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A NUMERICAL STUDY OF RANDOM HOMOGENIZATION

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1 An example of random heterogeneous media

Let each point $(m, n) \in \mathbb{Z}^2$ be the centre of a circle of radius $1/4$. In this way we have split $\mathbb{R}^2$ into two sets, namely points inside the circles and points outside the circles. Let us now assume that $\mathbb{R}^2$ consists of two different isotropic materials, material 1 in the circles and the rest of material 2. Material 1 has conductivity a_1 and material 2 has conductivity a_2 .

Let us now randomly move each circle a distance r in the direction α , where $0 \leq r \leq 1/4$ and $0 \leq \alpha \leq 2\pi$. On the set $S_{mn} = (0, 1/4) \times (0, 2\pi)$ ($(m, n) \in \mathbb{Z}^2$) we define the measure $\lambda_S = 4dr \times 1/2\pi d\alpha$. Let $\Gamma = \prod_{(m,n) \in \mathbb{Z}^2} S_{mn}$ and λ_Γ the product measure on Γ . Γ can now be identified with the set of functions Γ^* , where

$$\Gamma^* = \left\{ \begin{array}{l} \gamma = 1 \text{ on each moved circle,} \\ \gamma = 2 \text{ otherwise.} \end{array} \right.$$

To the set Γ^* we now add all the functions which are obtained by a shift. In this way we obtain a new set Ω of functions, namely

$$\Omega = \{ \omega : \omega(t) = \gamma(t + \eta), \gamma \in \Gamma^*, \eta \in \mathbb{R}^2 \}.$$

The set Ω is naturally associated with $\Gamma \times \mathbb{R}^2/\mathbb{Z}^2$ and we can define a measure μ on Ω as $\mu = \lambda_\Gamma \times dx$, where dx stands for the Lebesgue measure. By construction Ω is translationally invariant, i.e. contains all functions of the form $\omega(\cdot + x)$. We now introduce the dynamical system $T_x : \Omega \rightarrow \Omega$ defined as

$$(T_x \omega)(t) = \omega(t + x).$$

Let the function $g : \Omega \rightarrow \mathbb{R}$ be defined as

$$g(\omega) = \left\{ \begin{array}{l} a_1 \text{ if } \omega(0) = 1, \\ a_2 \text{ if } \omega(0) = 2. \end{array} \right.$$

We now define the random field $a(x, \omega)$ in terms of realizations of the function g in the following way

$$a(x, \omega) = g(T_x \omega) = \left\{ \begin{array}{l} a_1 \text{ if } \omega(x) = 1, \\ a_2 \text{ if } \omega(x) = 2. \end{array} \right.$$

To sum up, we have now constructed a random field which may be used to model certain 2-phase composite materials with circular inclusions. Moreover, it is clear that the dynamical system is ergodic. See Figure 1 where we have extracted a 16×16 subcell from the whole structure. For more facts concerning random media we refer to [4].

2 Homogenization of random operators

Let $(\Omega, \mathcal{F}, \mu)$ be a probability space and T_x ($x \in \mathbb{R}^N$) an ergodic dynamical system on Ω . Let $A = A(\omega)$ be a measurable matrix function such that

$$\begin{aligned} \langle A(\omega)\xi, \xi \rangle &\geq \alpha |\xi|^2, \quad \xi \in \mathbb{R}^N, \quad \alpha > 0, \\ a_{ij}(\omega) &\leq \alpha^{-1}, \end{aligned}$$

for every $\xi \in \mathbb{R}^N$ and for $\omega \in \Omega_0$, where Ω_0 is a measurable subset of Ω such that $\mu(\Omega_0) = 1$. Let (ε) be a sequence of positive numbers converging to 0 and Q an open bounded subset of $\mathbb{R}^N$. Given $f \in W^{-1,2}(Q)$ let us consider the following Dirichlet boundary value problem:

$$\left\{ \begin{array}{l} -\operatorname{div}(A(x/\varepsilon, \omega)\nabla u_\varepsilon) = f \text{ on } Q, \\ u_\varepsilon \in W_0^{1,2}(Q). \end{array} \right. \quad (1)$$

By the Lax-Milgram lemma there exists a unique solution $u_\varepsilon \in W_0^{1,2}(Q)$ for each ε .

We have the following homogenization result:

Theorem 1 *Let (u_ε) be the solutions of (1). Then for a.e. $\omega \in \Omega$ we have that*

$$\begin{aligned} u_\varepsilon &\rightharpoonup u \quad \text{weakly in } W_0^{1,2}(Q), \\ A(x/\varepsilon, \nabla u_\varepsilon) &\rightharpoonup A_{\text{hom}} \nabla u \quad \text{weakly in } [L^2(Q)]^N, \end{aligned}$$

where u is the unique solution of the homogenized equation

$$\left\{ \begin{array}{l} -\operatorname{div}(A_{\text{hom}} \nabla u) = f \quad \text{on } Q, \\ u \in W_0^{1,2}(Q). \end{array} \right.$$

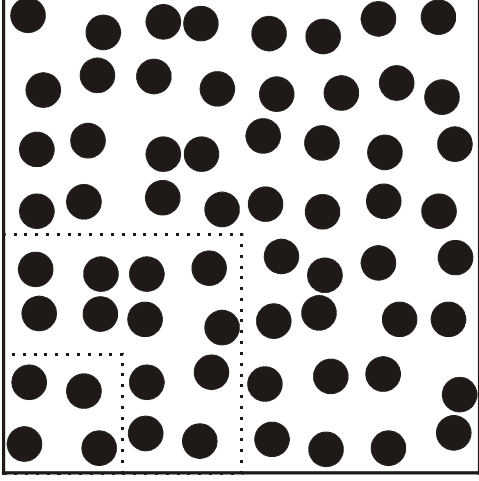


Figure 1: Periodic random 16×16 cell with two sub-cells indicated.

The operator $A_{\text{hom}} : \mathbb{R}^N \rightarrow \mathbb{R}^N$ is defined for every $\eta \in \mathbb{R}^N$ by

$$A_{\text{hom}}\eta = \int_{\Omega} A(\omega)(\eta + v_{\eta}(\omega)) d\mu,$$

where v_{η} is the solution of the auxiliary problem: Find $v_{\eta} \in V_{\text{pot}}^2(\Omega)$ such that

$$A(\omega)(\eta + v_{\eta}(\omega)) \in L_{\text{sol}}^2(\Omega).$$

3 Periodic approximation of A_{hom}

Let the matrix $A(x, \omega)$ be restricted to the cube $Y_{\delta} = [0, \delta]^N$ and then extended by periodicity to $\mathbb{R}^N$. We denote the extension $A_{\delta, \text{per}}(x, \omega)$. For each ω we then have a family of equations

$$\begin{cases} -\operatorname{div}(A_{\delta, \text{per}}(x/\varepsilon, \omega)\nabla u_{\varepsilon}) = f \text{ on } Q, \\ u_{\varepsilon} \in W_0^{1,2}(Q), \end{cases} \quad (2)$$

with periodic coefficients. Thus (2) can be homogenized in the standard way. The so defined homogenized matrix $A_{\delta, \text{hom}}$ is given by

$$A_{\delta, \text{hom}}\eta = \frac{1}{|Y_{\delta}|} \int_{Y_{\delta}} A(y)(\eta + v_{\eta}(y)) dy,$$

where v_{η} is the solution of

$$-\operatorname{div}(A(y)(\eta + v_{\eta}(y))) = 0 \quad \text{on } Y_{\delta}, \quad v_{\eta}(y) \in W_{\text{per}}^{1,2}(Y_{\delta}).$$

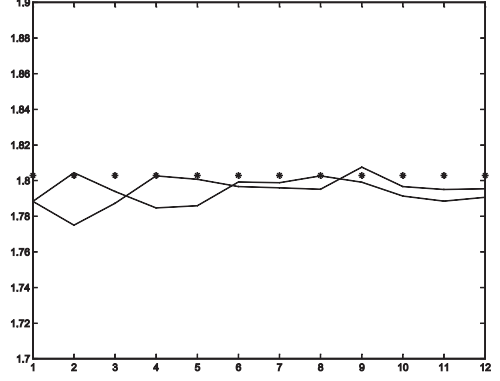


Figure 2: Effective properties for two sequences of subcells compared with the effective properties in the random limit.

For more information concerning periodic homogenization see e.g. [3]. Recently it was proved in [1] that

$$A_{\delta, \text{hom}} \rightarrow A_{\text{hom}} \quad \text{as } \delta \rightarrow \infty,$$

(i.e. $a_{ij}^{\delta, \text{hom}} \rightarrow a_{ij}^{\text{hom}}$ as $\delta \rightarrow \infty$, $i, j = 1, \dots, N$).

4 Numerical analysis

In the talk we will present several different numerical studies of this convergence. We will also compare random homogenization with periodic, see [2]. For the example described above, we have plotted two realizations in Figure 2. The picture shows the convergence of the effective properties for sequences of $n \times n$ subcells extracted from a 16×16 random cell compared with the effective properties in the random limit.

References

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