

Introduction

This thesis deals with the theory and applications of the *homogenization method*. The theory of homogenization dates back to the late sixties, was very rapidly developed during the last two decades and is now established as a distinct discipline within mathematics. The homogenization theory has important applications in the mathematical analysis of different physical and mechanical phenomena in composites, perforated media, dispersed media, porous media and similar situations. The main problem is to determine effective properties for e.g. heat transfer, elasticity, electric conductivity, magnetic permeability, flow etc. of strongly heterogeneous multiphase materials. A common feature for such problems is that the governing equations involve rapidly oscillating functions due to the heterogeneity of the underlying material. These rapid oscillations render a direct numerical treatment impossible. Therefore one has to do some type of averaging or asymptotic analysis, which immediately leads to the concept of homogenization.

This thesis is written in a bottom-up type of style. We start purely mathematically by presenting the basic ideas in the area of homogenization and give some theoretical results. It is now natural to introduce the concept of bounds for the homogenized operator. Having this foundation, a good way to get a feeling for the homogenization principles is then to consider the computational aspects of homogenization. We finish by showing some cases where the homogenization theory can be used in the science of composites engineering.

The following pages are meant to give an elementary presentation of the basic ideas in the homogenization theory. Moreover, they also serve as an overview of the field and points out where the results contained in this thesis fit in.

A Analytical and theoretical homogenization results

In this section we present the basic ideas and some of the most important theoretical concepts of the homogenization theory. The physical problems that we have considered in this thesis (elasticity, stationary heat conductivity) are elliptic, so we concentrate on the homogenization of such problems. One should however keep in mind that the theory of homogenization has

successfully been applied to a number of other equations such as the wave equation, Helmholtz' equation, Navier-Stokes' equation, the porous medium equation, the biharmonic equation, Maxwell's equations, Hamilton-Jacobi's equation and the non-stationary diffusion (heat) equation, see e.g. [8], [16], [26], [31], [32], [40], [42], [46], [51], [60], [66], [86], [90], [93], [94] and also to other material structures, see e.g. [25]. We also mention the collection of classical papers in [20], where the introduction gives a good selection of references. Some of these applications have been treated in the following theses by other members of our research group in homogenization, see [12], [17], [18], [50], [69], [73], [96], [98], [107] and [108].

A.1 The basic ideas in the homogenization theory

Suppose that we would like to know the stationary temperature distribution in a homogeneous body of shape $\Omega \subset \mathbf{R}^3$ with an internal heat source f , heat conductivity a and zero temperature on the boundary $\partial\Omega$. Or maybe we are interested in the deflection of a membrane of shape $\Omega \subset \mathbf{R}^2$ made of a homogeneous material with stiffness a , loaded by a vertical force f and clamped on the boundary $\partial\Omega$. These problems can be described by the following boundary value problem: Find $u \in W_0^{1,p}(\Omega)$, $1 < p < \infty$, such that

$$-\operatorname{div}(a(Du)) = f \quad \text{on } \Omega, \quad (1)$$

where $\Omega \subset \mathbf{R}^n$ is an open bounded subset, f a given function on Ω and $a : \mathbf{R}^n \rightarrow \mathbf{R}^n$ satisfies suitable continuity and monotonicity conditions of such type that we have existence and uniqueness of the solution of the above problem. Here $W_0^{1,p}(\Omega)$ is the usual Sobolev space of functions u with boundary values 0 such that $u \in L^p(\Omega)$, $Du \in [L^p(\Omega)]^n$.

Now that suppose we would like to be able to model physical situations where the underlying material is heterogeneous, that is, Ω consists of a material with different properties in different positions of Ω . Then we would have to replace a in (1) with a map of the type $a : \Omega \times \mathbf{R}^n \rightarrow \mathbf{R}^n$ and we get the equation

$$\begin{cases} -\operatorname{div}(a(x, Du)) = f \quad \text{on } \Omega, \\ u \in W_0^{1,p}(\Omega). \end{cases} \quad (2)$$

Because of the dependence of x , equation (2) is much more difficult to handle than equation (1). One interesting special case is a two-phase composite

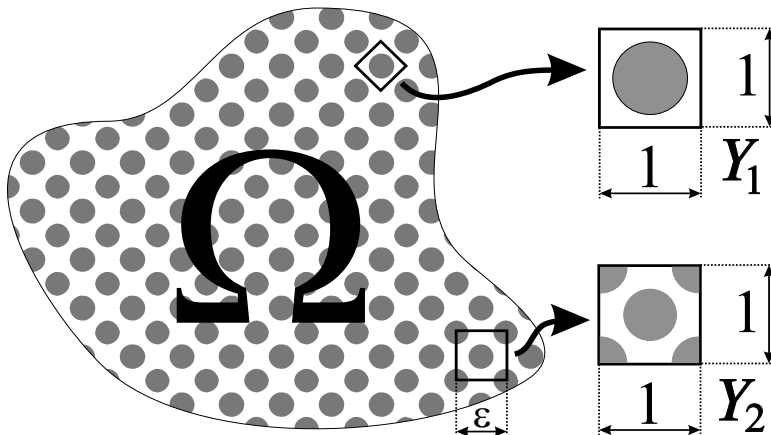


Figure 1: A periodic composite with two different Y -cells

where one material is periodically distributed in the other. In this case the underlying periodic inclusions are often microscopic with respect to the overall body Ω . Because of the periodicity, we can now divide Ω into periodic cells which we call Y , see Figure 1. Note that the microstructure of a given periodic material can be described by several different periodic cells.

Mathematically this is usually described by maps of the form $a_h(x, \xi) = a(x/\varepsilon_h, \xi)$, where $a(\cdot, \xi)$ is assumed to be Y -periodic and ε_h is a small parameter. This means that ε_h is a parameter which varies the frequency of a_h , or in other words, the fineness of the periodic structure. Equation (2) then takes the form

$$\begin{cases} -\operatorname{div}(a_h(x, Du_h)) = f \text{ on } \Omega, \\ u_h \in W_0^{1,p}(\Omega). \end{cases} \quad (3)$$

Now let (ε_h) be a sequence of positive real numbers such that $\varepsilon_h \rightarrow 0$ as $h \rightarrow \infty$. In this way we obtain a sequence of problems, one for each value of h . The larger the value of h gets, the finer the microstructure becomes, see Figure 2.

The natural question that now arises is if we have some type of convergence of the solutions u_h . Assume that we can establish convergence in some appropriate sense, that is,

$$u_h \rightarrow u \text{ as } h \rightarrow \infty.$$

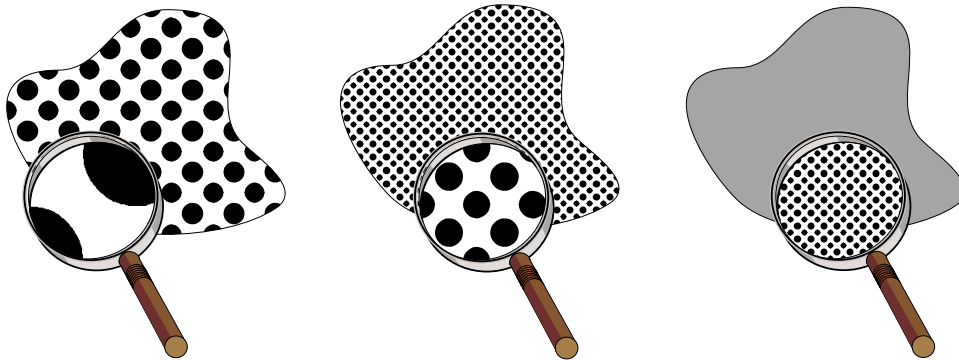


Figure 2: The microstructure of Ω as h tends to infinity.

The second step would then be to investigate whether u satisfies an equation of a similar type, that is

$$\begin{cases} -\operatorname{div}(b(x, Du)) = f \text{ on } \Omega, \\ u \in W_0^{1,p}(\Omega). \end{cases}$$

If this is the situation, how do we then find b ? From Figure 2, we see that the material macroscopically appears to behave like a homogeneous material for large values of h , even though the material is strongly heterogeneous on the microlevel. We say that b is the *effective* conductivity, stiffness, etc. This makes it reasonable to assume that b is independent of x , which means that u satisfies an homogenized equation of the type

$$\begin{cases} -\operatorname{div}(b(Du)) = f \text{ on } \Omega, \\ u \in W_0^{1,p}(\Omega). \end{cases}$$

The subject that deals with these types of questions is known as homogenization. In particular, the convergence of partial differential operators of the type above is an important case of *G-convergence* of monotone operators, see e.g. [4], [10], [24], [54] and [87]. Part A of this thesis deals with *G-convergence* of monotone operators and degenerate monotone operators.

Remark 1 *We will use the notion G-convergence for operators of the type above. Some authors do however distinguish between G-convergence (introduced by Spagnolo [97] in 1967) and H-convergence (introduced by Tartar*

in 1977, see [81], [101] and also [20]), where H -convergence initially was introduced as an extension of G -convergence to a wider class of problems.

Another approach to study different physical phenomena in heterogeneous materials is by using the fact that the state of the material u often can be found as the solution of a minimization problem of the type

$$E_h = \min_{u \in W_0^{1,p}(\Omega)} \left\{ \int_{\Omega} g\left(\frac{x}{\varepsilon_h}, Du(x)\right) dx - \int_{\Omega} f u dx \right\},$$

where the local *energy density function* $g(\cdot, \xi)$ is periodic and is assumed to satisfy the so called natural growth conditions. The convergence of this type of integral functionals is called Γ -convergence (introduced by De Giorgi, see [34], [35], [36] and also [6], [14], [15], [29], [54]). From the theory of Γ -convergence it follows that $E_h \rightarrow E_{\text{hom}}$ as $h \rightarrow \infty$, where

$$E_{\text{hom}} = \min_{u \in W_0^{1,p}(\Omega)} \left\{ \int_{\Omega} g_{\text{hom}}(Du(x)) dx - \int_{\Omega} f u dx \right\}.$$

Here the homogenized energy density function g_{hom} is given by

$$g_{\text{hom}}(\xi) = \frac{1}{|Y|} \min_{u \in W_{\square}^{1,p}(Y)} \int_Y g(x, \xi + Du) dx, \quad (4)$$

where $W_{\square}^{1,p}(Y)$ is the set of all functions $u \in W^{1,p}(Y)$ which are Y -periodic and have mean value zero. Again we note that the limit problem does not depend on x , that is, g_{hom} is the energy density function of a homogeneous material.

Remark 2 *The partial differential equations above should be interpreted in the weak sense. The model problem (3), for instance, should be read as*

$$\begin{cases} \int_{\Omega} \langle a_h(x, Du_h), D\phi \rangle dx = \int_{\Omega} f \phi dx \text{ for every } \phi \in W_0^{1,p}(\Omega), \\ u_h \in W_0^{1,p}(\Omega), \end{cases} \quad (5)$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product.

A.2 Some basic homogenization results

In order to demonstrate some of the techniques and difficulties encountered in the homogenization procedure we consider homogenization of the one dimensional Poisson equation. The homogenization process is much simpler in $\mathbf{R}^1$ than in higher dimensions, however it reveals the main difficulty. We will use C to denote positive constants that may differ from place to place. Moreover, let $\Omega = (x_1, x_2)$, $f \in L^2(\Omega)$ and $a \in L^\infty(\Omega)$ be a measurable and periodic function with period 1 satisfying

$$0 < \beta_1 \leq a(x) \leq \beta_2 < \infty \text{ for a.e. } x \in \mathbf{R}. \quad (6)$$

Moreover, we define $a_h(x) = a\left(\frac{x}{\varepsilon_h}\right)$. Then equation (5) takes the form

$$\begin{cases} \int_{\Omega} a_h(x) Du_h D\phi dx = \int_{\Omega} f\phi dx \text{ for every } \phi \in W_0^{1,2}(\Omega), \\ u_h \in W_0^{1,2}(\Omega). \end{cases} \quad (7)$$

By a standard result in the existence theory of partial differential equations, there exists a unique solution of these problems for each h . By choosing $\phi = u_h$ in (7) and taking (6) into account, we obtain by Hölder's inequality that

$$\beta_1 \|Du_h\|_{L^2(\Omega)}^2 \leq \int_{\Omega} a_h |Du_h|^2 dx = \int_{\Omega} f u_h dx \leq \|f\|_{L^2(\Omega)} \|u_h\|_{L^2(\Omega)}.$$

The Poincaré inequality for functions with zero boundary values states that there is a constant K only depending on Ω such that

$$\|u_h\|_{L^2(\Omega)} \leq K \|Du_h\|_{L^2(\Omega)}.$$

This implies that

$$\|u_h\|_{W_0^{1,2}(\Omega)}^2 \leq C, \quad (8)$$

where C is a constant independent of h . Since $W_0^{1,2}(\Omega)$ is reflexive, the Eberlein-Šmuljan theorem then states that there exists a subsequence, still denoted by (u_h) , such that

$$u_h \rightharpoonup u_* \text{ weakly in } W_0^{1,2}(\Omega). \quad (9)$$

Since $W^{1,2}(\Omega)$ is compactly embedded in $L^2(\Omega)$, we have by the Rellich embedding theorem that $u_h \rightarrow u_*$ strongly in $L^2(\Omega)$. In general, however, we only have that $Du_h \rightarrow Du_*$ weakly in $L^2(\Omega)$. Moreover, since a is 1-periodic, we have that (a_h) weakly* converges to its arithmetic mean $\langle a \rangle$, that is,

$$a_h \rightarrow \langle a \rangle = \int_0^1 a \, dx \text{ weakly* in } L^\infty(\Omega). \quad (10)$$

From (7), (9) and (10), it is then reasonable to assume that we in the limit have

$$\begin{cases} \int_{\Omega} \langle a \rangle Du_* D\phi \, dx = \int_{\Omega} f \phi \, dx \text{ for every } \phi \in W_0^{1,2}(\Omega), \\ u_* \in W_0^{1,2}(\Omega). \end{cases}$$

Unfortunately this is not true in general, since $(a_h Du_h)$ is a product of sequences which only converge weakly. This is the prime obstacle in the limit process of the homogenization theory. To obtain the correct solution we proceed in the following way: First we note that $(a_h Du_h)$ is bounded in $L^2(\Omega)$ because of (6) and (8). We also have that $D(a_h Du_h) = f \in L^2(\Omega)$ by (7). Hence there exists a constant C , independent of h , such that

$$\|a_h Du_h\|_{W^{1,2}(\Omega)} \leq C.$$

Since $W^{1,2}(\Omega)$ is reflexive and compactly embedded in $L^2(\Omega)$, the Eberlein-Šmuljan theorem and the Rellich embedding theorem then state that there exists a subsequence, still denoted by $(a_h Du_h)$, and an $\eta \in L^2(\Omega)$ such that

$$a_h Du_h \rightarrow \eta \text{ strongly in } L^2(\Omega).$$

This combined with the fact that (a_h^{-1}) converges to $\langle a^{-1} \rangle$ weakly* in $L^\infty(\Omega)$ (hence weakly in $L^2(\Omega)$) gives us

$$Du_h = a_h^{-1} a_h Du_h \rightarrow \langle a^{-1} \rangle \eta \text{ weakly in } L^2(\Omega). \quad (11)$$

Thus in view of (9) and (11), we see that

$$\eta = \langle a^{-1} \rangle^{-1} Du_*.$$

Now, by passing to the limit in (7) we obtain that

$$\begin{cases} \int_{\Omega} b Du_* D\phi \, dx = \int_{\Omega} f \phi \, dx \text{ for every } \phi \in W_0^{1,2}(\Omega), \\ u_* \in W_0^{1,2}(\Omega), \end{cases}$$

where the homogenized operator is given by $b = \langle a^{-1} \rangle^{-1}$, the harmonic mean of a . Now since

$$\frac{1}{\beta_2} \leq \left\langle \frac{1}{a} \right\rangle \leq \frac{1}{\beta_1},$$

we conclude that the homogenized equation has a unique solution and thus that the whole sequence (u_h) converges. We want to remark that the main difficulty was the passage to the limit in $(a_h Du_h)$, which is the product of only weakly convergent sequences. However, one shall refrain from drawing the hasty conclusion that weak* convergence in $(L^\infty(\Omega))^{n \times n}$ of the inverse matrices A_h^{-1} of A_h is the key to the understanding of the problem in higher dimensions. In this case the problem of passing to the limit is rather delicate and requires introduction of new techniques. One of the main tools to overcome this difficulty is the *compensated compactness* method introduced by Murat and Tartar (see [82] and [102]). This method shows that under some additional assumptions, the product of two weakly convergent sequences in $L^2(\Omega)$ converges in the sense of distributions to the product of their limits. Another related concept is that of *H-measures* (or *microlocal defect measures*), introduced independently by Gérard and Tartar around 1990, see e.g. [43], [45] and [105].

Remark 3 *The corresponding homogenization problem for the one-dimensional p -Poisson equation*

$$\begin{cases} \int_{\Omega} a_h(x) |Du_h|^{p-2} Du_h D\phi \, dx = \int_{\Omega} f \phi \, dx \text{ for every } \phi \in W_0^{1,p}(\Omega), \\ u_h \in W_0^{1,p}(\Omega), \end{cases}$$

gives the homogenized operator $b = \langle a^{1/(1-p)} \rangle^{1-p}$.

We now turn to homogenization in $\mathbf{R}^n$. In Section A.1 we described the basic ideas in different homogenization problems. Assume that a satisfies

suitable structure conditions, see e.g. Remark 4. Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. The main result for the homogenization problem connected to the Dirichlet problems

$$\begin{cases} -\operatorname{div}(a_h(x, Du_h)) = f \text{ on } \Omega, \\ u_h \in W_0^{1,p}(\Omega), \end{cases}$$

is that

$$\begin{aligned} u_h &\rightharpoonup u \text{ weakly in } W_0^{1,p}(\Omega), \\ a_h(x, Du_h) &\rightharpoonup b(Du) \text{ weakly in } [L^q(\Omega)]^n, \end{aligned} \tag{12}$$

where u is the solution of the homogenized equation

$$\begin{cases} -\operatorname{div}(b(Du)) = f \text{ on } \Omega, \\ u \in W_0^{1,p}(\Omega). \end{cases}$$

The homogenized operator $b : \mathbf{R}^n \rightarrow \mathbf{R}^n$ is defined by

$$b(\xi) = \frac{1}{|Y|} \int_Y a(x, \xi + Dw^\xi(x)) dx, \tag{13}$$

where w^ξ is the solution of the local problem on Y

$$\begin{cases} \int_Y \langle a(x, \xi + Dw^\xi), D\phi \rangle dx = 0 \text{ for every } \phi \in W_{\square}^{1,p}(Y), \\ w^\xi \in W_{\square}^{1,p}(Y). \end{cases} \tag{14}$$

For a proof of this result see e.g. [22] or [44]. A common technique to prove this theorem is *Tartar's method of oscillating test functions* [103] related to the notion of compensated compactness mentioned above. Another technique is the *two-scale convergence method*, see Remark 9 in section D.2. This homogenization result is extended to cover a_h of the form $a(x, x/\varepsilon_h, \xi)$ in paper A1 of this thesis. This type of equations can be used to model nonlinear physical phenomena in composite materials, where the typical cell of periodicity may vary with the position in the overall composite.

Remark 4 A common assumption is that $a(x, \xi)$ satisfies the conditions

$$\begin{aligned} |a(x, \xi_1) - a(x, \xi_2)| &\leq c_1 \lambda(x) (1 + |\xi_1| + |\xi_2|)^{p-1-\alpha} |\xi_1 - \xi_2|^\alpha, \\ \langle a(x, \xi_1) - a(x, \xi_2), \xi_1 - \xi_2 \rangle &\geq c_2 \lambda(x) (1 + |\xi_1| + |\xi_2|)^{p-\beta} |\xi_1 - \xi_2|^\beta, \end{aligned}$$

for $c_1, c_2 > 0$, $0 \leq \alpha \leq \min(1, p-1)$ and $\max(p, 2) \leq \beta < \infty$. These conditions are for instance satisfied by the p -Poisson operator

$$a(x, \xi) = \lambda(x) |\xi|^{p-2} \xi,$$

where the constants c_1 and c_2 can be chosen as $c_1 = \max(p-1, (2\sqrt{2})^{2-p})$, $c_2 = \min(p-1, (2\sqrt{2})^{2-p})$.

A.3 Correctors

Returning to (12), we see that $u_h - u$ converges to 0 weakly in $W_0^{1,p}(\Omega)$. The Rellich imbedding theorem then implies that $u_h - u$ converges to 0 strongly in $L^p(\Omega)$. Unfortunately we do not have strong convergence of $Du_h - Du$ to 0 in $[L^p(\Omega)]^n$ in general. However, it is possible to express Du_h in terms of Du up to a rest which strongly converges to 0 in $[L^p(\Omega)]^n$. Results of this type are called *corrector* results. Indeed, let $P_h : \Omega \times \mathbf{R}^n \rightarrow \mathbf{R}^n$ be defined by

$$P_h(x, \xi) = \xi + Dw_\xi\left(\frac{x}{\varepsilon_h}\right),$$

where w_ξ is the periodic extension of the solution to the local problem (14). Now we define the step function $M_h\phi : \mathbf{R}^n \rightarrow \mathbf{R}^n$ by

$$(M_h\phi)(x) = \sum_{i \in I_h} \chi_{Y_h^i}(x) \frac{1}{|Y_h^i|} \int_{Y_h^i} \phi(y) dy$$

for every $\phi \in [L^p(\Omega)]^n$, where $Y_h^i = \varepsilon_h(i + Y)$ and $I_h = \{i \in \mathbf{Z}^n : Y_h^i \subset \Omega\}$. Then

$$Du_h - P_h(\cdot, M_h Du) \rightarrow 0 \text{ in } [L^p(\Omega)]^n.$$

This result was proved in [30] and is extended to cover a of the form $a(x, x/\varepsilon_h, \xi)$ in paper A1 of this thesis. In particular, both convergence results and corrector results are obtained.

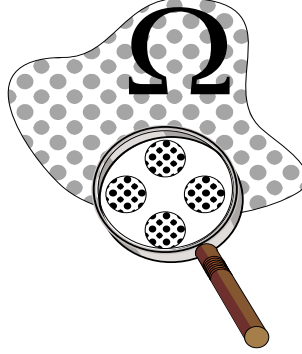


Figure 3: A composite with two local scales.

A.4 Reiterated homogenization

One natural extension of the homogenization problem described above is to include several microscopic scales, see Figure 3. Let us consider the limit processes corresponding to the problems

$$-\operatorname{div} \left(a \left(\frac{x}{\varepsilon_h}, \dots, \frac{x}{\varepsilon_h^m}, Du_h \right) \right) = f \text{ on } \Omega,$$

and

$$E_h = \min_{u \in W_0^{1,p}(\Omega)} \left\{ \int_{\Omega} g \left(\frac{x}{\varepsilon_h}, \dots, \frac{x}{\varepsilon_h^m}, Du_h(x) \right) dx - \int_{\Omega} fu dx \right\}.$$

Homogenization problems with more than one microscopic scale are known as *reiterated homogenization* problems.

Let us consider the reiterated homogenization problem with m different local scales where we have the following sequence of linear problems

$$\begin{cases} -\operatorname{div} \left(a \left(\frac{x}{\varepsilon_h}, \dots, \frac{x}{\varepsilon_h^m}, Du_h \right) \right) = f \text{ on } \Omega, \\ u_h \in W_0^{1,p}(\Omega). \end{cases}$$

It was shown in [67] and [68] that

$$\begin{aligned} u_h &\rightharpoonup u \text{ weakly in } W_0^{1,p}(\Omega), \\ a \left(\frac{x}{\varepsilon_h}, \dots, \frac{x}{\varepsilon_h^m}, Du_h \right) &\rightharpoonup b(Du) \text{ weakly in } [L^q(\Omega)]^n, \end{aligned}$$

where u is the solution of the homogenized equation

$$\begin{cases} -\operatorname{div}(b(Du)) = f \text{ on } \Omega, \\ u \in W_0^{1,p}(\Omega). \end{cases}$$

The homogenized operator $b : \mathbf{R}^n \rightarrow \mathbf{R}^n$ is defined by

$$b(\xi) = \psi_m(\xi),$$

where ψ_m is computed according to the following iterative procedure:

$$\begin{aligned} \psi_0(y_1, \dots, y_m, \xi) &= a(y_1, \dots, y_m, \xi), \\ \psi_i(y_1, \dots, y_{m-i}, \xi) &= \frac{1}{|Y|} \int_Y \psi_{i-1}(y_1, \dots, y_{m-i+1}, \xi + Dw_i^\xi(y_{m-i+1})) dy_{m-i+1}. \end{aligned}$$

Here w_i^ξ is the solution of the following local problem on Y : Find w_i^ξ such that

$$\begin{cases} \int_Y \left\langle \psi_{i-1}(y_1, \dots, y_{m-i+1}, \xi + Dw_i^\xi(y_{m-i+1})), D\phi \right\rangle dy_{m-i+1} = 0, \\ w_i^\xi \in W_{\square}^{1,p}(Y), \end{cases}$$

for every $\phi \in W_{\square}^{1,p}(Y)$.

The homogenization problems connected to monotone operators of the type described above have been generalized in many directions. Extensions have been made to for example vector valued problems, almost periodic problems and problems with stiff and soft inclusions. Paper A2 in this thesis deals with the reiterated homogenization of nonlinear degenerated monotone operators. Reiterated homogenization techniques are also used in papers D1, D2 and E2 to compute the elastic properties of woven composites. Moreover, we also consider computational aspects of reiterated homogenization in paper C1.

A.5 Some special cases with closed form expressions for the homogenized operator b

As we saw in section A.2, the homogenized operator (13) depends on the solution of a cell problem (14). There are however some special cases when

we can get closed form expressions for b . We give a few of them below. Some of these special cases are intimately connected to the concept of bounds, see Section B. For simplicity, we concentrate on linear heat conductivity in the plane, that is, we consider (3) with $p = 2$ and $a(y, \xi) = \lambda(y) \xi$ where $\xi \in \mathbf{R}^2$.

A.5.1 The Hashin structure

In the early sixties, Hashin and Shtrikman [49] investigated bounds for the effective properties of isotropic three dimensional mixtures with arbitrary phase geometry. Hashin and Rosen [48] later showed that these bounds were optimal, meaning that one can find phase geometries such that they are obtained. The so called Hashin structure, which consists of coated spheres, is one such case. We study a three-phase composite consisting of three isotropic materials, let us call them material 1, 2 and 3, with conductivity

$$\lambda(x) I = [\alpha_1 \chi_{\Omega_1}(x) + \alpha_2 \chi_{\Omega_2}(x) + \alpha_3 \chi_{\Omega_3}(x)] I$$

where χ_{Ω_i} is the characteristic function for the set Ω_i and I the unit matrix. Let the unit cell geometry be described by

$$\begin{aligned} \Omega_1 &= \{x : |x| \leq r_1\}, \quad \Omega_2 = \left\{x : r_1 \leq |x| \leq r_2 < \frac{1}{2}\right\}, \\ \Omega_3 &= \left\{x : |x_i| < \frac{1}{2} \wedge |x| \geq r_2, \quad i = 1, 2\right\}. \end{aligned}$$

In order to compute the homogenized coefficients (13), we need to solve the cell problem (14)

$$-\operatorname{div}(\lambda(x) D\phi^\xi(x)) = 0 \text{ on } Y, \quad (15)$$

where $\phi^\xi(x) = \langle \xi, x \rangle + w^\xi(x)$ and $w^\xi(x)$ is Y -periodic. In the case $\xi = e_1 = [1 \ 0]^T$, we look for a solution of the type

$$\phi^{e_1}(x) = \begin{cases} C_1 x_1, & x \in \Omega_1, \\ x_1 \left(C_2 + \frac{K_2}{|x|^2}\right), & x \in \Omega_2, \\ x_1, & x \in \Omega_3, \end{cases} \quad (16)$$

see Figure 4. It is easily seen that (16) satisfies (15) on Y . By physical reasons, the solution $\phi^\xi(x)$ as well as the flux $\lambda(x) \frac{\partial \phi^\xi}{\partial n}$ must be continuous over the boundaries $\Omega_1 \cap \Omega_2$ and $\Omega_2 \cap \Omega_3$. This gives four equations to solve for the

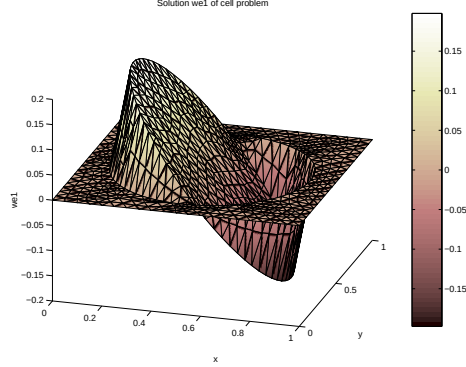


Figure 4: The solution w^{e_1} of the cell problem when α_3 is chosen according to (17).

three unknowns C_1 , C_2 and K_2 . In order to get a consistent solution, we get that α_3 must be

$$\alpha_3 = \alpha_2 \left(C_2 - \frac{K_2}{r_2^2} \right) = \frac{\alpha_2 \left(1 + \frac{\alpha_1}{\alpha_2} + m_1 \left(\frac{\alpha_1}{\alpha_2} - 1 \right) \right)}{1 + \frac{\alpha_1}{\alpha_2} - m_1 \left(\frac{\alpha_1}{\alpha_2} - 1 \right)}, \quad (17)$$

where $m_1 = \frac{r_1^2}{r_2^2}$, the volume fraction of material 1 in material 2. Since we now know the solution $w^{e_1}(y) = \phi^{e_1}(y) - \langle e_1, y \rangle$ of the cell problem, we can compute the homogenized coefficients

$$b(e_1) = \int_Y \lambda(x) (e_1 + Dw^{e_1}) dx = [\alpha_3 \ 0]^T$$

and similarly

$$b(e_2) = \int_Y \lambda(x) (e_2 + Dw^{e_1}) dx = [0 \ \alpha_3]^T.$$

This means that we can put the coated cylinder consisting of material 1 coated by material 2 into the homogeneous isotropic material 3 without changing the effective properties, see Figure 4. By filling the whole cell with such homothetically coated cylinders, we get an isotropic two-phase composite with conductivity a_3 , see Figure 5. We also mention the fact that

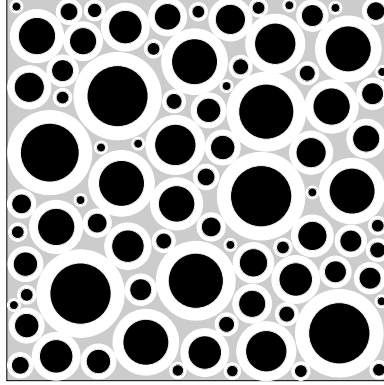


Figure 5: The Hashin structure.

if $\alpha_1 \geq \alpha_2$ then α_3 above is the lower Hashin-Shtrikman bound. By letting material 1 and 2 change places and solving a similar problem as above, we get the upper Hashin-Shtrikman bound. We refer to Section B for more on this topic.

A.5.2 The Mortola-Steffe structure

We define our unit cell $Y = (0, 1)^2$ and divide it into four equal parts

$$\begin{aligned} Y_1 &= \left(0, \frac{1}{2}\right) \times \left(\frac{1}{2}, 1\right), \quad Y_2 = \left(\frac{1}{2}, 1\right) \times \left(\frac{1}{2}, 1\right), \\ Y_3 &= \left(0, \frac{1}{2}\right) \times \left(0, \frac{1}{2}\right), \quad Y_4 = \left(\frac{1}{2}, 1\right) \times \left(0, \frac{1}{2}\right). \end{aligned}$$

We study a four-phase composite consisting of four isotropic materials, let us call them material 1, 2, 3 and 4, with conductivity

$$\lambda(x) I = [\alpha \chi_{Y_1}(x) + \beta \chi_{Y_2}(x) + \gamma \chi_{Y_3}(x) + \delta \chi_{Y_4}(x)] I,$$

where χ_{Y_i} is the characteristic function for the set Y_i and I the unit matrix, see Figure 6.

In 1985, Mortola and Steffe [79] conjectured that the homogenized conductivity coefficients of this structure are

$$(\bar{\lambda}_{ij}) = \begin{pmatrix} \bar{\lambda}_{11} & 0 \\ 0 & \bar{\lambda}_{22} \end{pmatrix},$$

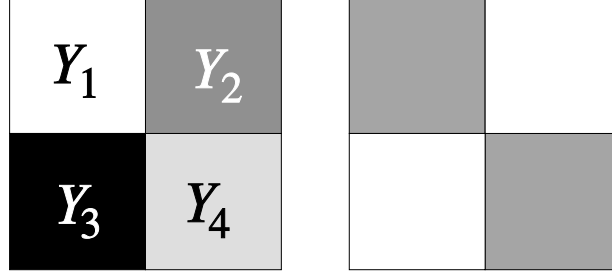


Figure 6: Unit cells for the Mortola-Steffe structure on the left and the checkerboard structure on the right.

where

$$\begin{aligned}\bar{\lambda}_{11} &= \sqrt{\frac{\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta}{\alpha + \beta + \gamma + \delta} \frac{(\alpha + \gamma)(\beta + \delta)}{(\alpha + \beta)(\gamma + \delta)}}, \\ \bar{\lambda}_{22} &= \sqrt{\frac{\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta}{\alpha + \beta + \gamma + \delta} \frac{(\alpha + \beta)(\gamma + \delta)}{(\alpha + \gamma)(\beta + \delta)}}.\end{aligned}$$

This conjecture was proven to be true by Craster and Obnosov [28] and independently by Milton [76] in 2000.

If we let $\delta = \alpha$ and $\gamma = \beta$, we get the so called checkerboard structure, see Figure 6. We immediately see that the homogenized conductivity coefficients for the checkerboard structure are

$$\bar{\lambda}_{11} = \bar{\lambda}_{22} = \sqrt{\alpha\beta},$$

the geometric mean. This was proved already in 1970 by Dykhne [39].

A.5.3 The case when λ is separable

Let $Y = (0, 1)^n$ and consider the case when $\lambda(y)$ is of the form

$$\lambda(x) = \lambda_1(x_1) \lambda_2(x_2) \cdots \lambda_n(x_n),$$

where $\lambda(x_i)$ are $(0, 1)$ -periodic and

$$0 < \alpha_i \leq \lambda_i(x_i) \leq \beta_i < \infty, \text{ for a.e. } y_i \in \mathbf{R}, i = 1, \dots, n. \quad (18)$$

We want to solve the corresponding cell problem

$$-\operatorname{div}(\lambda(x)(\xi + Dw^\xi)) = 0 \text{ for } y \in Y.$$

By a standard result in the existence theory of partial differential equations, there exists a unique solution of this problem. Hence if we can find one solution then it must be the only solution. Let us look for a separable classic solution of the form

$$w^\xi(x) = w_1^\xi(x_1) w_2^\xi(x_2) \cdots w_n^\xi(x_n).$$

This implies that we must solve n one-dimensional problems

$$\frac{d}{dx_i} \left(\lambda_i(x_i) \left(\xi_i + w_1^\xi(x_1) \cdots w_{i-1}^\xi(x_{i-1}) \frac{dw_i^\xi}{dx_i} w_{i+1}^\xi(x_{i+1}) \cdots w_n^\xi(x_n) \right) \right) = 0,$$

where $\xi_i = \langle \xi, e_i \rangle$. By inserting the solution $w^\xi(x)$ in (13) we see that the i :th component of the homogenized operator in this case is given by

$$b(\xi)_i = \xi_i \langle \lambda_i^{-1} \rangle^{-1} \langle \lambda_1 \cdots \lambda_{i-1} \lambda_{i+1} \cdots \lambda_n \rangle.$$

Another way to obtain this result is by looking at some special bounds for the p -Poisson equation that coincides for this case, see Section B.1.

One interesting special case is when $\lambda(x)$ only depends on one variable, for instance when $\lambda(x_1, x_2) = \lambda_1(x_1)$. Then

$$b(e_1)_1 = \langle \lambda_1^{-1} \rangle^{-1} = \mathcal{H}, \quad b(e_2)_2 = \langle \lambda_1 \rangle = \mathcal{A},$$

the harmonic and arithmetic means, respectively. It can be shown that this case is the most extreme in that meaning that any other geometric structure will have effective conductivities $b(e_i)_i$ that are bounded by $\mathcal{H}$ and $\mathcal{A}$, i.e.

$$\mathcal{H} \leq b(e_i)_i \leq \mathcal{A}.$$

We refer to Section B for more on this topic.

A.6 Homogenization of degenerated equations

Assume that we have a material where the heat conductivity is allowed to go to zero or infinity in a prescribed way. Will then the effective conductivity of

the material be zero, infinity or something else? Questions like these occur when dealing with homogenization of *degenerated* equations. In article [33] the homogenization problem of degenerated elliptic equations was treated. The authors consider the problems

$$\begin{cases} -\operatorname{div}(a(x/\varepsilon_h, Du_h)) = f \text{ on } \Omega, \\ u_h \in W_0^{1,p}(\Omega, \lambda_{\varepsilon_h}), \end{cases}$$

where $a(y, \xi)$ satisfies suitable structure conditions and the weight $\lambda_{\varepsilon_h}(x) = \lambda(x/\varepsilon_h)$ is a positive function belonging to the Muckenhoupt class A_p introduced by Muckenhoupt [80] in connection with the study of the maximal function operator between weighted L^p -spaces. The main result is that

$$\begin{aligned} u_h &\rightharpoonup u \text{ weakly in } W_0^{1,1}(\Omega), \\ a(x/\varepsilon_h, Du_h) &\rightharpoonup b(Du_0) \text{ weakly in } [L^1(\Omega)]^n, \end{aligned}$$

where u_0 is the solution of the homogenized equation

$$\begin{cases} -\operatorname{div}(b(Du_0)) = f \text{ on } \Omega, \\ u_0 \in W_0^{1,p}(\Omega). \end{cases}$$

The homogenized operator $b : \mathbf{R}^n \rightarrow \mathbf{R}^n$ is defined by

$$b(\xi) = \frac{1}{|Y|} \int_Y a(y, \xi + Dw^\xi(y)) dy,$$

where w^ξ is the solution of the local problem on Y

$$\begin{cases} \int_Y \langle a(y, \xi + Dw^\xi), D\phi \rangle dy = 0 \text{ for every } \phi \in W_{\square}^{1,p}(Y, \lambda), \\ w^\xi \in W_{\square}^{1,p}(Y, \lambda). \end{cases}$$

This result is generalized to reiterated homogenization in paper A2 of this thesis, where we study the limit behavior as the natural number $h \rightarrow \infty$ for the sequence of solutions $(u_h)_{h \in \mathbf{N}}$ of the Dirichlet boundary value problems

$$\begin{cases} -\operatorname{div}(a(hx, h^2x, Du_h)) = f \text{ on } \Omega, \\ u_h \in W_0^{1,p}(\Omega, \lambda_h), \end{cases} \quad (19)$$

where $\lambda_h(x)$ is a positive weight belonging to the Muckenhoupt class A_p .

B Bounds for the homogenized operator

In this section we will present the basic ideas and results in the field of finding upper and lower bounds on the homogenized operator. The proofs of the results we present are collected in the book [54], see also the articles [21], [61], [75] and [83]. We will consider the linear problem, i.e., when a is of the form $a_h(x, \xi) = A\left(\frac{x}{\varepsilon_h}\right)\xi$, A is a Y -periodic matrix. Moreover, we assume that $A(x) = \alpha I$, where $\alpha : \mathbf{R}^n \rightarrow \mathbf{R}$ satisfies that $0 < \beta_1 \leq \alpha(x) \leq \beta_2 < \infty$. By a periodic two-phase composite we mean that α only takes two values, i.e., α can be written as

$$\alpha(x) = \alpha_1 \chi_{\Omega_1}(x) + \alpha_2 \chi_{\Omega_2}(x), \quad \alpha_1 > \alpha_2,$$

where Ω_i is the periodic extension of $\{x \in Y : \alpha(x) = \alpha_i\}$ and χ_{Ω_i} is the characteristic function of the set Ω_i . Moreover, let ν_i be the volume fraction of the material occupying Ω_i .

The *Reuss-Voigt bounds* (or *Wiener bounds*) then states that the homogenized matrix b satisfies the estimates

$$\mathcal{H}I \leq b \leq \mathcal{A}I, \tag{20}$$

where I is the identity matrix, $\mathcal{H}$ is the harmonic mean of α over a cell of periodicity and $\mathcal{A}$ is the arithmetic mean value of α over a cell of periodicity. Note that if A and B are two matrices, the notation $A \leq B$ means that $B - A$ has positive eigenvalues, which implies that (20) can be written as

$$\mathcal{H} \leq \lambda_i \leq \mathcal{A},$$

where λ_i are the eigenvalues of the homogenized matrix b . Note that for the special case of a two-phase composite, the Reuss-Voigt bounds implies that

$$\frac{1}{\frac{\nu_1}{\alpha_1} + \frac{\nu_2}{\alpha_2}} \leq \lambda_i \leq \nu_1 \alpha_1 + \nu_2 \alpha_2. \tag{21}$$

In Figure 7a we have plotted the Reuss-Voigt bounds for an example in $\mathbf{R}^2$.

An improvement of the Reuss-Voigt bounds are the *trace bounds* (or *Hashin-Shtrikman bounds*, see [49]), which state that

$$L \leq \frac{\lambda_1 + \dots + \lambda_n}{n} \leq U,$$

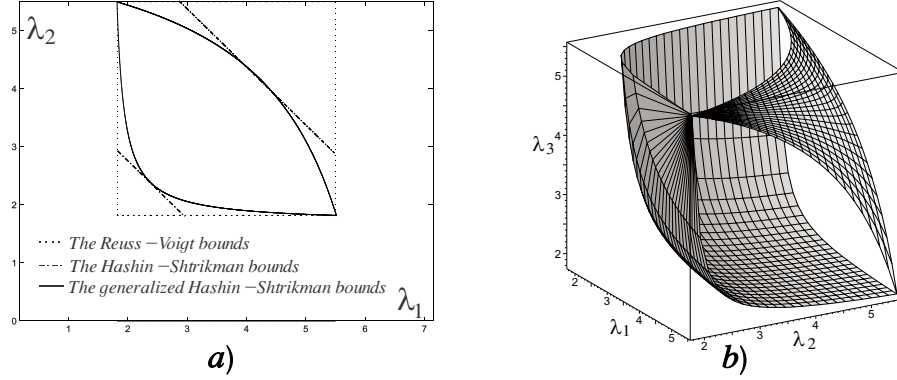


Figure 7: a) The G-closure in $\mathbf{R}^2$. b) The G-closure in $\mathbf{R}^3$.

where

$$L = \mathcal{A} - \frac{\langle (\alpha - \mathcal{A})^2 \rangle}{n \inf \alpha + \langle (\alpha - \mathcal{A})^2 \rangle (\mathcal{A} - \inf \alpha)^{-1}},$$

$$U = \mathcal{A} - \frac{\langle (\alpha - \mathcal{A})^2 \rangle}{n \sup \alpha - \langle (\alpha - \mathcal{A})^2 \rangle (\sup \alpha - \mathcal{A})^{-1}}.$$

Here $\langle \cdot \rangle$ denotes the arithmetic mean. In the special case of a two-phase material, L and U are given by

$$L = \alpha_2 + \frac{\nu_1}{\frac{1}{\alpha_1 - \alpha_2} + \frac{\nu_2}{n\alpha_2}}, \quad (22)$$

$$U = \alpha_1 + \frac{\nu_2}{\frac{1}{\alpha_2 - \alpha_1} + \frac{\nu_1}{n\alpha_1}},$$

respectively. In Figure 7a we have also plotted the Hashin-Shtrikman bounds for an example in $\mathbf{R}^2$.

Remark 5 *The Hashin-Shtrikman bounds are known in physics, mechanics or chemistry under various other names such as Maxwell-Garnett, Clausius-Mossotti or Lorentz-Lorenz bounds, see e.g. [95].*

It is possible to obtain even better estimates, namely the so called *generalized Hashin-Shtrikman bounds*:

$$\text{tr} (b - \inf \alpha I)^{-1} \leq \frac{\left\langle \frac{1}{\alpha + (n-1) \inf \alpha} \right\rangle}{n^{-1} - \inf \alpha \left\langle \frac{1}{\alpha + (n-1) \inf \alpha} \right\rangle},$$

$$\mathrm{tr}(\sup \alpha I - b)^{-1} \leq \frac{\left\langle \frac{1}{\alpha + (n-1) \sup \alpha} \right\rangle}{\sup \alpha \left\langle \frac{1}{\alpha + (n-1) \sup \alpha} \right\rangle - n^{-1}},$$

where $\mathrm{tr}(C)$ stands for the trace of the matrix C . In particular, for the special case of a two-phase composite we obtain that

$$\begin{aligned} \mathrm{tr}(b - \alpha_1 I)^{-1} &= \sum_{i=1}^n \frac{1}{\lambda_i - \alpha_2} \leq \frac{n}{\nu_1(\alpha_1 - \alpha_2)} + \frac{\nu_2}{\alpha_2 \nu_1}, \\ \mathrm{tr}(\alpha_2 I - b)^{-1} &= \sum_{i=1}^n \frac{1}{\alpha_1 - \lambda_i} \leq \frac{n}{\nu_2(\alpha_1 - \alpha_2)} - \frac{\nu_1}{\alpha_1 \nu_2}. \end{aligned} \quad (23)$$

Figure 7a also contains a plot of the generalized Hashin-Shtrikman bounds for an example in $\mathbf{R}^2$.

As seen in Figure 7 the classical and generalized Hashin-Shtrikman bounds coincide in the isotropic case, i.e., when $\lambda_1 = \lambda_2$. The bounds presented above holds true for even more general situations, i.e., without any assumption about periodicity. For example, the bounds are valid for reiterated homogenization problems, see [54].

The research on finding methods for obtaining bounds has attracted the attention of many authors and is an intense research area. Various generalizations relative the situation described above has been and are still studied. Some examples are problems with N phases and anisotropic behavior of the constituents. Moreover, the methods has been extended in such a way that it is possible to study other types of equations as well, for example the equations describing linear elasticity. Another challenging and much less developed area is to find bounds corresponding to nonlinear problems, see e.g. [69], [92], [99] and [100].

A natural question to ask is whether the bounds above are *optimal* or not, that is, whether one can find a function α such that the bounds are attained. Another related question is whether one can find a function α such that any possible choice of $(\lambda_1, \dots, \lambda_n)$ can be obtained. Questions like these are called *G-closure* problems. By G-closure we mean the set of all possible values of the homogenized matrix b that correspond to arbitrary microstructures with fixed volume fraction of materials. In $\mathbf{R}^2$, the G-closure is the set of points bounded by (23) whereas in $\mathbf{R}^3$ they must be completed by the Reuss-Voigt bounds (21), see Figure 7.

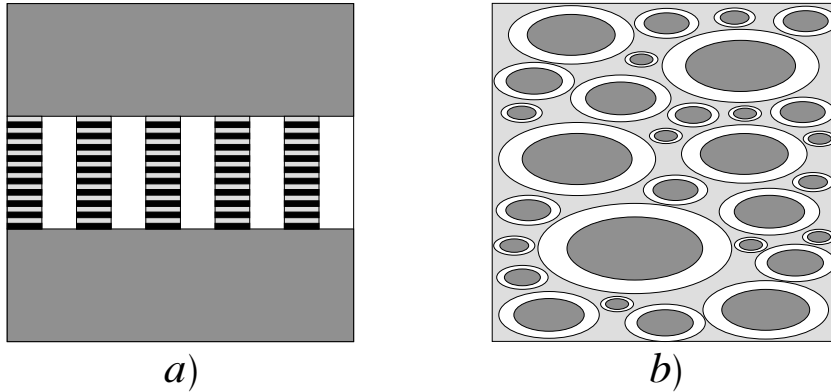


Figure 8: *a)* A rank-3 laminate. *b)* The ellipsoidal Hashin structure.

It was shown by Lurie and Cherkaev [21] and independently Tartar [83], [104], around 1985 that G-closure problems are intimately connected to structures made by successive lamination (so called *rank- m laminates*, see Figure 8) for the linear conductivity problem with isotropic phases. In $\mathbf{R}^2$ for instance, it is possible to obtain all possible pairs of eigenvalues on the lower and upper bounds in (23) by using rank-2 laminates. These bounds can also be obtained by the ellipsoidal *Hashin structure*, consisting of space filling coated ellipsoids, see Figure 8. Moreover, it is possible to obtain every pair of eigenvalues (λ_1, λ_2) in the set of points bounded by (23) by a rank-3 laminate.

There are still several open G-closure problems, for example the case when we have $N > 2$ isotropic phases and the case with two phases where at least one of them is anisotropic. In this connection, we also mention that results from the theory of homogenization and optimal bounds have proven to be useful in the field of structural optimization, see e.g. [3], [9], [19], [20], [55], [56], [58] and [77].

B.1 Bounds for the p -Poisson equation

The bounds above hold for any composites with given volume fractions and conductivities. Needless to say, you can construct countless different composites with highly differing effective properties for a fixed set of conductivities and volume fractions. It is not hard to imagine that we would get tighter

bounds by taking the geometric properties of the phases into account. In particular we mention the bounds obtained by Lukkassen et al. [71] for the p -Poisson equation

$$-\operatorname{div} \left(\alpha \left(\frac{x}{\varepsilon} \right) |Du|^{p-2} Du \right) = f \text{ on } \Omega \subset \mathbf{R}^n$$

where $\alpha(x)$ is 1-periodic. In the case when $\{e_i\}$, $i = 1, \dots, n$, is the canonical basis in $\mathbf{R}^n$, these bounds state that the *effective energy* $\langle b_{\text{hom}} e_i, e_i \rangle$ satisfies the bounds

$$\lambda_i^- \leq \langle b_{\text{hom}} e_i, e_i \rangle \leq \lambda_i^+,$$

where

$$\begin{aligned} \lambda_i^- &= \int_0^1 \cdots \int_0^1 \left(\int_0^1 \alpha(x)^{1/(1-p)} dx_i \right)^{1-p} dx_1 \cdots dx_{i-1} dx_{i+1} \cdots dx_n, \\ \lambda_i^+ &= \left(\int_0^1 \left(\int_0^1 \cdots \int_0^1 \alpha(x) dx_1 \cdots dx_{i-1} dx_{i+1} \cdots dx_n \right)^{1/(1-p)} dx_i \right)^{1-p}. \end{aligned}$$

Together with the Hashin-Shtrikman bounds they can provide enough information to get a good approximation of the homogenized properties. These bounds are generalized in paper B1 of this thesis to cover the degenerated p -Poisson equation.

Remark 6 *It is easy to see that these bounds coincide when $\alpha(x)$ is of the type*

$$\alpha(x) = f_1(x_i) g(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n), \quad \alpha(x) \text{ is } Y\text{-periodic}.$$

In this case the bounds are equal to the effective energy.

B.2 Equations of Halpin-Tsai type

Let us consider the expression

$$\bar{a}(\xi, \nu_1) = \frac{\alpha_2 \left(\xi + \frac{\alpha_1}{\alpha_2} + \nu_1 \left(\frac{\alpha_1}{\alpha_2} - 1 \right) \xi \right)}{\xi + \frac{\alpha_1}{\alpha_2} - \nu_1 \left(\frac{\alpha_1}{\alpha_2} - 1 \right)},$$

for $0 \leq \xi < \infty$. This expression can be rewritten as a Halpin-Tsai type equation, see e.g. [1], [47], by

$$\frac{\bar{a}(\xi, \nu_1)}{\alpha_2} = \frac{1 + \xi\eta\nu_1}{1 - \eta\nu_1},$$

where

$$\eta = \frac{\frac{\alpha_1}{\alpha_2} - 1}{\frac{\alpha_1}{\alpha_2} + \xi}$$

and $\xi \geq 0$ is the fitting parameter. The function $\bar{a}(\xi, \nu_1)$ has the following interesting properties:

1. $\bar{a}(\xi, \nu_1)$ is an increasing function in ν_1 for every fixed ξ and

$$\begin{cases} \bar{a}(\xi, 0) = \alpha_2, \\ \bar{a}(\xi, 1) = \alpha_1. \end{cases}$$

2. $\bar{a}(\xi, \nu_1)$ is an increasing function in ξ for every fixed ν_1 .
3. The Reuss-Voigt bounds are attained at the extreme points of ξ , that is,

$$\begin{aligned} \bar{a}(0, \nu_1) &= \mathcal{H} = \frac{1}{\frac{\nu_1}{\alpha_1} + \frac{1-\nu_1}{\alpha_2}}, \\ \lim_{\xi \rightarrow \infty} \bar{a}(\xi, \nu_1) &= \mathcal{A} = \nu_1\alpha_1 + (1 - \nu_1)\alpha_2. \end{aligned}$$

4. The Hashin-Shtrikman bounds are attained at $\xi = n - 1$ and $\xi = (n - 1)\frac{\alpha_1}{\alpha_2}$, that is,

$$\begin{aligned} \bar{a}(n - 1, \nu_1) &= \alpha_2 + \frac{\nu_1}{\frac{1}{\alpha_1 - \alpha_2} + \frac{1 - \nu_1}{n\alpha_2}}, \\ \bar{a}\left((n - 1)\frac{\alpha_1}{\alpha_2}, \nu_1\right) &= \alpha_1 + \frac{1 - \nu_1}{\frac{1}{\alpha_2 - \alpha_1} + \frac{\nu_1}{n\alpha_1}}, \end{aligned}$$

compare with equation (17).

In paper E1 of this thesis, the function $\bar{a}(\xi, \nu_1)$ is further investigated as a tool for determining the effective conductivity of some particular problems.

C Computational aspects of homogenization

We will in this section present some of the ideas that are used in the computation of homogenized properties. A simple procedure to compute the homogenized coefficients (13) is by making the substitution $u^{e_k}(x) = \langle e_k, x \rangle + w^{e_k}(x)$ in (14) and instead solve the problem

$$\int_Y \langle a(x, Du^{e_k}), D\phi \rangle dx = 0$$

with periodic boundary conditions on all faces except for those having normal vectors parallel to e_k , where the difference between the boundary values on opposite faces should be equal to the length l_k of the cell in the direction of e_k . It is now easy to see that the homogenized coefficients (13) is equal to the mean flow $\langle a(x, Du^{e_k}) \rangle$ over one cell.

In the case when the periodic cell is symmetric, the boundary conditions are even simpler. We put Dirichlet boundary conditions $u^{e_k} = 0$ and $u^{e_k} = l_k$ on the faces having normal vectors parallel to e_k and Neumann boundary conditions with zero flow over the boundaries on the remaining faces.

The corresponding procedure when we study the problem of determining the effective elastic properties of a composite is described in [72], where it is shown that the effective conductivity tensor is given by the average stress over one cell subjected to unit displacements on the boundaries.

Remark 7 *Other possible techniques for computation of effective properties include wavelet (or multiresolution) methods (see e.g. [13], [37] and [41]) and Fourier methods such as Bloch waves (see e.g. [5], [10], [27]) and series solutions (see e.g. [38], [65] and [78]) as well as complex variable methods (see e.g. [28], [76] and [85]).*

C.1 Numerical bounds of the conductivity

Let us consider the linear stationary heat conductivity problem, that is, when a is of the form $a_h(x, \xi) = A\left(\frac{x}{\varepsilon_h}\right)\xi$, A is a Y -periodic matrix. Moreover, we assume that $A(x) = \alpha I$, where $\alpha : \mathbf{R}^n \rightarrow \mathbf{R}$ satisfies that $0 < \beta_1 \leq \alpha(x) \leq \beta_2 < \infty$. Suppose that the matrix $A(x) = \alpha(x)I$ possesses the *condition of cubic symmetry*, that is

$$A(x) = \sigma^{-1} A(\sigma x) \sigma$$

where σ is the rotation by $\pi/2$ in the plane of the coordinates $x_i, x_j, i \neq j, i, j = 1, \dots, n$. Furthermore, assume that $n = 2$. In this case the homogenized matrix B is given by

$$B = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix},$$

where b_i are found by either of the following minimization problems (cf. (4)):

$$b_i = \frac{1}{|Y|} \min_{v \in W_{\square}^{1,2}(Y)} \left\{ \int_Y \alpha(x) |e_i + Dv|^2 dx \right\},$$

$$\frac{1}{b_i} = \frac{1}{|Y|} \min_{v \in W_{\square}^{1,2}(Y)} \left\{ \int_Y \alpha(x) |e_j + Dv|^2 dx \right\},$$

and e_j is orthogonal to e_i , see e.g. [54].

This implies that we can obtain upper and lower numerical estimates of b_i . Let V be any Finite-Element subspace of $W_{\square}^{1,2}(Y)$ and denote by b_i^+ and b_i^- the values defined by

$$b_i^+ = \frac{1}{|Y|} \min_{v \in V} \left\{ \int_Y \alpha(x) |e_i + Dv|^2 dx \right\},$$

$$\frac{1}{b_i^-} = \frac{1}{|Y|} \min_{v \in V} \left\{ \int_Y \alpha(x) |e_j + Dv|^2 dx \right\}.$$

These values then trivially satisfies

$$b_i^- \leq b_i \leq b_i^+,$$

that is, we get lower and upper numerical bounds of the homogenized coefficients.

C.2 Iterated honeycombs

In paper C1 of this thesis, we consider iterated *honeycombs*. By honeycombs one usually means two-dimensional two-component periodic structures where the interior part consists of polygons (typically hexagons or rectangles). Iterated honeycombs are, roughly speaking, “honeycombs in honeycombs”, see Figure 9. Let us consider the iterated square honeycomb structure defined by the characteristic function $\chi_{m,k,h}$ on $\mathbf{R}^2$, where k, h, m are positive integers and

$$\chi_{m,k,h}(x) = \chi^{(m)}(hx_1) \chi^{(m)}(hx_2).$$

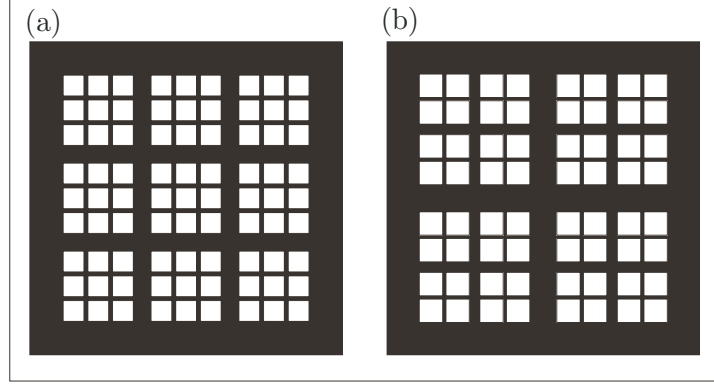


Figure 9: Two iterated honeycombs with $\nu = 0.36$. a) $m = 3$, $k = 3$. b) $m = 4$, $k = 2$.

Here $\chi^{(m)}$ is found iteratively by letting $\chi^{(i+1)}$ be the $[0, 1]$ -periodic extension of the function

$$\chi(\cdot) \chi^{(i)} \left(k \cdot \nu^{1/(2m)} - (1 - \nu^{1/(2m)}) / 2 \right), \quad \chi^{(0)} = 1,$$

where χ is the characteristic function of the interval $[0, \nu^{1/(2m)}]$ and $0 \leq \nu \leq 1$ is the volume fraction of the interior part. We then define the conductivity $\lambda_{m,k,h}(x)$ by

$$\lambda_{m,k,h}(x) = \lambda_{\text{in}} \chi_{m,k,h}(x) + \lambda_{\text{con}} (1 - \chi_{m,k,h}(x)),$$

where λ_{in} and λ_{con} are the conductivities of the interior and the connected part of the iterated square honeycomb-structure, respectively.

In order to be able to speak about effective conductivity for the above structure we consider the following two limit-cases:

1. fix m and k and let h approach infinity. The corresponding effective conductivity is in this case denoted $\lambda_{m,k}$.
2. fix m and let $h = k$ approach infinity. The corresponding effective conductivity is in this case denoted $\lambda_{m,\infty}$.

For iterated square honeycombs the approximation formula of Lukkassen (see [69], [70]) reduces to

$$\frac{1}{\lambda_{m,\infty} - \lambda_{\text{con}}} = \frac{1}{\lambda_{\text{in}} - \lambda_{\text{con}}} \frac{1}{v} + \frac{r_m}{1 + v^{\frac{1}{2m}}} \frac{1 - v}{\lambda_{\text{con}} v},$$

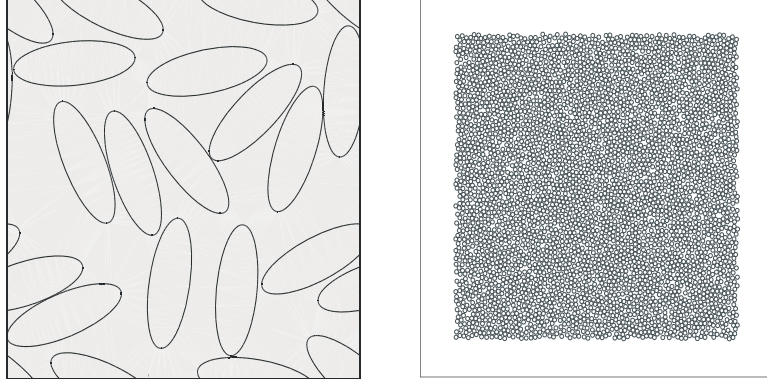


Figure 10: To the left a cell with 16 randomly oriented ellipses and to the right a unit square honeycomb containing 4096 disks.

where $v^{1/(2m)} \leq r_m \leq 1$. Note that this yields the convergence $\lambda_{m,\infty} \rightarrow \lambda_{\infty,\infty}$ as $m \rightarrow \infty$, where

$$\lambda_{\infty,\infty} = \lambda_{\text{con}} + \left(\frac{1}{\lambda_{\text{in}} - \lambda_{\text{con}}} \frac{1}{v} + \frac{1-v}{2\lambda_{\text{con}}v} \right)^{-1},$$

which is precisely the Hashin-Shtrikman upper bound if $\lambda_{\text{in}} < \lambda_{\text{con}}$ and the Hashin-Shtrikman lower bound if $\lambda_{\text{in}} > \lambda_{\text{con}}$.

C.3 Random structures

In paper C2 of this thesis, we compare the effective conductivities of composites having random unit cells with composites having rectangular or hexagonal unit cells. The random cells are generated by a Monte Carlo procedure of Metropolis type, see [74] and [106]. This is done by first placing every inclusion in a rectangular or hexagonal array. Each disk is then given a small tentative displacement in a random direction. The move is accepted or rejected whether or not the move will cause the disk to overlap a neighboring disk. One iteration step consists of trying to move each disk once. The length of the displacement is chosen such that the ratio of acceptance is 30-50%. This process is repeated several times until an equilibrium state is obtained, see Figure 10. In paper C1 we consider iterated square honeycombs containing up to 4000 small disks randomly distributed inside each square.

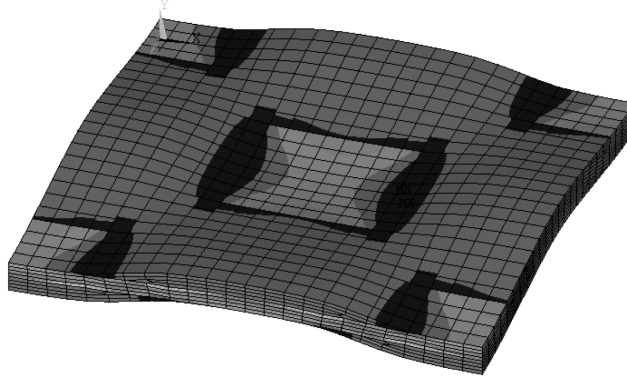


Figure 11: One representative volume element for a single layer woven composite. The weave curves under loading yielding a nonlinear stiffening effect because of tow straightening.

Remark 8 *In this connection we remark that random homogenization is a distinct and well developed discipline of homogenization. Since this discipline is out of the scope of this thesis, we refer the reader to e.g. the classical articles [62], [63], [64] and [88], the more recent articles [59], [89] and [91] as well as the books [10], [11], [54] and the references given therein.*

D Determination of elastic properties by homogenization methods

This section demonstrates some applications of homogenization techniques in engineering sciences. There are also other frequently used methods in composite engineering for computing the effective properties of composites, see e.g. [1], [23], [95] and the references given there.

D.1 Woven composites and reiterated homogenization

In papers D1, D2 and E2 of this thesis, reiterated homogenization is used to compute the elastic properties of woven composites. In the first paper, D1, we evaluate two common semianalytical models, see [52], [53] and [57], that are used to compute the elastic properties of woven composites by homogenization techniques. In the second paper, D2, we investigate the effect of the

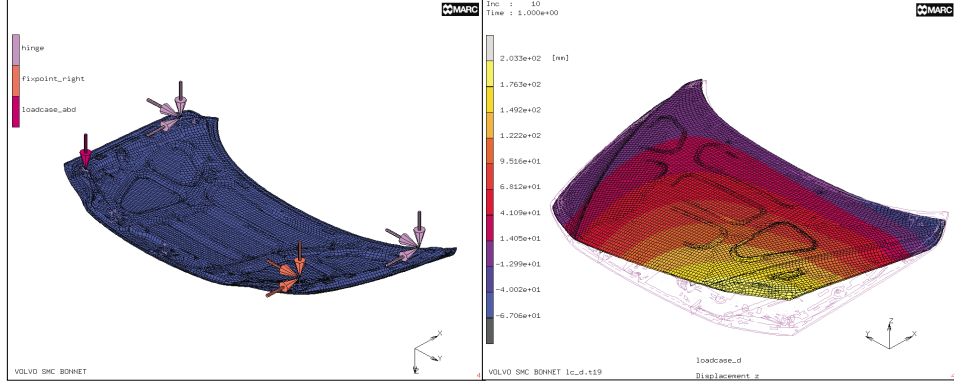


Figure 12: Boundary conditions and deformations of an unreinforced VOLVO engine hood for one particular load case.

number of stacked fabric layers on the elastic properties of woven composites by comparing reiterated homogenization with several micromechanical models based on classical beam theory, see Figure 11. Finally, the third paper, E2, is a joint project with VOLVO Car Corporation where we used reiterated homogenization to compute the elastic properties of reinforcements for tailgates and engine hoods of VOLVO cars, see Figure 12.

D.2 Asymptotic expansion

The reiterated homogenization expressions in paper D1 are achieved by the method of multiple scales asymptotic expansion. The idea of reiterated homogenization using multiple scales expansion can for instance be found in [10]. In order to derive the governing equations, we consider a linear elastic body which occupies a region Ω in $\mathbf{R}^3$ and introduce a Cartesian coordinate system x_i . Moreover, let f_i and u_i be the internal force field and the displacement field, respectively. The equilibrium equations for the body are

$$\begin{cases} -\frac{\partial}{\partial x_j} (C_{ijkl} e_{kl}(u)) = f_i & \text{in } \Omega, \\ u_i = 0 & \text{on } \partial\Omega, \end{cases}$$

where e_{kl} is the strain tensor defined as

$$e_{kl} = \frac{1}{2} \left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k} \right),$$

and C_{ijkl} is the tensor of elastic moduli (or stiffness tensor) with the symmetric properties $C_{ijkl} = C_{ijlk} = C_{jikl} = C_{klij}$. Let us now assume that the body consists of two or more different linear elastic materials which are periodically distributed in the sense that we can define unit cells which are periodically repeated. In this case, we also assume that we have two different scales of periodicity. Furthermore, assume that $C_{ijkl}(y, z)$ is Y -periodic in y and Z -periodic in z . We introduce the local variables $y = x/\varepsilon$ and $z = x/\varepsilon^2$ and put $C_{ijkl}^\varepsilon = C_{ijkl}(x/\varepsilon, x/\varepsilon^2) = C_{ijkl}(y, z)$. We also assume that the functions C_{ijkl}^ε satisfy

$$\alpha \xi_{ij} \xi_{ij} \leq C_{ijkl}^\varepsilon \xi_{ij} \xi_{kl} \leq \beta \xi_{ij} \xi_{ij} \text{ a.e. in } \Omega,$$

for every symmetric real-valued tensors ξ_{ij} , where $0 < \alpha \leq \beta < \infty$.

Consider the following class of problems, one problem for each choice of ε ,

$$\begin{cases} -\frac{\partial}{\partial x_j} (C_{ijkl}^\varepsilon e_{kl}(u^\varepsilon)) = f_i \text{ in } \Omega, \\ u_i^\varepsilon = 0 \text{ on } \Gamma_1. \end{cases} \quad (24)$$

We want to approximate the solutions u^ε by means of a function u which solves the problem corresponding to a homogenized material,

$$\begin{cases} -\frac{\partial}{\partial x_j} (\bar{C}_{ijkl} e_{kl}(u)) = f_i \text{ in } \Omega, \\ u_i = 0 \text{ on } \Gamma, \end{cases} \quad (25)$$

where $\bar{C}_{ijkl}$ is a constant tensor whose response is close (in some context) to the response of the actual heterogeneous tensor C_{ijkl} . To find this tensor, we assume a multiple scales expansion of $u^\varepsilon(x)$ on the form

$$u_i^\varepsilon(x) = u_i^{(0)}(x, y, z) + \varepsilon u_i^{(1)}(x, y, z) + \varepsilon^2 u_i^{(2)}(x, y, z) + \dots,$$

where $u_i^{(n)}$, $n = 0, 1, 2, \dots$, are Y -periodic in the coordinate y and Z -periodic in the coordinate z . Insertion of $u^\varepsilon(x)$ into (24) and solution of the corre-

sponding lowest order (in ε) equations gives the homogenized tensor

$$\overline{C}_{ijkl} = \frac{1}{|Y||Z|} \iint_{Y \times Z} C_{ijmn} [\delta_{mk} \delta_{nl} + e_{mn}(\zeta^{kl} + \eta^{kl}) + e_{mn}(\zeta^{rs}) e_{rs}(\eta^{kl})] dy dz,$$

where $\zeta^{mn} = \zeta^{mn}(y, z)$ are the solutions of the Z -cell problems

$$\begin{cases} -\frac{\partial}{\partial z_j} (C_{ijkl}(y, z) e_{kl}(\zeta^{mn})) = \frac{\partial C_{ijmn}}{\partial z_j} \text{ on } Z, \\ \zeta_i^{mn} \text{ is } Z\text{-periodic,} \end{cases} \quad (26)$$

and $\eta^{mn} = \eta^{mn}(y)$ are the solutions of the Y -cell problems

$$\begin{cases} -\frac{\partial}{\partial y_j} (\widehat{C}_{ijkl}(y) e_{kl}(\eta^{mn})) = \frac{\partial \widehat{C}_{ijmn}}{\partial y_j} \text{ on } Y, \\ \eta_i^{mn} \text{ is } Y\text{-periodic,} \end{cases} \quad (27)$$

respectively. The Z -cell homogenized tensor $\widehat{C}_{ijkl}(y)$ in (27) is given by

$$\widehat{C}_{ijkl}(y) = \frac{1}{|Z|} \int_Z C_{ijkl}(y, z) + C_{ijmn}(y, z) e_{mn}(\zeta^{kl}) dz. \quad (28)$$

By letting $\overline{C}_{ijkl}^Y$ denote the Y -cell homogenized tensor

$$\overline{C}_{ijkl}^Y = \frac{1}{|Y|} \int_Y \widehat{C}_{ijkl}(y) + \widehat{C}_{ijmn}(y) e_{mn}(\eta^{kl}) dy = \overline{C}_{ijkl}, \quad (29)$$

we see that the overall homogenized tensor $\overline{C}_{ijkl}$ is achieved by first solving the Z -cell problem (26), then compute the Z -cell homogenized tensor (28), followed by solving the Y -cell problem (27) and finally compute the overall homogenized tensor by (29). The procedure is easily generalized to problems with m scales of periodicity. In this case, the homogenized tensor is obtained by iterating the homogenization procedure m times, starting with the lowest scale. See also section A.4.

Remark 9 Note that the asymptotic expansion method is mainly formal. However, it can be mathematically justified by the so called two-scale convergence method (not treated in this thesis), where a sequence (u_ε) of functions in $L^p(\Omega)$ is said to two-scale converge to a limit $u_0 \in L^p(\Omega \times Y)$ if

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} u_\varepsilon(x) \phi\left(x, \frac{x}{\varepsilon}\right) dx = \frac{1}{|Y|} \int_{\Omega} \int_Y u_0(x, y) \phi(x, y) dy dx$$

for every $\phi \in \mathfrak{D}(\Omega; C_{\text{per}}^\infty(Y))$. This method was introduced by Nguetseng [84] and developed by Allaire [2].

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