left[\frac{L}{2(E_1 + E_2)} + \frac{\lambda - 2L}{4E_2} \right]^{-1}, \quad (\text{A21})$$

where the final correction must be weighted by the volumetric ratio of slice/tow. Thus, corrected version of Eq. (A16) is

$$E_x(h_0) = \frac{L\alpha}{4h_{\text{tow}}} [1 - \gamma(h_0)a(h_0)] + E_p, \quad (\text{A22})$$

$$E_p = \left(\frac{\lambda}{2L} - 1 \right) E_{\text{slice}}.$$

A.3. The case of biaxial loading

The model described earlier may simply be modified for biaxial loading of a single fabric layer. One may find the constitutive equations by minimizing the energy of the whole structure with respect to out-of-plane displacements at maxima points ΔZ

$$\Delta U_{\text{RUC}} = 2L[\alpha(\Delta X - a_1\Delta Z)^2 + \alpha(\Delta Y - a_2\Delta Z)^2 + 2\beta\Delta Z^2], \quad (\text{A23})$$

where ΔX and ΔY are in-plane displacements. Minimization gives the result

$$\begin{pmatrix} \Delta\sigma_{xx} \\ \Delta\sigma_{yy} \end{pmatrix} = \frac{L\alpha}{4h_{\text{tow}}} \begin{bmatrix} 1 - a_1q_{11} & -a_2q_{12} \\ -a_1q_{21} & 1 - a_2q_{22} \end{bmatrix} \begin{pmatrix} \Delta\epsilon_{xx} \\ \Delta\epsilon_{yy} \end{pmatrix} + E_p \begin{pmatrix} \Delta\epsilon_{xx} \\ \Delta\epsilon_{yy} \end{pmatrix}, \quad (\text{A24})$$

where the matrix q is related to the out-of-plane displacement

$$\Delta Z \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \gamma_1 \Delta X \\ \gamma_2 \Delta Y \end{pmatrix} = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix} \begin{pmatrix} \Delta X \\ \Delta Y \end{pmatrix} \quad (\text{A25})$$

and is given by

$$\begin{aligned} q_{11} &= q_{21} = a_1(2\beta/\alpha + a_1^2 + a_2^2)^{-1}, \\ q_{12} &= q_{22} = a_2(2\beta/\alpha + a_1^2 + a_2^2)^{-1}. \end{aligned} \quad (\text{A26})$$

One can see that for unidirectional tensile load $\sigma_{yy} = 0$, Eqs. (A26), (A24) and (A15), (A13) coincides.

The geometric variable $a(h_0)$ for X - and Y -oriented beams differs, hence straightening of the warp yarn leads to over-bending of the fill yarn, i.e. we now need two variables a_1 and a_2 . Thus, we have for the initial non-twisted sine-shaped plain weave RUC in the undeformed state that $h_0 = h_{\text{tow}}/2$ and

$$a_1 = a = \frac{\pi^2}{2\lambda} h_0, \quad a_2 = \frac{\pi^2}{2\lambda} (h_{\text{tow}} - h_0), \quad (\text{A27})$$

which leads to a possible increase of h_0 during the load if $\Delta X < \Delta Y$ and $\Delta X > 0$, $\Delta Y > 0$.

A.4. Final notes

All the expressions above assume that the warp and fill tows are identical. It is rather straight-forward to do the necessary modifications if those two sets of tows are different with respect to their mechanic properties. For higher accuracy, the main characteristics of beams, such as α , β and a may be obtained numerically from 2D FEM models by simulations of different load modes of the beam.

Appendix B. The reiterated homogenization method

The determination of macroscopic (or effective) properties for materials with microscopic heterogeneous structure (such as composites) has interested scientists for a long time. Some of the first results in this field are due to physicists, such as Einstein, Maxwell, and Rayleigh. In the late 1960s, the problem of describing the effective properties of composites was reformulated in such a way that this field became interesting from a mathematical point of view. Since the problems studied involve partial differential equations with rapidly oscillating coefficients, a direct numerical solution of the governing equations is in general impossible. Therefore, one has to do some type of averaging or asymptotic analysis. The homogenization theory is the branch of mathematics that deals with the existence, uniqueness, convergence, etc. of this kind of problems.

By considering materials with several subscales of heterogeneities, reiterated homogenization was introduced on the physical level by Bruggeman [12] in the 1930s and mathematically justified by Bensoussan et al. [13] in 1978, where the iterated homogenization theorem for linear operators was proved. This theorem has been an indispensable tool in the construction of structures with extreme effective material properties. The iterated homogenization theorem was generalized by Lions et al. [14] in 2000 to cover non-linear monotone operators. A good introduction to this field of mathematics is given in the review article [15] by Lukkassen and Milton. We also mention the papers [16] and [4] as examples of applications of reiterated homogenization to woven composites.

B.1. The main result

In this section, we state the main reiterated homogenization theorem for the elasticity problem. We refer to the appendix in paper [4] for a proof of this result, where the ideas from Ref. [13] was used.

Let us consider a linear elastic body which occupies a region Ω in $\mathbf{R}^3$ and introduce a Cartesian coordinate system x_i . Assume that the body consists of two or more different linear elastic materials which are periodically distributed in

the sense that we can define unit cells which are periodically repeated. We also assume that we have two different scales of periodicity. We define the strain tensor e_{ij} by

$$e_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right),$$

where u is the displacement in Ω . Moreover, we assume that the stiffness tensor $C_{ijkl} = C_{ijkl}(y, z)$ is Y -periodic in its first argument, Z -periodic in its second argument and satisfies

$$\alpha \xi_{ij} \xi_{ij} \leq C_{ijkl} \xi_{ij} \xi_{kl} \leq \beta \xi_{ij} \xi_{ij} \text{ a.e. in } \Omega,$$

for every symmetric real-valued tensors ξ_{ij} where $0 < \alpha \leq \beta < \infty$. The reiterated homogenization theorem then states that the effective stiffness tensor $\bar{C}_{ijmn}$ is given by

$$\bar{C}_{ijmn} = \frac{1}{|Y|} \int_Y \hat{C}_{ijmn}(y) + \hat{C}_{ijkl}(y) e_{kl}(U^{mn}) dy, \quad (B1)$$

where $U^{mn} = U^{mn}(y)$ are the weak solutions of the Y -cell problems

$$\begin{cases} -\frac{\partial}{\partial y_j} (\hat{C}_{ijkl}(y) e_{kl}(U^{mn})) = \frac{\partial \hat{C}_{ijmn}}{\partial y_j} \text{ on } Y, \\ U_i^{mn} \text{ is } Y\text{-periodic.} \end{cases} \quad (B2)$$

The Z -cell homogenized tensor $\hat{C}_{ijkl} = \hat{C}_{ijkl}(y)$ is given by

$$\hat{C}_{ijkl}(y) = \frac{1}{|Z|} \int_Z C_{ijkl}(y, z) + C_{ijmn}(y, z) e_{mn}(V^{kl}) dz, \quad (B3)$$

where $V^{kl} = V^{kl}(y, z)$ are the weak solutions of the Z -cell problems

$$\begin{cases} -\frac{\partial}{\partial z_j} (C_{ijmn}(y, z) e_{mn}(V^{kl})) = \frac{\partial C_{ijkl}}{\partial z_j} \text{ on } Z, \\ V_i^{kl} \text{ is } Z\text{-periodic.} \end{cases} \quad (B4)$$

B.2. Numerical solution

The woven composite can be considered to have two different subscales, the first being on the tow level and the second being on the bundle level. To use the reiterated homogenization method, we therefore let the RUC be the Y -cell and assume that the fibers are periodically distributed in each tow in order to define a Z -cell. To be more specific, we assume that the fibers in the tows are arranged in a periodic hexagonal array so the Z -cell homogenized tensor becomes isotropic.

We first solve the Z -cell problems (Eq. (B4)) numerically in ANSYS, using a mesh with about 10,000 2D elements. We then use the solution to compute the Z -cell homogenized tensor ((B3)), which in turn gives the tow stiffnesses. Knowing the tow stiffnesses, we model the single-layer woven composite RUC in ANSYS and discretize the RUC with about 22,000 3D elements. By solving the Y -cell problems (Eq. (B2)), see Fig. 10, we finally get the effective

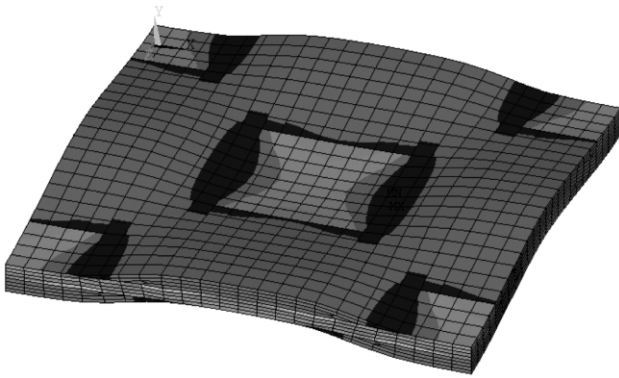


Fig. 10. Solution of the Y-cell problem under applied load. The RUC curves during loading due to tow straightening leading to an increase in the effective stiffness.

stiffness tensor by Eq. (B1). One should, however, note that since we model a single-layer composite, we cannot use actual Y -periodic boundary conditions in Eq. (B2) since the surfaces with out-of-plane normals must be free by nature. Instead, the solutions U^{mm} are only supposed to be periodic in the XY -plane.

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