

Periodic Approximation of Elastic Properties of Random Media

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Abstract. Most papers on stochastic homogenization either deal with theoretical aspects or with questions regarding computational issues. Since the theoretical analysis involves the solution of a problem which is stated in an abstract probability space, it is not clear how the two areas are connected. In previous works this problem has not been considered. However, recently Bourgeat and Piatnitski investigated this connection in the scalar case for second order operators of divergence form. They proved that in the limit, the method of periodic approximation gives the same effective properties as in stochastic homogenization. In this paper we prove similar results for the vector valued case, which appears in e.g. the theory of elasticity. Moreover, we provide a numerical analysis of the results.

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1. Introduction

Using the rigorous theory of stochastic homogenization one can homogenize the characteristics of a random material which are involved in models of different physical phenomena described by e.g. a second order operator with random stationary coefficients. However, since stochastic homogenization involves the need to solve a problem which is stated in an abstract probability space it is not clear how one should construct (or even approximate) the effective coefficients. This is of course in contrast to the technique of periodic homogenization which gives us a clear recipe on how to find the effective coefficients, see e.g. [7], [8] and [10]. Thus one has to use other methods to achieve this for a random material. One example of such a method is periodic approximation, see [2], which means that one takes a finite part, e.g. $Y_\rho = [0, \rho]^n \subset \mathbb{R}^n$ of a realization, periodically extends it to the whole $\mathbb{R}^n$ and then uses periodic homogenization. Such methods have

been used by many authors, but there is no indication at all how these computations are connected to the rigorous mathematical results in stochastic homogenization, see [8], [10], [9], [5] and [6]. However, in [2] this connection was investigated in the scalar case for second order operators of divergence form. It was rigorously proved that in the limit periodic approximation gives the same effective coefficients as in stochastic homogenization. For a numerical analysis of the results in [2], see e.g. [1]. In this paper we prove similar results for the vector valued case, which appears in e.g. the theory of elasticity. Moreover, we provide some numerical convergence experiments and illustrations.

The paper is organized in the following way: In the second section we give some preliminaries and set up the notation used. In the Section 3 and 4 we have the homogenization results corresponding to stochastic homogenization and periodic approximation. We also prove that periodic approximation gives the same effective coefficients as stochastic homogenization. In Section 4 we rigorously define a certain type of random, heterogeneous material and provide a numerical analysis where we compare periodic approximation, developed in this paper, with other methods.

2. Notation and Preliminaries

Let $Q \subset \mathbb{R}^n$ denote an open regular bounded region. The space of vector valued functions $\mathbf{u} = \{u_1, \dots, u_n\}$ with components in $L^2(Q)$ is denoted by $\mathbf{L}^2(Q)$. For matrix valued functions $\eta = \{\eta_{ij}\}$ with components in $L^2(Q)$ the same notation is used but it should be clear from the context which space we consider. The notation of $\mathbf{H}^{1,2}(Q)$ is equally obvious, i.e. it consists of vector valued functions $\mathbf{u} = \{u_1, \dots, u_n\}$ with $u_i \in L^2(Q)$ and $Du_i \in [L^2(Q)]^n$ where Du_i denotes the gradient of u_i . The space $\mathbf{H}_0^{1,2}(Q)$ consists of the functions in $\mathbf{H}^{1,2}(Q)$ which are zero on the boundary of Q . Also, the space $\mathbf{H}_{per}^{1,2}(Y_\rho)$ is the set of all Y_ρ -periodic functions $\mathbf{u}$ with mean value zero such that $\mathbf{u} \in \mathbf{H}^{1,2}(Y_\rho)$. A symmetric matrix valued function $\mathbf{v} = \{v_{ij}\}$, $v_{ij} \in L^2(Q)$ is called a potential matrix if $\mathbf{v} = \mathbf{e}(\mathbf{u})$ for some $\mathbf{u} \in \mathbf{H}^{1,2}(Q)$, where $\mathbf{e}(\mathbf{u})$ is the strain tensor

$$\{\mathbf{e}(u)_{ij}\} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right).$$

Let $(\Omega, \mathcal{F}, \mu)$ be a probability space. An n -dimensional dynamical system T_x , $x \in \mathbb{R}^n$ on Ω is a family of invertible mappings $T_x : \Omega \rightarrow \Omega$, $x \in \mathbb{R}^n$ such that:

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- $T_{x+y} = T_x T_y$, $T_0 = I$ (identity operator).
- T_x is measure preserving on Ω , i.e. $\mu(\mathcal{A}) = \mu(T_x(\mathcal{A}))$ for any $\mathcal{A} \in \mathcal{F}$ and any $x \in \mathbb{R}^n$.
- For any measurable function f on Ω , the function $f(T_x \omega)$ defined on $\mathbb{R}^n \times \Omega$ is measurable.

Now a random field can be introduced as follows: for a random variable $f = f(\omega)$, we define a random field on $\mathbb{R}^n$ by $f(T_x \omega)$. It is well known that a random field defined this way is statistically homogeneous. Next we turn our attention to ergodicity. A measurable function f defined on Ω is called invariant if for a.e. $\omega \in \Omega$ the equality $f(\omega) = f(T_x \omega)$ holds for every x in $\mathbb{R}^n$. A dynamical system is called ergodic if any invariant function is constant a.e. in Ω . From now on the dynamical system is assumed to be ergodic.

For the readers convenience we recall the following compensated compactness result (for a proof see [8]):

Lemma 1. *Consider two sequences of matrix valued functions $\mathbf{p}_\varepsilon, \mathbf{v}_\varepsilon \in \mathbf{L}^2(Q)$. Assume that the matrices $\mathbf{v}_\varepsilon$ are potential, and*

$$\begin{aligned} \mathbf{p}_\varepsilon &\rightarrow \mathbf{p} \text{ weakly in } \mathbf{L}^2(Q), \\ \mathbf{v}_\varepsilon &\rightarrow \mathbf{v} \text{ weakly in } \mathbf{L}^2(Q), \\ \operatorname{div} \mathbf{p}_\varepsilon &= \mathbf{f} \in \mathbf{H}^{-1}(Q). \end{aligned}$$

Then $\int_Q (\mathbf{p}_\varepsilon \cdot \mathbf{v}_\varepsilon) \phi dx \rightarrow \int_Q (\mathbf{p} \cdot \mathbf{v}) \phi dx$ for all $\phi \in C_0^\infty(Q)$.

3. Stochastic Homogenization

In this section we briefly recall some facts concerning stochastic homogenization. Let $\mathcal{S}^n$ denote the set of all symmetric $n \times n$ matrices (i.e. matrices $\eta = \{\eta_{ij}\}$ such that $\eta_{ij} = \eta_{ji}$). Let a tensor function $\mathbf{A}(\omega) = \{a_{ijkl}(\omega)\}$ be given such that

$$\begin{aligned} \mathbf{A}\eta \cdot \eta &\geq \alpha |\eta|^2, \quad \eta \in \mathcal{S}^n, \\ |\mathbf{A}\eta| &\leq \beta |\eta|, \quad \eta \in \mathcal{S}^n, \end{aligned}$$

where α and β are positive constants. Here $\cdot$ is the usual scalar product between matrices, i.e. $\eta \cdot \xi = \eta_{ij} \xi_{ij}$, and $|\eta|^2 = \eta_{ij} \eta_{ij}$. Define the following

family of problems

$$(1) \quad \begin{cases} -\operatorname{div}(\mathbf{A}(T_{x/\varepsilon}\omega) \mathbf{e}(\mathbf{u}_\varepsilon)) = \mathbf{f}, \\ \mathbf{u}_\varepsilon \in \mathbf{H}_0^{1,2}(Q), \end{cases}$$

where $\mathbf{u}_\varepsilon$ is the displacement vector and $\mathbf{f}$ the body force. By the results in [8] the following convergencies hold a.s. (when $\varepsilon \rightarrow 0$):

$$(2) \quad \begin{aligned} \mathbf{u}_\varepsilon &\rightarrow \mathbf{u} \text{ weakly in } \mathbf{H}_0^{1,2}(Q), \\ \mathbf{A}(T_{x/\varepsilon}\omega) \mathbf{e}(\mathbf{u}_\varepsilon) &\rightarrow \mathbf{B}\mathbf{e}(\mathbf{u}) \text{ weakly in } \mathbf{L}^2(Q) \end{aligned}$$

where $\mathbf{u}$ is the solution of

$$(3) \quad \begin{cases} -\operatorname{div}(\mathbf{B}\mathbf{e}(\mathbf{u})) = \mathbf{f}, \\ \mathbf{u} \in \mathbf{H}_0^{1,2}(Q). \end{cases}$$

The tensor $\mathbf{B} = \{b_{ijkl}\}$ is constant and satisfies similar structure conditions as $\mathbf{A}$ and thus problem (3) indeed has a unique solution. Let us point out that the effective coefficients can be "computed" using the solutions of the following auxiliary problem: given $\eta \in \mathcal{S}^n$ find $\mathbf{v}^\eta \in \mathcal{V}_{pot}^2(\Omega)$ such that

$$(4) \quad \int_{\Omega} \mathbf{A}(T_{x/\varepsilon}\omega) (\eta + \nu^\eta) \cdot \phi d\mu = 0$$

for all $\phi \in \mathcal{V}_{pot}^2(\Omega)$. Here $\mathcal{V}_{pot}^2(\Omega)$ is the space of potential matrices having zero mean value. Note that a symmetric matrix $\mathbf{v}(\omega) = \{v_{ij}(\omega)\}$, $v_{ij} \in L^2(\Omega)$ is said to be potential if almost all realizations $\mathbf{v}(T_x\omega)$ are potential matrices on $\mathbb{R}^n$. Then $\mathbf{B}$ can be computed by

$$\mathbf{B}\eta = \int_{\Omega} \mathbf{A}(\omega) (\eta + \nu^\eta) d\mu,$$

where ν^η is the solution of (4).

4. Periodic Approximation

The coefficients $\mathbf{A}(T_x\omega)$ are first restricted onto the cube $Y_\rho = [0, \rho]^n$ and then extended from Y_ρ to $\mathbb{R}^n$ periodically with period ρ in each coordinate direction so that

$$\mathbf{A}_{per}^\rho(x, \omega) = \begin{cases} \mathbf{A}(T_x\omega) & \text{if } x \in Y_\rho \\ \mathbf{A}(T_{x(\bmod Y_\rho)}\omega) & \text{otherwise.} \end{cases}$$

The problems

$$\begin{cases} -\operatorname{div} \left(\mathbf{A}_{per}^\rho \left(\frac{x}{\varepsilon}, \omega \right) \mathbf{e}(\mathbf{u}_\varepsilon) \right) = \mathbf{f}, \\ \mathbf{u}_\varepsilon \in \mathbf{H}_0^{1,2}(Q), \end{cases}$$

can now be homogenized using periodic homogenization (see e.g. [8]) and the effective tensor $\mathbf{B}^\rho = \{b_{ijkl}^\rho\}$ is given by (a.s.)

$$(5) \quad \mathbf{B}^\rho \eta = \frac{1}{\rho^n} \int_{Y_\rho} \mathbf{A}_{per}^\rho(x, \omega) (\eta + \mathbf{e}(\chi_\eta^\rho)) dx, \text{ for all } \eta \in \mathbb{R}^n,$$

where $\chi_\eta^\rho = \chi_\eta^\rho(x, \omega)$ is the solution of the local problem:

$$(6) \quad \begin{cases} -\operatorname{div} (\mathbf{A}_{per}^\rho(x, \omega) (\eta + \mathbf{e}(\chi_\eta^\rho))) = \mathbf{0}, \\ \chi_\eta^\rho \in \mathbf{H}_{per}^{1,2}(Y_\rho). \end{cases}$$

We note especially that $\mathbf{B}^\rho$ in general depends on ω . A natural question is now to ask if this periodic approximation gives the same result as stochastic homogenization, i.e. if $\mathbf{B}^\rho \rightarrow \mathbf{B}$ as $\rho \rightarrow \infty$ a.s.?

In order to have problem (6) stated in Y_1 for all ρ the rescaling $y = x/\rho$ is made and (6) translates to

$$(7) \quad \begin{cases} -\operatorname{div} (\mathbf{A}_{per}^\rho(\rho x, \omega) (\eta + \mathbf{e}(\bar{\chi}_\eta^\rho))) = \mathbf{0}, \\ \bar{\chi}_\eta^\rho \in \mathbf{H}_{per}^{1,2}(Y_1), \end{cases}$$

where $\bar{\chi}_\eta^\rho(x, \omega) = \frac{1}{\rho} \chi_\eta^\rho(\rho x, \omega)$. Since $\mathbf{A}_{per}^\rho(\rho x, \omega) = \mathbf{A}(T_{\rho x} \omega)$ for $x \in Y_1$,

$$(8) \quad \mathbf{B}^\rho \eta = \int_{Y_1} \mathbf{A}(T_{\rho x} \omega) (\eta + \mathbf{e}(\bar{\chi}_\eta^\rho)) dx, \text{ for all } \eta \in \mathbb{R}^n.$$

By the Korn inequality, (7) gives the a priori estimates

$$\begin{aligned} \|\bar{\chi}_\eta^\rho\|_{\mathbf{H}_{per}^{1,2}(Y_1)} &\leq C, \\ \|\mathbf{A}_{per}^\rho(\rho x, \omega) (\eta + \mathbf{e}(\bar{\chi}_\eta^\rho))\|_{\mathbf{L}^2(Y_1)} &\leq C, \end{aligned}$$

and one can conclude that there exist $\bar{\chi}_\eta \in \mathbf{H}_{per}^{1,2}(Y_1)$ and $\mathbf{q} \in \mathbf{L}^2(Y_1)$ such that (for a subsequence)

$$(9) \quad \begin{aligned} \bar{\chi}_\eta^\rho(x) &\rightarrow \bar{\chi}_\eta(x) \text{ weakly in } \mathbf{H}_{per}^{1,2}(Y_1), \\ \mathbf{A}_{per}^\rho(\rho x, \omega) (\eta + \mathbf{e}(\bar{\chi}_\eta^\rho)) &\rightarrow \mathbf{q} \text{ weakly in } \mathbf{L}^2(Y_1), \end{aligned}$$

as $\rho \rightarrow \infty$.

Now we want to show that $\mathbf{q} = \mathbf{B}(\eta + \mathbf{e}(\bar{\chi}_\eta(x)))$. For this purpose, consider the following auxiliary problem

$$\begin{cases} \operatorname{div}(\mathbf{A}(T_{\rho x}\omega)(\mathbf{e}(\mathbf{u}^\rho))) = \operatorname{div}(\mathbf{B}\mathbf{e}(\phi)), \\ \mathbf{u}^\rho \in \mathbf{H}_0^{1,2}(Y_1), \end{cases}$$

where $\phi \in \mathbf{C}_0^\infty(Y_1)$. By (2) a.s. (as $\rho \rightarrow \infty$)

$$\begin{aligned} \mathbf{u}^\rho &\rightarrow \mathbf{u} \text{ weakly in } \mathbf{H}_0^{1,2}(Q), \\ \mathbf{A}(T_{\rho x}\omega)\mathbf{e}(\mathbf{u}^\rho) &\rightarrow \mathbf{B}\mathbf{e}(\mathbf{u}) \text{ weakly in } \mathbf{L}^2(Q), \end{aligned}$$

where $\mathbf{u}$ is the solution of the homogenized equation

$$\begin{cases} \operatorname{div}(\mathbf{B}(\mathbf{e}(\mathbf{u}))) = \operatorname{div}(\mathbf{B}\mathbf{e}(\phi)), \\ \mathbf{u} \in \mathbf{H}_0^{1,2}(Y_1). \end{cases}$$

Thus $\mathbf{u} = \phi$ and we obtain a.s. (as $\rho \rightarrow \infty$)

$$\begin{aligned} \mathbf{u}^\rho &\rightarrow \phi \text{ weakly in } \mathbf{H}_0^{1,2}(Y_1), \\ \mathbf{A}(T_{\rho x}\omega)\mathbf{e}(\mathbf{u}^\rho) &\rightarrow \mathbf{B}\mathbf{e}(\phi) \text{ weakly in } \mathbf{L}^2(Y_1), \end{aligned}$$

Since $\mathbf{A}(T_{\rho x}\omega)$ is symmetric, the following equality holds

$$(\eta + \mathbf{e}(\bar{\chi}_\eta^\rho)) \cdot \mathbf{A}(T_{\rho x}\omega)\mathbf{e}(\mathbf{u}^\rho) = \mathbf{e}(\mathbf{u}^\rho) \cdot \mathbf{A}(T_{\rho x}\omega)(\eta + \mathbf{e}(\bar{\chi}_\eta^\rho)) \text{ on } Y_1.$$

Using compensated compactness and passing to the limit, we obtain

$$(\eta + \mathbf{e}(\bar{\chi}_\eta)) \cdot \mathbf{B}\mathbf{e}(\phi) = \mathbf{e}(\phi) \cdot \mathbf{q}$$

which implies that $\mathbf{q} = \mathbf{B}(\eta + \mathbf{e}(\bar{\chi}_\eta))$ since $\phi \in \mathbf{C}_0^\infty(Y_1)$ was arbitrary. Using (8) and the second convergence in (9), we obtain

$$\mathbf{B}^\rho \eta = \int_{Y_1} \mathbf{A}(T_{\rho x}\omega)(\eta + \mathbf{e}(\bar{\chi}_\eta^\rho)) dx \rightarrow \int_{Y_1} \mathbf{B}(\eta + \mathbf{e}(\bar{\chi}_\eta)) dx = \mathbf{B}\eta.$$

Since this is true for every $\eta \in \mathcal{S}^n$ we have $\mathbf{B}^\rho \rightarrow \mathbf{B}$ as $\rho \rightarrow \infty$.

5. Numerical Analysis

In this section we construct a class of random elasticity tensors which admit homogenization. Moreover, we use the technique of periodic approximation to compute the effective properties for realizations belonging to this class.

5.1 A class of random media

We will construct a family of described by random fields.

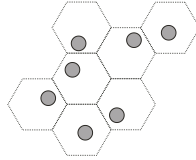


Figure 1. A piece of a realization

Consider a hexagon with side length h . In the center of this hexagon we put a circular inclusion with radius r . Assume that the hexagon consists of two different isotropic materials, material 1 in the circles and material 2 in the complement, with elasticity tensors $\mathbf{C}^1 = \{c_{ijkl}^1\}$ and $\mathbf{C}^2 = \{c_{ijkl}^2\}$ respectively. Now we periodically cover $\mathbb{R}^2$ with such hexagons. We proceed by randomly moving each circle inside every hexagon in such a way that they still are completely inside the corresponding hexagon, see Figure 1. Each hexagon can be identified with the set $S_{mn} = \{\mathcal{R}\}$, $m, n \in \mathbb{Z}$ (see Figure 2) and on this set we define the measure $\lambda_{S_{mn}} = (1/|\mathcal{R}|) dxdy$, where $|\mathcal{R}|$ is the area of $\mathcal{R}$.

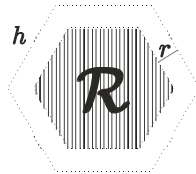


Figure 2. The marked region $\mathcal{R}$ consists of the admissible points for the center of a randomly moved circle inside the hexagon

Let $\Gamma = \prod_{(m,n) \in \mathbb{Z}^2} S_{mn}$ and λ_Γ be the product of the measures $\lambda_{S_{mn}}$. Then λ_Γ is a measure on Γ . Γ can now be identified with the set of functions Γ^* , where

$$\Gamma^* = \{\gamma : \gamma = 1 \text{ on each moved circle and } \gamma = 2 \text{ otherwise}\}.$$

To the set Γ^* we add all the functions which are obtained by a shift. In this way we obtain a new set Ω of functions, namely

$$\Omega = \{\omega : \omega(t) = \gamma(t + \eta), \gamma \in \Gamma^*, \eta \in \mathbb{R}^2\}.$$

The set Ω is naturally associated with $\Gamma \times \mathbb{R}^2/\mathbb{Z}^2$ and we can define a measure μ on Ω as $\mu = \lambda_\Gamma \times dx$, where dx stands for the Lebesgue measure. By construction Ω is translation invariant, i.e. contains all functions of the form $\omega(\cdot + x)$. We introduce the dynamical system $T_x : \Omega \rightarrow \Omega$ defined as

$$(T_x \omega)(t) = \omega(t + x).$$

Let the tensor $\mathbf{A}(\omega) = \{a_{ijkl}(\omega)\}$ be defined as

$$\mathbf{A}(\omega) = \begin{cases} \mathbf{C}^1 & \text{if } \omega(0) = 1, \\ \mathbf{C}^2 & \text{if } \omega(0) = 2. \end{cases}$$

Finally, we define a random field in terms of realizations of this tensor, i.e. $\mathbf{A}(x, \omega) = \mathbf{A}(T_x \omega)$. This field will be the coefficients in the linear elasticity equation.

To sum up, we have now constructed a random field which may be used to model certain two-phase composite materials with circular inclusions. Moreover, it is clear that the dynamical system is ergodic (see e.g. the book [4]).

5.2 Numerical results

Here we use the technique described in Section 4 (see also [2]) to compute effective properties of an example of random composite media. There is also another frequently used method for computing effective properties of random composite media, see e.g. [3] where it is applied to an electrostatic problem. We will refer to it as the mean value method and it works as follows: Assume that we for a realization cut out a cell of periodicity (e.g. Y_4). Then use standard periodic homogenization to compute the effective properties. Repeat this procedure N times (for equally large cells, e.g. Y_4 , but different realizations) and compute the average of the effective properties. This average is then used as a numerical approximation of the effective properties for the random media. Contrary to the method described in Section 4, there is no indication on how this method is related to the theoretical results in stochastic homogenization. We refer to the book [11] for more information concerning the computation of effective properties of random materials.

Let us consider a stochastic two-phase composite in $\mathbb{R}^2$, generated as described in Section 5.1, with circles of radii $r = 0.35$ and hexagon side length $h = 1$. In Figure 3 we show a finite part of a specific realization, where the dotted lines show different cells. Note that in general the volume

fraction of circles is not constant in Y_ρ for different values of ρ and for small cells (i.e. when ρ is small) this might influence the result of the calculations.

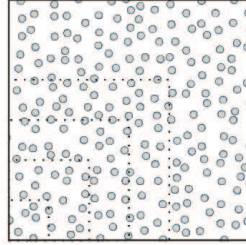


Figure 3. The cell Y_{24} for one realization. The cells Y_4 , Y_8 , Y_{12} and Y_{16} are also indicated

We assume that the two different constituent materials are isotropic, i.e. their elasticity tensors are fully described by Young's modulus E and Poisson's ratio ν . Material 1 (the circles) has $E_1 = 70$ GPa, $\nu_1 = 0.2$ while material 2 has $E_2 = 3.5$ GPa, $\nu_2 = 0.35$. For a fixed realization we have computed the elasticity tensor B^ρ (see Section 4). The results are shown in Figure 4, where the two continuous curves are the components b_{1111}^ρ and b_{2222}^ρ respectively for $\rho = 1, 2, \dots, 14$. As expected the two curves seem to converge to the same value. These values are compared with the corresponding effective value of a hexagonal periodic media (marked with crosses, $+$) and the one given by the mean value method for $N = 200$ realizations of Y_6 (marked with circles, $\circ$).

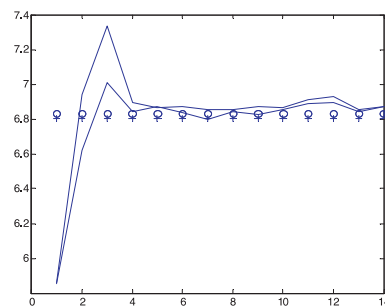


Figure 4. The components b_{1111}^ρ and b_{2222}^ρ for different values of ρ compared with the corresponding value for a hexagonal periodic medium and the one given by the mean value method

When we use periodic approximation, the resulting elasticity tensor is in general not isotropic. However, from physical reasons, one could expect the elasticity tensor to be isotropic in the limit. We remark that if the tensor is isotropic the components b_{1111}^ρ and b_{2222}^ρ will be equal and $b_{1212}^\rho = (b_{1111}^\rho - b_{1122}^\rho)/2$. In Figure 4 we can see that the difference between b_{1111}^ρ and b_{2222}^ρ tends to zero.

All the computations have been done by finite element methods in FEM-LAB because they are simple to use and offer sufficiently good accuracy.

6. Concluding Remarks

Most papers on stochastic homogenization either deal with theoretical aspects or with questions regarding computational issues. The main contribution of this paper is that we have connected these two areas.

We have presented the main ideas and results on how to homogenize random media. As shown in Section 3, the effective properties of a random media are expressed in terms of a solution of an auxiliary problem. Unfortunately this auxiliary problem is stated in an abstract probability space making it impossible to numerically compute the effective properties. However, as shown in this paper it is possible to find a converging sequence of periodic approximations that easily can be computed.

In this paper we have also compared the method described in Section 4 (see also [2]) with the mean value method described above. They seem to give same results but the theoretical reason for this is unclear since only the first method is based on a rigorous mathematical foundation. We also point out that in many works concerning the estimation of effective properties of random media, the precise description of random material is vague.

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