7 Appendix

7.1 Proof of Lemma 2

Proof. (of Lemma 2.)

(a) (6) and (8) immediately gives

$$|a(x,y,\xi)| \le c_1 (1+|\xi|)^{p-1-\alpha} |\xi|^\alpha \le c_1 (1+|\xi|)^{p-1} \le c_a \left(1+|\xi|^{p-1}\right).$$

(b) Young's inequality yields that

$$\begin{aligned} |\xi|^p &= (1+|\xi|)^{\frac{p}{\beta}(p-\beta)} |\xi|^p (1+|\xi|)^{\frac{p}{\beta}(\beta-p)} \\ &\le \frac{p}{\beta}\eta^{-\frac{\beta}{p}} (1+|\xi|)^{p-\beta} |\xi|^\beta + \frac{\beta-p}{\beta}\eta^{\frac{\beta}{\beta-p}} (1+|\xi|)^p \\ &\le \frac{1}{c_2}\frac{p}{\beta}\eta^{-\frac{\beta}{p}} \langle a(x,y,\xi),\xi\rangle + 2^{p-1}\frac{\beta-p}{\beta}\eta^{\frac{\beta}{\beta-p}} (1+|\xi|^p), \end{aligned}$$

that is,

$$|\xi|^p \left(1 - 2^{p-1}\frac{\beta-p}{\beta}\eta^{\frac{\beta}{\beta-p}}\right) \le \frac{1}{c_2}\frac{p}{\beta}\eta^{-\frac{\beta}{p}} \langle a(x,y,\xi),\xi\rangle + 2^{p-1}\frac{\beta-p}{\beta}\eta^{\frac{\beta}{\beta-p}}.$$

Now choose η sufficiently small and the inequality follows.

(c) By (11), (2), (10) and Young's inequality we have that

$$\begin{aligned} &\int_Y \left|\xi + Dv^{\xi,x_i}\right|^p dy \\ \le\ & c_b \int_Y 1 + \left\langle a\left(x,y,\xi+Dv^{\xi,x_i}\right), \xi + Dv^{\xi,x_i}\right\rangle dy \\ =\ & c_b \int_Y 1 + \left\langle a\left(x,y,\xi+Dv^{\xi,x_i}\right), \xi\right\rangle dy \\ \le\ & c_b \int_Y 1 + \left|a\left(x,y,\xi+Dv^{\xi,x_i}\right)\right| |\xi|\, dy \\ \le\ & c_b \int_Y 1 + c_a\left(1 + \left|\xi+Dv^{\xi,x_i}\right|^{p-1}\right)|\xi|\, dy \\ \le\ & c_b + c_b c_a \left(\frac{2^{q-1}}{q}\eta \int_Y \left(1+\left|\xi+Dv^{\xi,x_i}\right|^p\right) dy + \frac{1}{p}\eta^{-(p-1)} \int_Y |\xi|^p\, dy\right) \\ \le\ & C + C\eta \int_Y \left|\xi+Dv^{\xi,x_i}\right|^p dy + C\eta^{-(p-1)}|\xi|^p. \end{aligned}$$

By choosing η sufficiently small, it follows that

$$\int_Y \left|\xi + Dv^{\xi,x_i}\right|^p dy \leq c_c \left(1 + |\xi|^p\right).$$

□

7.2 Proof of Lemma 3

Before we prove Lemma 3, we state and prove the following auxiliary inequality first.

Lemma 9 *Let ϕ_1 and ϕ_2 be two functions in $L^p(\Omega, \mathbf{R}^n)$. Then we have that*

$$\begin{aligned}\left(\int_\Omega |\phi_1 - \phi_2|^p \, dx\right)^{\frac{1}{p}} &\leq \left(\int_\Omega |\phi_1 - \phi_2|^\beta \left(1 + |\phi_1| + |\phi_2|\right)^{p-\beta} dx\right)^{\frac{1}{\beta}} \\ &\quad \times \left(|\Omega|^{\frac{1}{p}} + \left(\int_\Omega |\phi_1|^p \, dx\right)^{\frac{1}{p}} + \left(\int_\Omega |\phi_2|^p \, dx\right)^{\frac{1}{p}}\right)^{\frac{\beta-p}{\beta}}. \quad (53)\end{aligned}$$

Proof. We have by Hölder's inequality that

$$\begin{aligned}\int_\Omega |\phi_1 - \phi_2|^p \, dx &= \int_\Omega \frac{|\phi_1 - \phi_2|^p}{\left(1 + |\phi_1| + |\phi_2|\right)^\lambda} \left(1 + |\phi_1| + |\phi_2|\right)^\lambda dx \\ &\leq \left(\int_\Omega \frac{|\phi_1 - \phi_2|^\beta}{\left(1 + |\phi_1| + |\phi_2|\right)^{\frac{\lambda\beta}{p}}} dx\right)^{\frac{p}{\beta}} \left(\int_\Omega \left(1 + |\phi_1| + |\phi_2|\right)^{\frac{\lambda\beta}{\beta-p}} dx\right)^{\frac{\beta-p}{\beta}}\end{aligned}$$

for every $\lambda \geq 0$. In particular, for $\lambda = \dfrac{p(\beta - p)}{\beta}$ we get

$$\begin{aligned}&\left(\int_\Omega |\phi_1 - \phi_2|^p \, dx\right)^{\frac{1}{p}} \\ &\leq \left(\int_\Omega |\phi_1 - \phi_2|^\beta \left(1 + |\phi_1| + |\phi_2|\right)^{p-\beta} dx\right)^{\frac{1}{\beta}} \left(\int_\Omega \left(1 + |\phi_1| + |\phi_2|\right)^p dx\right)^{\frac{\beta-p}{\beta p}}\end{aligned}$$

$$\leq \left(\int_\Omega |\phi_1 - \phi_2|^\beta \left(1 + |\phi_1| + |\phi_2|\right)^{p-\beta} dx \right)^{\frac{1}{\beta}}$$
$$\times \left(\left(\int_\Omega 1 dx \right)^{\frac{1}{p}} + \left(\int_\Omega |\phi_1|^p \, dx \right)^{\frac{1}{p}} + \left(\int_\Omega |\phi_2|^p \, dx \right)^{\frac{1}{p}} \right)^{\frac{\beta-p}{\beta}},$$

where the last step follows by Minkowski's inequality. □

Proof. (of Lemma 3.) By (53), the strict monotonicity assumption (7), (2) and (12) we get

$$\int_Y \left|\xi_1 + Dv^{\xi_1,x} - \xi_2 - Dv^{\xi_2,x}\right|^p dy$$
$$\leq \left(\int_Y \left|\xi_1 + Dv^{\xi_1,x} - \xi_2 - Dv^{\xi_2,x}\right|^\beta \left(1 + \left|\xi_1 + Dv^{\xi_1,x}\right| + \left|\xi_2 + Dv^{\xi_2,x}\right|\right)^{p-\beta} dx \right)^{\frac{p}{\beta}}$$
$$\times \left(1 + \left(\int_Y \left|\xi_1 + Dv^{\xi_1,x}\right|^p dx \right)^{\frac{1}{p}} + \left(\int_Y \left|\xi_2 + Dv^{\xi_2,x}\right|^p dx \right)^{\frac{1}{p}} \right)^{\frac{p(\beta-p)}{\beta}}$$
$$\leq \left[\int_Y \frac{1}{c_2} \left\langle a\left(x, y, \xi_1 + Dv^{\xi_1,x}\right) - a\left(x, y, \xi_2 + Dv^{\xi_2,x}\right), \xi_1 + Dv^{\xi_1,x} - \xi_2 - Dv^{\xi_2,x} \right\rangle dx \right]^{\frac{p}{\beta}}$$
$$\times \left[1 + c_c^{\frac{1}{p}} \left(1 + |\xi_1|^p\right)^{\frac{1}{p}} + c_c^{\frac{1}{p}} \left(1 + |\xi_2|^p\right)^{\frac{1}{p}} \right]^{\frac{p(\beta-p)}{\beta}}$$
$$\leq C \left[\int_Y \left\langle a\left(x, y, \xi_1 + Dv^{\xi_1,x}\right) - a\left(x, y, \xi_2 + Dv^{\xi_2,x}\right), \xi_1 - \xi_2 \right\rangle dx \right]^{\frac{p}{\beta}} \left(1 + |\xi_1|^p + |\xi_2|^p\right)^{\frac{\beta-p}{\beta}}.$$

Now by applying first the continuity assumption (6) and successively Hölder's inequality and (12) we obtain

$$\left[\int_Y \left\langle a\left(x, y, \xi_1 + Dv^{\xi_1,x}\right) - a\left(x, y, \xi_2 + Dv^{\xi_2,x}\right), \xi_1 - \xi_2 \right\rangle dx \right]^{\frac{p}{\beta}}$$
$$\leq \left[\int_Y \left| a\left(x, y, \xi_1 + Dv^{\xi_1,x}\right) - a\left(x, y, \xi_2 + Dv^{\xi_2,x}\right) \right| |\xi_1 - \xi_2| \, dx \right]^{\frac{p}{\beta}}$$

$$
\begin{aligned}
&\leq \left[c_1 \int_Y \left(1 + \left|\xi_1 + Dv^{\xi_1,x}\right| + \left|\xi_2 + Dv^{\xi_2,x}\right|\right)^{p-1-\alpha} \right. \\
&\quad \left. \times \left|\xi_1 + Dv^{\xi_1,x} - \xi_2 - Dv^{\xi_2,x}\right|^{\alpha} |\xi_1 - \xi_2| \, dx \right]^{\frac{p}{\beta}} \\
&\leq C \left(\int_Y \left(1 + \left|\xi_1 + Dv^{\xi_1,x}\right| + \left|\xi_2 + Dv^{\xi_2,x}\right|\right)^{p} dx \right)^{\frac{p-1-\alpha}{\beta}} \\
&\quad \times \left(\int_Y \left|\xi_1 + Dv^{\xi_1,x} - \xi_2 - Dv^{\xi_2,x}\right|^{p} dx \right)^{\frac{\alpha}{\beta}} \left(\int_Y |\xi_1 - \xi_2|^p \, dx \right)^{\frac{1}{\beta}} \\
&\leq C \left[3^{p-1} \left(1 + c_c \left(1 + |\xi_1|^p\right) + c_c \left(1 + |\xi_2|^p\right)\right)\right]^{\frac{p-1-\alpha}{\beta}} \\
&\quad \times \left(\int_Y \left|\xi_1 + Dv^{\xi_1,x} - \xi_2 - Dv^{\xi_2,x}\right|^{p} dx \right)^{\frac{\alpha}{\beta}} \left(\int_Y |\xi_1 - \xi_2|^p \, dx \right)^{\frac{1}{\beta}}.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
&\int_Y \left|\xi_1 + Dv^{\xi_1,x} - \xi_2 - Dv^{\xi_2,x}\right|^{p} dy \\
&\leq C \left(1 + |\xi_1|^p + |\xi_2|^p\right)^{\left(\frac{\beta-p}{\beta} + \frac{p-1-\alpha}{\beta}\right)\frac{\beta}{\beta-\alpha}} |\xi_1 - \xi_2|^{\frac{p}{\beta}\frac{\beta}{\beta-\alpha}} \\
&= C \left(1 + |\xi_1|^p + |\xi_2|^p\right)^{\frac{\beta-\alpha-1}{\beta-\alpha}} |\xi_1 - \xi_2|^{\frac{p}{\beta-\alpha}}.
\end{aligned}
$$

□

7.3 Proof of Theorem 7

Before we present the proof of this theorem we recall some results and prove some lemmas which will be useful in the proof. We also remark that the proof is influenced by [7] where the uniform periodicity problem was treated and [15] where the case $p = 2$ was treated.

Theorem 10 *Let $u \in W^{1,p}(\Omega)$ be a weak solution of the equation*

$$
-\operatorname{div}(a(x, Du)) = 0 \text{ on } \Omega.
$$

Then there exists $s > 0$ such that for each open set Q with compact closure in Ω we have that $u \in W^{1,p+s}(Q)$ and

$$
\|u\|_{W^{1,p+s}(Q)} \leq C \|u\|_{W^{1,p}(\Omega)}.
$$

Proof. See [11]. □

Lemma 11 *Let a, b, u_h and u be defined as in Theorem 5. Then*

$$\int_\Omega \left\langle a\left(x, \frac{x}{\varepsilon_h}, Du_h\right), Du_h \right\rangle dx \to \int_\Omega \left\langle b\left(x, Du\right), Du \right\rangle dx.$$

Proof. By (16) and (21) we have that

$$\int_\Omega \left\langle a\left(x, \frac{x}{\varepsilon_h}, Du_h\right), Du_h \right\rangle dx = \left\langle f_h \mid u_h \right\rangle$$

and

$$\int_\Omega \left\langle b\left(x, Du\right), Du \right\rangle dx = \left\langle f \mid u \right\rangle,$$

respectively. Since $f_h \to f$ in $W^{-1,q}(\Omega)$ and $u_h \to u$ weakly in $W_0^{1,p}(\Omega)$, we have that $\left\langle f_h \mid u_h \right\rangle \to \left\langle f \mid u \right\rangle$ and the lemma is proved. □

Lemma 12 *Let a_i, u and $v^{M_h Du, x_i}$ be defined as in Theorem 5. Then*

$$\begin{aligned} &\int\limits_{\Omega_i} \left\langle a_i\left(\frac{x}{\varepsilon_h}, M_h Du + Dv^{M_h Du, x_i}\left(\frac{x}{\varepsilon_h}\right)\right), M_h Du + Dv^{M_h Du, x_i}\left(\frac{x}{\varepsilon_h}\right) \right\rangle dx \\ \to &\int\limits_{\Omega_i} \left\langle b_i\left(Du\right), Du \right\rangle dx. \end{aligned}$$

Proof. First we note that

$$\begin{aligned} &\int\limits_{\Omega i} \left\langle a_i\left(\frac{x}{\varepsilon_h}, M_h Du + Dv^{M_h Du, x_i}\left(\frac{x}{\varepsilon_h}\right)\right), M_h Du + Dv^{M_h Du, x_i}\left(\frac{x}{\varepsilon_h}\right) \right\rangle dx \\ = &\sum_{j \in J_h^i} \int\limits_{Y_h^j} \left\langle a_i\left(\frac{x}{\varepsilon_h}, \xi_h^j + Dv^{\xi_h^j, x_i}\left(\frac{x}{\varepsilon_h}\right)\right), \xi_h^j + Dv^{\xi_h^j, x_i}\left(\frac{x}{\varepsilon_h}\right) \right\rangle dx \qquad (54) \\ &+ \int\limits_{\Omega_i \setminus \Omega_i^h} \left\langle a_i\left(\frac{x}{\varepsilon_h}, M_h Du + Dv^{M_h Du, x_i}\left(\frac{x}{\varepsilon_h}\right)\right), M_h Du + Dv^{M_h Du, x_i}\left(\frac{x}{\varepsilon_h}\right) \right\rangle dx. \end{aligned}$$

It is obviously sufficient to prove (a) - that the first term in equation (54) converges to $\int_{\Omega_i} \left\langle b_i\left(Du\right), Du \right\rangle dx$ and (b) - that the second term in (54) converges to 0. Indeed, by the definition of b_i together with the cell problem we have that

$$\begin{aligned} &\sum_{j \in J_h^i} \int\limits_{Y_h^j} \left\langle a_i\left(\frac{x}{\varepsilon_h}, \xi_h^j + Dv^{\xi_h^j, x_i}\left(\frac{x}{\varepsilon_h}\right)\right), \xi_h^j + Dv^{\xi_h^j, x_i}\left(\frac{x}{\varepsilon_h}\right) \right\rangle dx \\ = &\sum_{j \in J_h^i} \left(\varepsilon_h^n\right) \int\limits_{Y} \left\langle a_i\left(y, \xi_h^j + Dv^{\xi_h^j, x_i}\left(y\right)\right), \xi_h^j + Dv^{\xi_h^j, x_i}\left(y\right) \right\rangle dy \end{aligned}$$

$$
\begin{aligned}
&= \sum_{j\in J_h^i} (\varepsilon_h^n) \left\langle b_i\left(\xi_h^j\right), \xi_h^j \right\rangle = \sum_{j\in J_h^i} \int_{Y_h^j} \left\langle b_i\left(\xi_h^j\right), \xi_h^j \right\rangle \, dx \\
&= \int_{\Omega_i^h} \left\langle b_i\left(M_h Du\right), M_h Du \right\rangle \, dx.
\end{aligned}
$$

We will prove that the last integral converges to $\int_{\Omega_i} \langle b_i(Du), Du\rangle \, dx$. In fact, elementary estimates and the continuity property of b, see Remark 3, implies that

$$
\begin{aligned}
&\left| \int_{\Omega_i^h} \left\langle b_i\left(M_h Du\right), M_h Du \right\rangle \, dx - \int_{\Omega_i} \left\langle b_i\left(Du\right), Du \right\rangle \, dx \right| \\
\leq\ & \left| \int_{\Omega_i^h} \left\langle b_i\left(M_h Du\right), M_h Du \right\rangle - \left\langle b_i\left(Du\right), Du \right\rangle \, dx \right| + \left| \int_{\Omega_i \setminus \Omega_i^h} \left\langle b_i\left(Du\right), Du \right\rangle \, dx \right| \\
=\ & \left| \int_{\Omega_i^h} \left\langle b_i\left(M_h Du\right) - b_i\left(Du\right), M_h Du \right\rangle + \left\langle b_i\left(Du\right), M_h Du - Du \right\rangle \, dx \right| \\
&+ \left| \int_{\Omega_i \setminus \Omega_i^h} \left\langle b_i\left(Du\right), Du \right\rangle \, dx \right| \\
\leq\ & \int_{\Omega_i^h} \left| b_i\left(M_h Du\right) - b_i\left(Du\right) \right| \left| M_h Du \right| + \left| b_i\left(Du\right) \right| \left| M_h Du - Du \right| \, dx \\
&+ \int_{\Omega_i \setminus \Omega_i^h} \left| b_i\left(Du\right) \right| \left| Du \right| \, dx \\
\leq\ & \widetilde{c}_1 \int_{\Omega_i^h} \left(1 + \left|M_h Du\right| + \left|Du\right|\right)^{p-1-\gamma} \left|M_h Du - Du\right|^{\gamma} \left|M_h Du\right| \, dx \\
&+ \widetilde{c}_1 \int_{\Omega_i^h} \left(1 + \left|Du\right|\right)^{p-1-\gamma} \left|Du\right|^{\gamma} \left|M_h Du - Du\right| \, dx + \widetilde{c}_1 \int_{\Omega_i \setminus \Omega_i^h} \left(1 + \left|Du\right|\right)^{p-1-\gamma} \left|Du\right|^{\gamma+1} \, dx
\end{aligned}
$$

$$
\begin{aligned}
&\leq C\left(\int_{\Omega_i^h}(1+|M_hDu|+|Du|)^p\,dx\right)^{\frac{p-1-\gamma}{p}}\left(\int_{\Omega_i^h}|M_hDu-Du|^p\,dx\right)^{\frac{\gamma}{p}}\left(\int_{\Omega_i^h}|M_hDu|^p\,dx\right)^{\frac{1}{p}}\\
&\quad\times\left(\int_{\Omega_i^h}|M_hDu-Du|^p\,dx\right)^{\frac{\gamma}{p}}\left(\int_{\Omega_i^h}|M_hDu|^p\,dx\right)^{\frac{1}{p}}\\
&\quad+C\left(\int_{\Omega_i^h}(1+|Du|)^p\,dx\right)^{\frac{p-1-\gamma}{p}}\left(\int_{\Omega_i^h}|Du|^p\,dx\right)^{\frac{\gamma}{p}}\left(\int_{\Omega_i^h}|M_hDu-Du|^p\,dx\right)^{\frac{1}{p}}\\
&\quad+C\int_{\Omega_i\setminus\Omega_i^h}(1+|Du|)^p\,dx\\
&=C\,\|1+|M_hDu|+|Du|\|^{p-1-\gamma}_{L^p(\Omega_i,\mathbf{R}^n)}\,\|M_hDu-Du\|^{\gamma}_{L^p(\Omega_i,\mathbf{R}^n)}\,\|M_hDu\|_{L^p(\Omega_i,\mathbf{R}^n)}\\
&\quad+C\,\|1+|Du|\|^{p-1-\gamma}_{L^p(\Omega_i,\mathbf{R}^n)}\,\|Du\|^{\gamma}_{L^p(\Omega_i,\mathbf{R}^n)}\,\|M_hDu-Du\|_{L^p(\Omega_i,\mathbf{R}^n)}+C\int_{\Omega_i\setminus\Omega_i^h}(1+|Du|)^p\,dx\,.
\end{aligned}
$$

Since $\|M_hDu\|_{L^p(\Omega_i,\mathbf{R}^n)}\leq\|Du\|_{L^p(\Omega_i,\mathbf{R}^n)}$ by Jensen's inequality, it follows from (4) and the fact that $\left|\Omega_i\setminus\Omega_i^h\right|\to 0$ that all three terms converges to 0 as $h\to\infty$ and (a) is proved. Next we note that by (6) and a change of variables in (12) we have that

$$
\begin{aligned}
&\int_{\Omega_i\setminus\Omega_i^h}\left\langle a_i\left(\frac{x}{\varepsilon_h},M_hDu+Dv^{M_hDu,x_i}\left(\frac{x}{\varepsilon_h}\right)\right),M_hDu+Dv^{M_hDu,x_i}\left(\frac{x}{\varepsilon_h}\right)\right\rangle dx\\
\leq&\int_{\Omega_i\setminus\Omega_i^h}\left|a_i\left(\frac{x}{\varepsilon_h},M_hDu+Dv^{M_hDu,x_i}\left(\frac{x}{\varepsilon_h}\right)\right)\right|\left|M_hDu+Dv^{M_hDu,x_i}\left(\frac{x}{\varepsilon_h}\right)\right|dx\\
\leq&\int_{\Omega_i\setminus\Omega_i^h}c_1\left(1+\left|M_hDu+Dv^{M_hDu,x_i}\left(\frac{x}{\varepsilon_h}\right)\right|\right)^{p-1-\alpha}\left|M_hDu+Dv^{M_hDu,x_i}\left(\frac{x}{\varepsilon_h}\right)\right|^{\alpha+1}dx\\
\leq&\sum_{j\in B_h^i}\int_{Y_h^j}c_1\left(1+\left|\xi_h^j+Dv^{\xi_h^j,x_i}\left(\frac{x}{\varepsilon_h}\right)\right|\right)^{p-1-\alpha}\left|\xi_h^j+Dv^{\xi_h^j,x_i}\left(\frac{x}{\varepsilon_h}\right)\right|^{\alpha+1}dx\\
\leq&\sum_{j\in B_h^i}\int_{Y_h^j}c_1 2^{p-1}\left(1+\left|\xi_h^j+Dv^{\xi_h^j,x_i}\left(\frac{x}{\varepsilon_h}\right)\right|^p\right)dx\leq C\sum_{j\in B_h^i}\left(1+c_c\left(1+\left|\xi_h^j\right|^p\right)\right)\left|Y_h^j\right|
\end{aligned}
$$

$$
\begin{aligned}
&\leq C \sum_{j \in B_h^i} \left|Y_h^j\right| + C \sum_{j \in B_h^i} \left|Y_h^j\right| \left| \frac{1}{\left|Y_h^j\right|} \int\limits_{Y_h^j} Du \, dx \right|^p \\
&\leq C \sum_{j \in B_h^i} \left|Y_h^j\right| + C \sum_{j \in B_h^i} \int\limits_{Y_h^j} |Du|^p \; dx \leq C \int\limits_{F_i^h} (1 + |Du|^p) \, dx \to 0,
\end{aligned}
$$

where we in the last step used Jensen's inequality and in the limit process the fact that $\left|F_i^h\right| \to 0$ since $|\partial \Omega_i| = 0$. Thus (b) is proved and we are done. □

Lemma 13 *Let $\Gamma_i : \Omega_i \to \mathbf{R}^n$ be a simple function of the form*

$$
\Gamma_i = \sum_{k=1}^{m} \chi_{A_k} \alpha_k,
$$

where $A_k \subset \Omega_i$ is compact, $|\partial A_k| = 0$ and $|A_k \cap A_l| = 0$ for $k \neq l$. Then

$$
\begin{aligned}
&\limsup_{h \to \infty} \left\| M_h Du + Dv^{M_h Du, x_i} - \left(\Gamma_i + Dv^{\Gamma_i, x_i} \right) \right\|_{L^p(\Omega_i, \mathbf{R}^n)} \\
\leq \; & C \left\| 1 + |Du| + |\Gamma_i| \right\|_{L^p(\Omega_i, \mathbf{R}^n)}^{\frac{\beta - \alpha - 1}{\beta - \alpha}} \left\| Du - \Gamma_i \right\|_{L^p(\Omega_i, \mathbf{R}^n)}^{\frac{1}{\beta - \alpha}}.
\end{aligned}
$$

Proof. Put $A_{m+1} = \Omega_i \backslash \cup_{k=1}^m A_k$, $\alpha_{m+1} = 0$ and define the following index sets

$$
\begin{aligned}
J_h^{i,k} &= \left\{ j \in J_h^i : \overline{Y}_h^j \subset A_k \text{ where } k = 1, \ldots, m+1 \right\}, \\
B_h^{i,k} &= \left\{ j \in J_h^i : \overline{Y}_h^j \cap A_k \; \neq \emptyset, \; \overline{Y}_h^j \backslash A_k \; \neq \emptyset \right\}.
\end{aligned}
$$

Moreover, we define $F_i^{h,k} = \cup_{j \in B_h^{i,k}} \overline{Y}_h^j$. Since $\cup_{k=1}^m A_k$ is compact, we have that $\cup_{k=1}^m A_k \subset \Omega_i^h \subset \Omega_i$ for h sufficiently large, which implies that

$$
\begin{aligned}
&\int\limits_{\Omega_i} \left| M_h Du + Dv^{M_h Du, x_i} \left(\frac{x}{\varepsilon_h} \right) - \left(\Gamma_i + Dv^{\Gamma_i, x_i} \left(\frac{x}{\varepsilon_h} \right) \right) \right|^p \; dx \\
= \; &\int\limits_{\Omega_i^h} \left| M_h Du + Dv^{M_h Du, x_i} \left(\frac{x}{\varepsilon_h} \right) - \left(\Gamma_i + Dv^{\Gamma_i, x_i} \left(\frac{x}{\varepsilon_h} \right) \right) \right|^p \; dx \\
&+ \int\limits_{\Omega_i \backslash \Omega_i^h} \left| M_h Du + Dv^{M_h Du, x_i} \left(\frac{x}{\varepsilon_h} \right) \right|^p \; dx. \qquad (55)
\end{aligned}
$$

Let us study the first term on the right hand side.

$$
\int\limits_{\Omega_i^h} \left| M_h Du + Dv^{M_h Du, x_i} \left(\frac{x}{\varepsilon_h} \right) - \left(\Gamma_i + Dv^{\Gamma_i, x_i} \left(\frac{x}{\varepsilon_h} \right) \right) \right|^p \; dx
$$

$$
\begin{aligned}
&\leq \sum_{k=1}^{m+1} \sum_{j \in J_h^{i,k}} \int\limits_{Y_h^j} \left| \xi_h^j + Dv^{\xi_h^j, x_i} - (\alpha_k + Dv^{\alpha_k, x_i}) \right|^p \, dx \\
&\quad + \sum_{k=1}^{m+1} \sum_{j \in B_h^{i,k}} \int\limits_{Y_h^j} \left| \xi_h^j + Dv^{\xi_h^j, x_i} - (\alpha_k + Dv^{\alpha_k, x_i}) \right|^p \, dx.
\end{aligned}
$$

By using (13) with a change of variables and the Jensen inequality, we obtain

$$
\begin{aligned}
&\int\limits_{\Omega_i^h} \left| M_h Du + Dv^{M_h Du, x_i}\left(\frac{x}{\varepsilon_h}\right) - \left(\Gamma_i + Dv^{\Gamma_i, x_i}\left(\frac{x}{\varepsilon_h}\right)\right) \right|^p \, dx \\
&\leq \sum_{k=1}^{m+1} \sum_{j \in J_h^{i,k}} C \left(1 + \left|\xi_h^j\right|^p + |\alpha_k|^p\right)^{\frac{\beta-\alpha-1}{\beta-\alpha}} \left|\xi_h^j - \alpha_k\right|^{\frac{p}{\beta-\alpha}} \left|Y_h^j\right| \\
&\quad + \sum_{k=1}^{m+1} \sum_{j \in B_h^{i,k}} C \left(1 + \left|\xi_h^j\right|^p + |\alpha_k|^p\right)^{\frac{\beta-\alpha-1}{\beta-\alpha}} \left|\xi_h^j - \alpha_k\right|^{\frac{p}{\beta-\alpha}} \left|Y_h^j\right| \\
&\leq C \int\limits_{\Omega_i} \left(1 + |M_h Du|^p + |\Gamma_i|^p\right)^{\frac{\beta-\alpha-1}{\beta-\alpha}} |M_h Du - \Gamma_i|^{\frac{p}{\beta-\alpha}} \, dx \\
&\quad + C \sum_{k=1}^{m+1} \int\limits_{F_i^{h,k}} \left(1 + |M_h Du|^p + |\alpha_k|^p\right)^{\frac{\beta-\alpha-1}{\beta-\alpha}} |M_h Du - \alpha_k|^{\frac{p}{\beta-\alpha}} \, dx \\
&\leq C \left(\int\limits_{\Omega_i} \left(1 + |M_h Du| + |\Gamma_i|\right)^p dx \right)^{\frac{\beta-\alpha-1}{\beta-\alpha}} \left(\int\limits_{\Omega_i} |M_h Du - \Gamma_i|^p \, dx \right)^{\frac{1}{\beta-\alpha}} \\
&\quad + \sum_{k=1}^{m+1} \left(\int\limits_{F_i^{h,k}} \left(1 + |M_h Du| + |\alpha_k|\right)^p dx \right)^{\frac{\beta-\alpha-1}{\beta-\alpha}} \left(\int\limits_{F_i^{h,k}} |M_h Du - \alpha_k|^p \, dx \right)^{\frac{1}{\beta-\alpha}} \\
&\leq C \left(\int\limits_{\Omega_i} \left(1 + |Du| + |\Gamma_i|\right)^p dx \right)^{\frac{\beta-\alpha-1}{\beta-\alpha}} \left(\int\limits_{\Omega_i} |Du - \Gamma_i|^p \, dx \right)^{\frac{1}{\beta-\alpha}} \\
&\quad + C \sum_{k=1}^{m+1} \left(\int\limits_{F_i^{h,k}} \left(1 + |Du| + |\alpha_k|\right)^p dx \right)^{\frac{\beta-\alpha-1}{\beta-\alpha}} \left(\int\limits_{F_i^{h,k}} |Du - \alpha_k|^p \, dx \right)^{\frac{1}{\beta-\alpha}}. \qquad (56)
\end{aligned}
$$

Now we note that $\left|F_i^{h,k}\right| \to |\partial A_k|$ and by assumption, that $|\partial A_k| = 0$ for $k = 1, \ldots, m$. Moreover, we have that $\left|\Omega_i \backslash \Omega_i^h\right| \to 0$. The lemma follows by (55) and (56). $\square$

Lemma 14 *Let $\Gamma_i : \Omega_i \to \mathbf{R}^n$ be a simple function of the form*

$$\Gamma_i = \sum_{k=1}^{m} \chi_{A_k} \alpha_k,$$

where $A_k \subset \Omega_i$ is compact, $|\partial A_k| = 0$ and $|A_k \cap A_l| = 0$ for $k \neq l$. Then

$$\begin{aligned} & \limsup_{h\to\infty} \left\| 1 + \left| M_h Du + Dv^{M_h Du, x_i} \right| + \left| \Gamma_i + Dv^{\Gamma_i, x_i} \right| \right\|_{L^p(\Omega_i, \mathbf{R}^n)} \\ \leq \; & C \left\| 1 + |Du| + |\Gamma_i| \right\|_{L^p(\Omega_i, \mathbf{R}^n)}. \end{aligned}$$

Proof. Put $A_{m+1} = \Omega_i \backslash \cup_{k=1}^{m} A_k$, $\alpha_{m+1} = 0$ and define the following index sets

$$\begin{aligned} J_h^{i,k} &= \left\{ j \in J_h^i : \overline{Y}_h^j \subset A_k \text{ where } k = 1, \ldots, m+1 \right\}, \\ B_h^{i,k} &= \left\{ j \in J_h^i : \overline{Y}_h^j \cap A_k \neq \emptyset, \; \overline{Y}_h^j \backslash A_k \neq \emptyset \right\}. \end{aligned}$$

Moreover, we define $F_i^{h,k} = \cup_{j \in B_h^{i,k}} \overline{Y}_h^j$. Since $\cup_{k=1}^m A_k$ is compact, we have that $\cup_{k=1}^m A_k \subset \Omega_i^h \subset \Omega_i$ for h sufficiently large, which implies that

$$\begin{aligned} & \int_{\Omega_i} \left(1 + \left| M_h Du + Dv^{M_h Du, x_i}\left(\frac{x}{\varepsilon_h}\right) \right| + \left| \Gamma_i + Dv^{\Gamma_i, x_i}\left(\frac{x}{\varepsilon_h}\right) \right| \right)^p dx \\ = \; & \int_{\Omega_i^h} \left(1 + \left| M_h Du + Dv^{M_h Du, x_i}\left(\frac{x}{\varepsilon_h}\right) \right| + \left| \Gamma_i + Dv^{\Gamma_i, x_i}\left(\frac{x}{\varepsilon_h}\right) \right| \right)^p dx \\ & + \int_{\Omega_i \backslash \Omega_i^h} \left(1 + \left| M_h Du + Dv^{M_h Du, x_i}\left(\frac{x}{\varepsilon_h}\right) \right| \right)^p dx. \end{aligned} \tag{57}$$

Let us study the first term on the right hand side.

$$\begin{aligned} & \int_{\Omega_i^h} \left(1 + \left| M_h Du + Dv^{M_h Du, x_i}\left(\frac{x}{\varepsilon_h}\right) \right| + \left| \Gamma_i + Dv^{\Gamma_i, x_i}\left(\frac{x}{\varepsilon_h}\right) \right| \right)^p dx \\ \leq \; & C \int_{\Omega_i^h} \left(1 + \left| M_h Du + Dv^{M_h Du, x_i}\left(\frac{x}{\varepsilon_h}\right) \right|^p + \left| \Gamma_i + Dv^{\Gamma_i, x_i}\left(\frac{x}{\varepsilon_h}\right) \right|^p \right) dx \\ \leq \; & C \sum_{k=1}^{m+1} \sum_{j \in J_h^{i,k}} \int_{Y_h^j} \left(1 + \left| \xi_h^j + Dv^{\xi_h^j, x_i} \right|^p + \left| \alpha_k + Dv^{\alpha_k, x_i} \right|^p \right) dx \\ & + C \sum_{k=1}^{m+1} \sum_{j \in B_h^{i,k}} \int_{Y_h^j} \left(1 + \left| \xi_h^j + Dv^{\xi_h^j, x_i} \right|^p + \left| \alpha_k + Dv^{\alpha_k, x_i} \right|^p \right) dx. \end{aligned}$$

By using (13) with a change of variables and Jensen's inequality, we obtain

$$
\begin{aligned}
&\int_{\Omega_i^h}\left(1+\left|M_hDu+Dv^{M_hDu,x_i}\left(\frac{x}{\varepsilon_h}\right)\right|+\left|\Gamma_i+Dv^{\Gamma_i,x_i}\left(\frac{x}{\varepsilon_h}\right)\right|\right)^p dx\\
\leq\ &\sum_{k=1}^{m+1}\sum_{j\in J_h^{i,k}}\left(1+C\left(1+\left|\xi_h^j\right|^p\right)^{\frac{\beta-\alpha-1}{\beta-\alpha}}\left|\xi_h^j\right|^{\frac{p}{\beta-\alpha}}+C\left(1+|\alpha_k|^p\right)^{\frac{\beta-\alpha-1}{\beta-\alpha}}|\alpha_k|^{\frac{p}{\beta-\alpha}}\right)\left|Y_h^j\right|\\
&+\sum_{k=1}^{m+1}\sum_{j\in B_h^{i,k}}\left(1+C\left(1+\left|\xi_h^j\right|^p\right)^{\frac{\beta-\alpha-1}{\beta-\alpha}}\left|\xi_h^j\right|^{\frac{p}{\beta-\alpha}}+C\left(1+|\alpha_k|^p\right)^{\frac{\beta-\alpha-1}{\beta-\alpha}}|\alpha_k|^{\frac{p}{\beta-\alpha}}\right)\left|Y_h^j\right|\\
\leq\ &C\sum_{k=1}^{m+1}\sum_{j\in J_h^{i,k}}\left(1+\left|\xi_h^j\right|^p+|\alpha_k|^p\right)\left|Y_h^j\right|+C\sum_{k=1}^{m+1}\sum_{j\in B_h^{i,k}}\left(1+\left|\xi_h^j\right|^p+|\alpha_k|^p\right)\left|Y_h^j\right|\\
\leq\ &C\int_{\Omega_i}\left(1+|M_hDu|^p+|\Gamma_i|^p\right)dx+C\sum_{k=1}^{m+1}\int_{F_i^{h,k}}\left(1+|M_hDu|^p+|\alpha_k|^p\right)\,dx\\
\leq\ &C\int_{\Omega_i}\left(1+|Du|^p+|\Gamma_i|^p\right)\,dx+C\sum_{k=1}^{m+1}\int_{F_i^{h,k}}\left(1+|Du|^p+|\alpha_k|^p\right)\,dx\\
\leq\ &C\int_{\Omega_i}\left(1+|Du|+|\Gamma_i|\right)^p dx+C\sum_{k=1}^{m+1}\int_{F_i^{h,k}}\left(1+|Du|+|\alpha_k|\right)^p\,dx. \qquad (58)
\end{aligned}
$$

Now we note that $\left|F_i^{h,k}\right|\to|\partial A_k|$ and by assumption, that $|\partial A_k|=0$ for $k=1,\ldots,m$. Moreover, we have that $\left|\Omega_i\backslash\Omega_i^h\right|\to 0$. The lemma follows by (57) and (58). □

Lemma 15 *There exists a $s'>0$ such that the functions $a_i\left(\frac{x}{\varepsilon_h},\xi+Dv^{\xi,x_i}\left(\frac{x}{\varepsilon_h}\right)\right)$ are uniformly bounded in $L^{q+s'}(\Omega_i,\mathbf{R}^n)$.*

Proof. Apply Meyer's estimate, see Theorem 10, with $Q=\left(-\frac{1}{2},\frac{1}{2}\right)^n$ and $\Omega=(-\frac{3}{2},\frac{3}{2})^n$. By the definition of v^{ξ,x_i} we have that $-\operatorname{div}\left(a_i\left(y,\xi+Dv^{\xi,x_i}(y)\right)\right)=0$ on Ω. Meyer's estimate implies that there exists a $s>0$ such that

$$\left\|(\xi,y)+v^{\xi,x_i}(y)\right\|_{W^{1,p+s}(Q)}\leq C\left\|(\xi,y)+v^{\xi,x_i}(y)\right\|_{W^{1,p}(\Omega)}.$$

This estimate together with Poincare's inequality and the periodicity of v^{ξ,x_i} imply that

$$\left\|\xi+Dv^{\xi,x_i}(y)\right\|_{L^{p+s}(Q,\mathbf{R}^n)}\leq C\left\|\xi+Dv^{\xi,x_i}(y)\right\|_{L^p(Q,\mathbf{R}^n)},$$

that is,

$$\left\|\xi+Dv^{\xi,x_i}(y)\right\|_{L^{p+s}(Y,\mathbf{R}^n)}\leq C\left\|\xi+Dv^{\xi,x_i}(y)\right\|_{L^p(Y,\mathbf{R}^n)}\leq C\left(1+|\xi|\right),$$

where we in the last inequality have used (13). By a change of variables we obtain

$$\left\| \xi + Dv^{\xi,x_i}\left(\frac{x}{\varepsilon_h}\right)\right\|^{p+s}_{L^{p+s}\left(Y_h^j,\mathbf{R}^n\right)} \leq C\left(1+|\xi|\right)^{p+s}\left|Y_h^j\right|. \tag{59}$$

This implies that $\xi + Dv^{\xi,x_i}\left(\frac{x}{\varepsilon_h}\right)$ is uniformly bounded in $L^{p+s}\left(\Omega_i,\mathbf{R}^n\right)$, and according to (6), we get that $a_i\left(\frac{x}{\varepsilon_h},\xi + Dv^{\xi,x_i}\left(\frac{x}{\varepsilon_h}\right)\right)$ is uniformly bounded in $L^{q+s'}\left(\Omega_i,\mathbf{R}^n\right)$, where $s' = s/(p-1)$. □

Lemma 16 *Let a_i, u, v^{M_hDu,x_i} and b_i be defined as in Theorem 5. Then*

$$\int_{\Omega_i}\left\langle a_i\left(\frac{x}{\varepsilon_h},M_hDu + Dv^{M_hDu,x_i}\left(\frac{x}{\varepsilon_h}\right)\right),Du_h\right\rangle dx \to \int_{\Omega_i}\left\langle b_i\left(Du\right),Du\right\rangle dx.$$

Proof. First we note that for every $\delta > 0$ there exists a simple function $\Gamma_i = \sum_{k=1}^m \chi_{A_k}\alpha_k$ satisfying the conditions in Lemma 13 such that

$$\left\|Du - \Gamma_i\right\|_{L^p(\Omega_i,\mathbf{R}^n)} \leq \delta.$$

By adding and subtracting $\int_{\Omega_i}\left\langle a_i\left(\frac{x}{\varepsilon_h},\Gamma_i + Dv^{\Gamma_i,x_i}\left(\frac{x}{\varepsilon_h}\right)\right),Du_h\right\rangle dx$ we have that

$$\begin{aligned}
&\left|\int_{\Omega_i}\left\langle a_i\left(\frac{x}{\varepsilon_h},M_hDu + Dv^{M_hDu,x_i}\left(\frac{x}{\varepsilon_h}\right)\right),Du_h\right\rangle - \left\langle b_i\left(Du\right),Du\right\rangle dx\right| \\
\leq\ &\int_{\Omega_i}\left|\left\langle a_i\left(\frac{x}{\varepsilon_h},M_hDu + Dv^{M_hDu,x_i}\right) - a_i\left(\frac{x}{\varepsilon_h},\Gamma_i + Dv^{\Gamma_i,x_i}\right),Du_h\right\rangle\right| dx \\
&+\left|\int_{\Omega_i}\left\langle a_i\left(\frac{x}{\varepsilon_h},\Gamma_i + Dv^{\Gamma_i,x_i}\right),Du_h\right\rangle - \left\langle b_i\left(Du\right),Du\right\rangle dx\right|.
\end{aligned} \tag{60}$$

Let us estimate the first integral on the right hand side in (60). By Schwarz' inequality, (6) and Hölder's inequality, we obtain that

$$\begin{aligned}
&\int_{\Omega_i}\left|\left\langle a_i\left(\frac{x}{\varepsilon_h},M_hDu + Dv^{M_hDu,x_i}\right) - a_i\left(\frac{x}{\varepsilon_h},\Gamma_i + Dv^{\Gamma_i,x_i}\right),Du_h\right\rangle\right| dx \\
\leq\ &\int_{\Omega_i}\left|a_i\left(\frac{x}{\varepsilon_h},M_hDu + Dv^{M_hDu,x_i}\right) - a_i\left(\frac{x}{\varepsilon_h},\Gamma_i + Dv^{\Gamma_i,x_i}\right)\right|\left|Du_h\right| dx
\end{aligned}$$

$$
\begin{aligned}
&\leq \int_{\Omega_i} c_1 \left(1+\left|M_h Du + Dv^{M_h Du, x_i}\right| + \left|\Gamma_i + Dv^{\Gamma_i, x_i}\right|\right)^{p-1-\alpha} \\
&\quad \times \left|M_h Du + Dv^{M_h Du, x_i} - \left(\Gamma_i + Dv^{\Gamma_i, x_i}\right)\right|^{\alpha} |Du_h| \, dx \\
&\leq C \left(\int_{\Omega_i} \left(1+\left|M_h Du + Dv^{M_h Du, x_i}\right| + \left|\Gamma_i + Dv^{\Gamma_i, x_i}\right|\right)^{p} dx \right)^{\frac{p-1-\alpha}{p}} \\
&\quad \times \left(\int_{\Omega_i} \left|M_h Du + Dv^{M_h Du, x_i} - \left(\Gamma_i + Dv^{\Gamma_i, x_i}\right)\right|^{p} dx \right)^{\frac{\alpha}{p}} \left(\int_{\Omega_i} |Du_h|^p \, dx \right)^{\frac{1}{p}}.
\end{aligned}
$$

Therefore, according to Lemma 13 and Lemma 14,

$$
\begin{aligned}
&\limsup_{h\to\infty} \int_{\Omega_i} \left| \left\langle a_i\left(\frac{x}{\varepsilon_h}, M_h Du + Dv^{M_h Du, x_i}\right) - a_i\left(\frac{x}{\varepsilon_h}, \Gamma_i + Dv^{\Gamma_i, x_i}\right), Du_h \right\rangle \right| dx \\
&\leq c \left\| 1 + |Du| + |\Gamma_i| \right\|^{p-1-\frac{\alpha}{\beta-\alpha}}_{L^p(\Omega_i, \mathbf{R}^n)} \left\| Du - \Gamma_i \right\|^{\frac{\alpha}{\beta-\alpha}}_{L^p(\Omega_i, \mathbf{R}^n)} \left\| Du_h \right\|_{L^p(\Omega_i, \mathbf{R}^n)} \leq C\delta^{\frac{\alpha}{\beta-\alpha}}, \qquad (61)
\end{aligned}
$$

where C is a constant independent of h, see (18). Let us now study the second integral on the right hand side in (60). We will prove that

$$
\int_{\Omega_i} \left\langle a_i\left(\frac{x}{\varepsilon_h}, \Gamma_i + Dv^{\Gamma_i, x_i}\right), Du_h \right\rangle - \left\langle b_i\left(Du\right), Du \right\rangle \, dx \to 0. \qquad (62)
$$

In fact, we note that the compensated compactness lemma, Lemma 1, implies that

$$
\int_{\Omega_i} \left\langle a_i\left(\frac{x}{\varepsilon_h}, \alpha_k + Dv^{\alpha_k, x_i}\right), Du_h \right\rangle \phi \, dx \to \int_{\Omega_i} \left\langle b_i\left(\alpha_k\right), Du \right\rangle \phi \, dx, \qquad (63)
$$

as $h \to \infty$ for every $\phi \in C_0^\infty\left(\Omega_i\right)$, and then show that

$$
\int_{\Omega_i} \left\langle a_i\left(\frac{x}{\varepsilon_h}, \alpha_k + Dv^{\alpha_k, x_i}\right), Du_h \right\rangle w \, dx \to \int_{\Omega_i} \left\langle b_i\left(\alpha_k\right), Du \right\rangle w \, dx
$$

for every $w \in L^t\left(\Omega_i\right)$, $t < \infty$. By Lemma 15 there exists $s' > 0$ such that

$$
\left\| a_i\left(\frac{x}{\varepsilon_h}, \alpha_k + Dv^{\alpha_k, x_i}\right) \right\|_{L^{q+s'}(\Omega_i, \mathbf{R}^n)} \leq C.
$$

For $1 < r = \dfrac{p^2 + p^2 s' - p s'}{p^2 + p s' - s'} \leq p$ we obtain that

$$
\left\| \left\langle a_i\left(\frac{x}{\varepsilon_h}, \alpha_k + Dv^{\alpha_k, x_i}\right), Du_h \right\rangle \right\|^r_{L^r(\Omega_i)} = \int_{\Omega_i} \left| \left\langle a_i\left(\frac{x}{\varepsilon_h}, \alpha_k + Dv^{\alpha_k, x_i}\right), Du_h \right\rangle \right|^r dx
$$

$$
\begin{aligned}
&\leq \int\limits_{\Omega_i} \left| a_i\left(\frac{x}{\varepsilon_h}, \alpha_k + Dv^{\alpha_k,x_i}\right)\right|^r |Du_h|^r \, dx \\
&\leq \left(\int\limits_{\Omega_i} \left| a_i\left(\frac{x}{\varepsilon_h}, \alpha_k + Dv^{\alpha_k,x_i}\right)\right|^{q+s'} dx\right)^{\frac{p(p-1)}{p^2+ps'-s'}} \left(\int\limits_{\Omega_i} |Du_h|^p \, dx\right)^{\frac{p+ps'-s'}{p^2+ps'-s'}} \leq C.
\end{aligned}
$$

Choose $t < \infty$ such that $\dfrac{1}{t} + \dfrac{1}{r} = 1$. Then there exists a sequence (ϕ_m), $\phi_m \in C_0^\infty(\Omega_i)$, such that

$$
\|w - \phi_m\|_{L^t(\Omega_i)} \to 0 \text{ for every } w \in L^t(\Omega_i).
$$

We have that

$$
\begin{aligned}
&\left| \int\limits_{\Omega_i} \left[\left\langle a_i\left(\frac{x}{\varepsilon_h}, \alpha_k + Dv^{\alpha_k,x_i}\right), Du_h \right\rangle - \langle b_i(\alpha_k), Du\rangle \right] w \, dx \right| \\
= &\left| \int\limits_{\Omega_i} \left[\left\langle a_i\left(\frac{x}{\varepsilon_h}, \alpha_k + Dv^{\alpha_k,x_i}\right), Du_h \right\rangle - \langle b_i(\alpha_k), Du\rangle \right] (w - \phi_m + \phi_m) \, dx \right| \\
\leq &\int\limits_{\Omega_i} \left| \left[\left\langle a_i\left(\frac{x}{\varepsilon_h}, \alpha_k + Dv^{\alpha_k,x_i}\right), Du_h \right\rangle - \langle b_i(\alpha_k), Du\rangle \right] (w - \phi_m) \right| dx \\
&+ \left| \int\limits_{\Omega_i} \left[\left\langle a_i\left(\frac{x}{\varepsilon_h}, \alpha_k + Dv^{\alpha_k,x_i}\right), Du_h \right\rangle - \langle b_i(\alpha_k), Du\rangle \right] \phi_m \, dx \right| \\
\leq &\left\| \left\langle a_i\left(\frac{x}{\varepsilon_h}, \alpha_k + Dv^{\alpha_k,x_i}\right), Du_h \right\rangle - \langle b_i(\alpha_k), Du\rangle \right\|_{L^r(\Omega_i)} \|w - \phi_m\|_{L^t(\Omega_i)} \\
&+ \left| \int\limits_{\Omega_i} \left[\left\langle a_i\left(\frac{x}{\varepsilon_h}, \alpha_k + Dv^{\alpha_k,x_i}\right), Du_h \right\rangle - \langle b_i(\alpha_k), Du\rangle \right] \phi_m \, dx \right|. \qquad (64)
\end{aligned}
$$

For every $\varepsilon > 0$ there exists an M such that

$$
m \geq M \Longrightarrow \|w - \phi_m\|_{L^t(\Omega_i)} \leq \varepsilon. \qquad (65)
$$

By using (63)-(65) we obtain that for $m \geq M$

$$
\begin{aligned}
&\lim_{h\to\infty} \left| \int\limits_{\Omega_i} \left[\left\langle a_i\left(\frac{x}{\varepsilon_h}, \alpha_k + Dv^{\alpha_k,x_i}\right), Du_h \right\rangle - \langle b_i(\alpha_k), Du\rangle \right] w \, dx \right| \\
\leq &\lim_{h\to\infty} \left\| \left\langle a_i\left(\frac{x}{\varepsilon_h}, \alpha_k + Dv^{\alpha_k,x_i}\right), Du_h \right\rangle - \langle b_i(\alpha_k), Du\rangle \right\|_{L^r(\Omega_i)} \varepsilon \leq C\varepsilon.
\end{aligned}
$$

Since ε was arbitrary, this implies that

$$\lim_{h\to\infty}\int_{\Omega_i}\left\langle a_i\left(\frac{x}{\varepsilon_h},\alpha_k+Dv^{\alpha_k,x_i}\right),Du_h\right\rangle w\,dx=\int_{\Omega_i}\left\langle b_i\left(\alpha_k\right),Du\right\rangle w\,dx$$

for every $w\in L^t\left(\Omega_i\right)$. By choosing $w\left(x\right)=1$ for $x\in A_k$ and $w\left(x\right)=0$ for $x\notin A_k$, we obtain that

$$\lim_{h\to\infty}\int_{\Omega_i}\left\langle a_i\left(\frac{x}{\varepsilon_h},\Gamma_i+Dv^{\Gamma_i,x_i}\right),Du_h\right\rangle dx=\int_{\Omega_i}\left\langle b_i\left(\Gamma_i\right),Du\right\rangle\,dx,$$

and the desired result follows by the continuity of b, see Remark 3. Indeed,

$$\begin{aligned}
&\lim_{h\to\infty}\left|\int_{\Omega_i}\left\langle a_i\left(\frac{x}{\varepsilon_h},\Gamma_i+Dv^{\Gamma_i,x_i}\right),Du_h\right\rangle-\int_{\Omega_i}\left\langle b_i\left(Du\right),Du\right\rangle dx\right|\\
\leq\ &\lim_{h\to\infty}\left|\int_{\Omega_i}\left\langle a_i\left(\frac{x}{\varepsilon_h},\Gamma_i+Dv^{\Gamma_i,x_i}\right),Du_h\right\rangle-\left\langle b_i\left(\Gamma_i\right),Du\right\rangle\,dx\right|\\
&+\lim_{h\to\infty}\int_{\Omega_i}\left|\left\langle b_i\left(\Gamma_i\right),Du\right\rangle-\left\langle b_i\left(Du\right),Du\right\rangle\right|\,dx\\
\leq\ &\lim_{h\to\infty}\int_{\Omega_i}\left|b_i\left(\Gamma_i\right)-b_i\left(Du\right)\right|\left|Du\right|\\
\leq\ &\lim_{h\to\infty}\int_{\Omega_i}\widetilde{c}_1\left(1+\left|\Gamma_i\right|+\left|Du\right|\right)^{p-1-\gamma}\left|\Gamma_i-Du\right|^{\gamma}\left|Du\right|\,dx\\
\leq\ &\lim_{h\to\infty}C\left\|1+\left|\Gamma_i\right|+\left|Du\right|\right\|^{p-1-\gamma}_{L^p(\Omega_i,\mathbf{R}^n)}\left\|\Gamma_i-Du\right\|^{\gamma}_{L^p(\Omega_i,\mathbf{R}^n)}\left\|Du\right\|_{L^p(\Omega_i,\mathbf{R}^n)}\leq C\delta^{\gamma}.
\end{aligned}$$

Since the δ was arbitrarily chosen, we have proved (62). The proof now follows by combining (60), (61) and (62). □

Lemma 17 *Let a_i, u, u_h, v^{M_hDu,x_i} and b_i be defined as in Theorem 5. Then*

$$\int_{\Omega_i}\left\langle a_i\left(\frac{x}{\varepsilon_h},Du_h\right),M_hDu+Dv^{M_hDu,x_i}\left(\frac{x}{\varepsilon_h}\right)\right\rangle dx\to\int_{\Omega_i}\left\langle b_i\left(Du\right),Du\right\rangle\,dx.$$

Proof. The proof is similar to that of Lemma 16, but for the readers convenience we present the details. For every $\delta>0$ there exists a simple function $\Gamma_i=\sum_{k=1}^m\chi_{A_k}\alpha_k$ satisfying the conditions in Lemma 13 such that

$$\left\|Du-\Gamma_i\right\|_{L^p(\Omega_i,\mathbf{R}^n)}\leq\delta.$$

By adding and subtracting $\int_{\Omega_i}\left\langle a_i\left(\frac{x}{\varepsilon_h},Du_h\right),\Gamma_i+Dv^{\Gamma_i,x_i}\left(\frac{x}{\varepsilon_h}\right)\right\rangle dx$ we have that

$$\left|\int_{\Omega_i}\left\langle a_i\left(\frac{x}{\varepsilon_h},Du_h\right),M_hDu+Dv^{M_hDu,x_i}\left(\frac{x}{\varepsilon_h}\right)\right\rangle-\int_{\Omega_i}\left\langle b_i\left(Du\right),Du\right\rangle dx\right|$$

$$\leq \int\limits_{\Omega_i} \left| \left\langle a_i\left(\frac{x}{\varepsilon_h}, Du_h\right), M_h Du + Dv^{M_h Du, x_i} - \left(\Gamma_i + Dv^{\Gamma_i, x_i}\right) \right\rangle \right| dx$$

$$+ \left| \int\limits_{\Omega_i} \left\langle a_i\left(\frac{x}{\varepsilon_h}, Du_h\right), \Gamma_i + Dv^{\Gamma_i, x_i} \right\rangle - \langle b_i(Du), Du\rangle \, dx \right|. \tag{66}$$

Let us estimate the first integral on the right hand side in (66). By Schwarz' inequality and Hölder's inequality we get that

$$\int\limits_{\Omega_i} \left| \left\langle a_i\left(\frac{x}{\varepsilon_h}, Du_h\right), M_h Du + Dv^{M_h Du, x_i} - \left(\Gamma_i + Dv^{\Gamma_i, x_i}\right) \right\rangle \right| dx$$

$$\leq \int\limits_{\Omega_i} \left| a_i\left(\frac{x}{\varepsilon_h}, Du_h\right) \right| \left| M_h Du + Dv^{M_h Du, x_i} - \left(\Gamma_i + Dv^{\Gamma_i, x_i}\right) \right| dx$$

$$\leq \left(\int\limits_{\Omega_i} \left| a_i\left(\frac{x}{\varepsilon_h}, Du_h\right) \right|^q dx \right)^{\frac{1}{q}} \left(\int\limits_{\Omega_i} \left| M_h Du + Dv^{M_h Du, x_i} - (\Gamma_i + Dv^{\Gamma_i, x_i}) \right|^p dx \right)^{\frac{1}{p}}.$$

Lemma 13 and the boundedness of $a_i\left(\frac{x}{\varepsilon_h}, Du_h\right)$ in $L^q(\Omega_i, \mathbf{R}^n)$, see (20), implies that

$$\limsup_{h\to\infty} \int_{\Omega_i} \left| \left\langle a_i\left(\frac{x}{\varepsilon_h}, Du_h\right), M_h Du + Dv^{M_h Du, x_i} - \left(\Gamma_i + Dv^{\Gamma_i, x_i}\right) \right\rangle \right| dx \leq C\delta, \tag{67}$$

where C is a constant independent of h. Let us now study the second integral on the right hand side in (66). We will show that

$$\int_{\Omega_i} \left\langle a_i\left(\frac{x}{\varepsilon_h}, Du_h\right), \Gamma_i + Dv^{\Gamma_i, x_i} \right\rangle - \langle b_i(Du), Du\rangle \, dx \to 0.$$

We note that according to Theorem 5, $a_i\left(\frac{x}{\varepsilon_h}, Du_h\right)$ converges weakly in $L^q(\Omega_i, \mathbf{R}^n)$ to $b_i(Du)$, $-\operatorname{div}\left(a_i\left(\frac{x}{\varepsilon_h}, Du_h\right)\right)$ converges to $-\operatorname{div} b_i(Du)$ in $W^{-1,q}(\Omega_i)$ and by periodicity $(\alpha_k, x) + \varepsilon_h v^{\alpha_k, x_i}\left(\frac{x}{\varepsilon_h}\right)$ converges weakly in $W^{1,p}(\Omega_i)$ to (α_k, x). The compensated compactness lemma, Lemma 1, implies that

$$\int_{\Omega_i} \left\langle a_i\left(\frac{x}{\varepsilon_h}, Du_h\right), \alpha_k + Dv^{\alpha_k, x_i} \right\rangle \phi \, dx \to \int_{\Omega_i} \langle b_i(Du), \alpha_k\rangle \, \phi \, dx, \tag{68}$$

as $h \to \infty$ for every $\phi \in C_0^\infty(\Omega_i)$. Next we will prove that

$$\int_{\Omega_i} \left\langle a_i\left(\frac{x}{\varepsilon_h}, Du_h\right), \alpha_k + Dv^{\alpha_k, x_i} \right\rangle w \, dx \to \int_{\Omega_i} \langle b_i(Du), \alpha_k\rangle \, w \, dx$$

for every $w \in L^t(\Omega_i)$ and some $t < \infty$. First we note that by (59), there exists $s > 0$ such that $\left\|\alpha_k + Dv^{\alpha_k,x_i}\left(\frac{x}{\varepsilon_h}\right)\right\|_{L^{p+s}(\Omega_i,\mathbf{R}^n)} \leq C$, where C is independent of h. For $1 < r = \frac{p(p+s)}{p^2+ps-s} \leq \frac{p}{p-1} = q$ we obtain that

$$\begin{aligned}
&\left\|\left\langle a_i\left(\frac{x}{\varepsilon_h}, Du_h\right), \alpha_k + Dv^{\alpha_k,x_i}\right\rangle\right\|^r_{L^r(\Omega_i)} \\
= &\int\limits_{\Omega_i}\left|\left\langle a_i\left(\frac{x}{\varepsilon_h}, Du_h\right), \alpha_k + Dv^{\alpha_k,x_i}\right\rangle\right|^r dx \leq \int\limits_{\Omega_i}\left|a_i\left(\frac{x}{\varepsilon_h}, Du_h\right)\right|^r \left|\alpha_k + Dv^{\alpha_k,x_i}\right|^r dx \\
\leq &\left(\int\limits_{\Omega_i}\left|a_i\left(\frac{x}{\varepsilon_h}, Du_h\right)\right|^q dx\right)^{\frac{(p-1)(p+s)}{p^2+ps-s}} \left(\int\limits_{\Omega_i}\left|\alpha_k + Dv^{\alpha_k,x_i}\right|^{p+s} dx\right)^{\frac{p}{p^2+ps-s}} \leq C. \qquad (69)
\end{aligned}$$

Choose $t < \infty$ such that $\frac{1}{t} + \frac{1}{r} = 1$. Then there exists a sequence (ϕ_m), $\phi_m \in C_0^\infty(\Omega_i)$, such that

$$\|w - \phi_m\|_{L^t(\Omega_i)} \to 0 \text{ for every } w \in L^t(\Omega_i).$$

We have that

$$\begin{aligned}
&\left|\int\limits_{\Omega_i}\left[\left\langle a_i\left(\frac{x}{\varepsilon_h}, Du_h\right), \alpha_k + Dv^{\alpha_k,x_i}\right\rangle - \langle b_i(Du), \alpha_k\rangle\right] w\, dx\right| \\
= &\left|\int\limits_{\Omega_i}\left[\left\langle a_i\left(\frac{x}{\varepsilon_h}, Du_h\right), \alpha_k + Dv^{\alpha_k,x_i}\right\rangle - \langle b_i(Du), \alpha_k\rangle\right] (w - \phi_m + \phi_m)\, dx\right| \\
\leq &\int\limits_{\Omega_i}\left|\left[\left\langle a_i\left(\frac{x}{\varepsilon_h}, Du_h\right), \alpha_k + Dv^{\alpha_k,x_i}\right\rangle - \langle b_i(Du), \alpha_k\rangle\right] (w - \phi_m)\right| dx \\
&+\left|\int\limits_{\Omega_i}\left[\left\langle a_i\left(\frac{x}{\varepsilon_h}, Du_h\right), \alpha_k + Dv^{\alpha_k,x_i}\right\rangle - \langle b_i(Du), \alpha_k\rangle\right] \phi_m\, dx\right| \\
\leq &\left\|\left\langle a_i\left(\frac{x}{\varepsilon_h}, Du_h\right), \alpha_k + Dv^{\alpha_k,x_i}\right\rangle - \langle b_i(Du), \alpha_k\rangle\right\|_{L^r(\Omega_i)} \|w - \phi_m\|_{L^t(\Omega_i)} \\
&+\left|\int\limits_{\Omega_i}\left[\left\langle a_i\left(\frac{x}{\varepsilon_h}, Du_h\right), \alpha_k + Dv^{\alpha_k,x_i}\right\rangle - \langle b_i(Du), \alpha_k\rangle\right] \phi_m\, dx\right|. \qquad (70)
\end{aligned}$$

For every $\varepsilon > 0$ there exists M such that

$$m \geq M \Longrightarrow \|w - \phi_m\|_{L^t(\Omega_i)} \leq \varepsilon. \qquad (71)$$

By using (68)-(71), we obtain for $m \geq M$,

$$\begin{aligned}
&\lim_{h\to\infty} \left| \int_{\Omega_i} \left(\left\langle a_i\left(\frac{x}{\varepsilon_h}, Du_h\right), \alpha_k + Dv^{\alpha_k,x_i} \right\rangle - \langle b_i(Du), \alpha_k\rangle \right) w\, dx \right| \\
\leq\ &\lim_{h\to\infty} \left\| \left\langle a_i\left(\frac{x}{\varepsilon_h}, Du_h\right), \alpha_k + Dv^{\alpha_k,x_i} \right\rangle - \langle b_i(Du), \alpha_k\rangle \right\|_{L^r(\Omega_i)} \varepsilon \leq C\varepsilon.
\end{aligned}$$

Since ε was arbitrary this implies that

$$\lim_{h\to\infty} \int_{\Omega_i} \left\langle a_i\left(\frac{x}{\varepsilon_h}, Du_h\right), \alpha_k + Dv^{\alpha_k,x_i} \right\rangle w\, dx = \int_{\Omega_i} \langle b_i(Du), \alpha_k\rangle\, w\, dx$$

for every $w \in L^t(\Omega_i)$. By choosing $w(x) = 1$ for $x \in A_k$ and $w(x) = 0$ for $x \notin A_k$, we obtain that

$$\lim_{h\to\infty} \int_{\Omega_i} \left\langle a_i\left(\frac{x}{\varepsilon_h}, Du_h\right), \Gamma_i + Dv^{\Gamma_i,x_i}\left(\frac{x}{\varepsilon_h}\right) \right\rangle dx = \int_{\Omega_i} \langle b_i(Du), \Gamma_i\rangle\, dx,$$

and the desired result follows by the continuity of b, see Remark 3. Indeed,

$$\begin{aligned}
&\lim_{h\to\infty} \left| \int_{\Omega_i} \left\langle a_i\left(\frac{x}{\varepsilon_h}, Du_h\right), \Gamma_i + Dv^{\Gamma_i,x_i}\left(\frac{x}{\varepsilon_h}\right) \right\rangle - \langle b_i(Du), Du\rangle\, dx \right| \\
\leq\ &\lim_{h\to\infty} \left| \int_{\Omega_i} \left\langle a_i\left(\frac{x}{\varepsilon_h}, Du_h\right), \Gamma_i + Dv^{\Gamma_i,x_i}\left(\frac{x}{\varepsilon_h}\right) \right\rangle - \langle b_i(Du), \Gamma_i\rangle\ dx \right| \\
&+ \lim_{h\to\infty} \int_{\Omega_i} |\langle b_i(Du), \Gamma_i\rangle - \langle b_i(Du), Du\rangle|\, dx \\
\leq\ &\lim_{h\to\infty} \int_{\Omega_i} |b_i(Du)|\, |\Gamma_i - Du|\, dx \\
\leq\ &\lim_{h\to\infty} \|b_i(Du)\|_{L^q(\Omega_i,\mathbf{R}^n)}\, \|\Gamma_i - Du\|_{L^p(\Omega_i,\mathbf{R}^n)} \leq C\delta. \qquad (72)
\end{aligned}$$

Since the δ was arbitrarily chosen, we conclude, according to (67) and (72), that both terms on the right hand side of (66) converges to zero and the convergence claimed in the lemma is proved. □

Lemma 18 *Let u_h, $v^{M_h Du,x_i}$ and P_h be defined as in Theorem 7. Then*

$$\int_\Omega \left(1 + |Du_h| + \left| \sum_{i=1}^N \chi_{\Omega_i}(x)\, P_h(x, M_h Du, x_i) \right| \right)^p dx \leq C_P. \qquad (73)$$

Proof. We have by (3), a change of variables in (12) and Jensen's inequality that

$$\int_{\Omega_i} |P_h(x, M_h Du, x_i)|^p\, dx = \int_{\Omega_i} \left| M_h Du + Dv^{M_h Du,x_i}\left(\frac{x}{\varepsilon_h}\right) \right|^p dx$$

$$
\begin{aligned}
&= \int\limits_{\Omega_i^h} \left| M_h Du + Dv^{M_h Du, x_i}\left(\frac{x}{\varepsilon_h}\right)\right|^p dx + \int\limits_{\Omega_i \setminus \Omega_i^h} \left| M_h Du + Dv^{M_h Du, x_i}\left(\frac{x}{\varepsilon_h}\right)\right|^p dx \\
&\leq \sum_{j\in J_i^h} \int\limits_{Y_h^j} \left| \xi_h^j + Dv^{\xi_h^j, x_i}\left(\frac{x}{\varepsilon_h}\right)\right|^p dx + \sum_{j\in B_i^h} \int\limits_{Y_h^j} \left| \xi_h^j + Dv^{\xi_h^j, x_i}\left(\frac{x}{\varepsilon_h}\right)\right|^p dx \\
&\leq \sum_{j\in J_i^h} c_c \left(1 + \left|\xi_h^j\right|^p\right) \left|Y_h^j\right| + \sum_{j\in B_i^h} c_c \left(1 + \left|\xi_h^j\right|^p\right) \left|Y_h^j\right| \\
&= C \sum_{j\in J_i^h} \int\limits_{Y_h^j} \left(1 + |M_h Du|^p\right) dx + C \sum_{j\in B_i^h} \int\limits_{Y_h^j} \left(1 + |M_h Du|^p\right) dx \\
&\leq C \sum_{j\in J_i^h} \int\limits_{Y_h^j} \left(1 + |Du|^p\right) dx + C \sum_{j\in B_i^h} \int\limits_{Y_h^j} \left(1 + |M_h Du|^p\right) dx \\
&= C \int\limits_{\Omega_i^h} \left(1 + |Du|^p\right) dx + C \int\limits_{F_i^h} \left(1 + |Du|^p\right) dx \\
&\leq C \int\limits_{\Omega_i} \left(1 + |Du|^p\right) dx + C \int\limits_{F_i^h} \left(1 + |M_h Du|^p\right) dx. \qquad (74)
\end{aligned}
$$

Hence we by (74) have that

$$
\begin{aligned}
&\int\limits_{\Omega} \left(1 + |Du_h| + \left|\sum_{i=1}^N \chi_{\Omega_i}(x)\, P_h\left(x, M_h Du, x_i\right)\right|\right)^p dx \\
&\leq 3^{p-1} \int\limits_{\Omega} \left(1 + |Du_h|^p + \left|\sum_{i=1}^N \chi_{\Omega_i}(x)\, P_h\left(x, M_h Du, x_i\right)\right|^p\right) dx \\
&\leq C \left(|\Omega| + \int\limits_{\Omega} |Du_h|^p\, dx + \sum_{i=1}^N \int\limits_{\Omega_i} \left|P_h\left(x, M_h Du, x_i\right)\right|^p dx\right) \\
&\leq C \left(|\Omega| + \int\limits_{\Omega} |Du_h|^p\, dx + C \sum_{i=1}^N \left(\int\limits_{\Omega_i} \left(1 + |Du|^p\right) dx + \int\limits_{F_i^h} \left(1 + |M_h Du|^p\right) dx\right)\right) \\
&\leq C \left(|\Omega| + \int\limits_{\Omega} |Du_h|^p\, dx + \int\limits_{\Omega} \left(1 + |Du|^p\right) dx + \sum_{i=1}^N \int\limits_{F_i^h} \left(1 + |M_h Du|^p\right) dx\right). \qquad (75)
\end{aligned}
$$

Now since $\left|F_i^h\right| \to 0$ as $h \to \infty$, we have for sufficently large h that

$$\sum_{i=1}^{N} \int_{F_i^h} \left(1 + |M_h Du|^p\right) dx \leq 1. \tag{76}$$

Hence we have that

$$\int_\Omega \left(1 + |Du_h| + \left|\sum_{i=1}^{N} \chi_{\Omega_i}(x)\, P_h\left(x, M_h Du, x_i\right)\right|\right)^p dx \leq C_P$$

by (18), (75) and (76). □

Proof. [of Theorem 7]

By the monotonicity assumption (7), (73) and Hölder's reversed inequality we have that

$$\begin{aligned}
&\left(\int_\Omega \left|Du_h - \sum_{i=1}^{N} \chi_{\Omega_i}(x)\, P_h\left(x, M_h Du, x_i\right)\right|^p dx\right)^{\frac{\beta}{p}} \\
\leq\ &\left(\frac{1}{C_P}\int_\Omega \left(1 + |Du_h| + \left|\sum_{i=1}^{N} \chi_{\Omega_i}(x)\, P_h\left(x, M_h Du, x_i\right)\right|\right)^p dx\right)^{\frac{p-\beta}{p}} \\
&\times \left(\int_\Omega \left|Du_h - \sum_{i=1}^{N} \chi_{\Omega_i}(x)\, P_h\left(x, M_h Du, x_i\right)\right|^p dx\right)^{\frac{\beta}{p}} \\
\leq\ &C\int_\Omega \left(1 + |Du_h| + \left|\sum_{i=1}^{N} \chi_{\Omega_i}(x)\, P_h\left(x, M_h Du, x_i\right)\right|\right)^{p-\beta} \\
&\times \left|Du_h - \sum_{i=1}^{N} \chi_{\Omega_i}(x)\, P_h\left(x, M_h Du, x_i\right)\right|^\beta dx \\
\leq\ &\frac{C}{c_2}\int_\Omega \left\langle a\left(x, \frac{x}{\varepsilon_h}, Du_h\right) - a\left(x, \frac{x}{\varepsilon_h}, \sum_{i=1}^{N} \chi_{\Omega_i}(x)\, P_h\left(x, M_h Du, x_i\right)\right),\right. \\
&\left. Du_h - \sum_{i=1}^{N} \chi_{\Omega_i}(x)\, P_h\left(x, M_h Du, x_i\right)\right\rangle dx \\
\leq\ &C\left[\int_\Omega \left\langle a\left(x, \frac{x}{\varepsilon_h}, Du_h\right), Du_h\right\rangle dx\right. \\
&\left. - \int_\Omega \left\langle a\left(x, \frac{x}{\varepsilon_h}, Du_h\right), \sum_{i=1}^{N} \chi_{\Omega_i}(x)\left(M_h Du + Dv^{M_h Du, x_i}\left(\frac{x}{\varepsilon_h}\right)\right)\right\rangle dx\right.
\end{aligned}$$

$$
\begin{aligned}
&-\int_{\Omega}\left\langle a\left(x,\frac{x}{\varepsilon_h},\sum_{i=1}^{N}\chi_{\Omega_i}(x)\left(M_hDu+Dv^{M_hDu,x_i}\left(\frac{x}{\varepsilon_h}\right)\right)\right),Du_h\right\rangle dx\\
&+\int_{\Omega}\left\langle a\left(x,\frac{x}{\varepsilon_h},\sum_{i=1}^{N}\chi_{\Omega_i}(x)\left(M_hDu+Dv^{M_hDu,x_i}\left(\frac{x}{\varepsilon_h}\right)\right)\right),\right.\\
&\left.\sum_{i=1}^{N}\chi_{\Omega_i}(x)\left(M_hDu+Dv^{M_hDu,x_i}\left(\frac{x}{\varepsilon_h}\right)\right)\right\rangle dx\Bigg].
\end{aligned}
$$

The theorem follows by applying Lemma 11 to the first term, Lemma 17 to the second term, Lemma 16 to the third term and Lemma 12 to the last term. □