TOPOLOGICAL SPACES ET CETERA

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Contents

1	Topological spaces	1
2	Open and closed sets	11
3	Continuity	19
4	Products	31
5	Quotient spaces	39
6	Compactness	47
7	Connectedness	57
8	Complete metric spaces	65
9	Exercises	75

Chapter 1

Topological spaces

We define the concept of a *topological space* and provide a few simple examples.

1.1 Let X be a set. By a *topology* τ on X we mean a collection of subsets which satisfies the following requirements:

T1: Every finite intersection of sets from τ is in τ.
T2: Every union of sets from τ is in τ.
T3: The empty set $\emptyset$ and X itself is in τ.

A *topological space* is a set X together with a topology. A subset of X is called *open* (for the topology τ) if it is a member of τ, and *closed* if its complement is open. The complement of a set $A \subseteq X$ is the set of points which do not lie in A.

1.2 We note that regarding propery **T1**, it is enough to require the intersection of two arbitrary open sets to be open. It then follows from the formula

$$U_1 \cup \ldots \cup U_{k+1} = (U_1 \cup \ldots \cup U_k) \cup U_{k+1}$$

that if the intersection property holds for $k \geq 2$ open sets, it will also hold for $k+1$ open sets. This means that if the property holds for all pair of open sets, then it also holds for all triplets of sets, all quadruples, et cetera. Thus the property holds for all finite selections of open sets.

1.3 Let X be an arbitrary set and let τ_{min} be the set $\{\emptyset, X\}$. τ_{min} satisfies properties **T1**-**T3** (check) and thus is a topology on X. This is the smallest topology we can have on a set, and it's called the *trivial* topology on X.
As another extremity we may consider the collection τ_{max} of *all* subsets in X. This too is a topology on X, which is called the *discrete* topology . If τ is an arbitrary topology on X, we have the relation $\tau_{min} \subseteq \tau \subseteq \tau_{max}$. In other words,

every member of τ_{min} is a member of τ, and every member of τ is a member of τ_{max}.
The trivial and the discrete topologies are simple to describe, but are not particularly interestering. Interesting topologies are rather tied to some extra structure on the set X. Let us take a look at the case $X = \mathbb{R}$.

1.4 Example Let $\mathbb{R}$ be the real line, and a,$b \in \mathbb{R}$, $a < b$. We write (a, b) and $[a, b]$ for the intervals $a < x < b$ and $a \leq x \leq b$. Similarly we define $(a, b]$, $[a, b)$ and the unbounded intervals (a, ∞), $[a, \infty)$, $(-\infty, b)$, $(-\infty, b]$ and $(-\infty, \infty) = \mathbb{R}$. For example, $[a, \infty)$ is the set of all $x \in \mathbb{R}$ such that $a \leq x$.
A subset $A \subseteq \mathbb{R}$ is defined to be open for the *standard topology* on $\mathbb{R}$ if for every point $x \in A$ there is an interval (a, b) such that $x \in (a, b)$ and the interval (a, b) is itself contained in A. This is the same as saying that for every $x \in A$ there exists an interval $(x - \epsilon, x + \epsilon)$ contained in A (check). A is said to be closed for the standard topology if the complement of A is open.

1.5 Exercise Verify the following.

- Intervals of the type (a, b), (a, ∞) and $(-\infty, b)$ are open sets for the standard topology.
- Intervals of the type $[a, b]$, $[a, \infty)$, $(-\infty, b]$ are closed sets.
- The unbounded interval $\mathbb{R}$ and the empty set $\emptyset$ are both open.
- The unbounded interval $\mathbb{R}$ and the empty set $\emptyset$ are both closed.
- The intervals $(a, b]$ and $[a, b)$ are neither open nor closed sets.
- All one-point-sets x in $\mathbb{R}$ are closed for the standardtopology.

The collection of sets in $\mathbb{R}$ which are open for the standard topology is denoted τ_{st}. Strictly speaking, we have not yet shown that τ_{st} in fact is a topology. Let us do so:

1.6 Exercise Show that τ_{st} is a topology on $\mathbb{R}$. That is, you must show

- Every finite intersection of sets from τ_{st} is in τ_{st}.
- Every union of sets from τ_{st} is in τ_{st}.
- $\emptyset$ and $\mathbb{R}$ both belong to τ_{st}.

1.7 It is necessary to restrict ourselves to *finite* intersections, because it's usually not correct that the intersection of infinitely many open sets from the standard topology is open. We illustrate this with a counterexample:

$$\bigcap_{n=1}^{\infty}(-\frac{1}{n},\frac{1}{n}) = (-1,1)\cap(-\frac{1}{2},\frac{1}{2})\cap\cdots = \{0\}$$

and $\{0\}$ is not an open set for the standard topology.

1.8 The real line $\mathbb{R}$ becomes a topological space in several ways. For instance, we may speak of the trivial topological space $\mathbb{R}$, or the discrete topological space $\mathbb{R}$ or the the standard topological space $\mathbb{R}$. All according to which topology we put on $\mathbb{R}$.
The trivial topology has only two sets and is an example of a finite topology. Finite topologies will not be of importance to us. Nonetheless, here is an exercise.

1.9 Let X be a finite set. Show the following.

- If X has at least two points, then τ_{min} and τ_{max} are different.
- If X has two points, then X has four topologies. Describe them.

1.10 Example The real n-space $\mathbb{R}^n$ has a natural topology which coincides with the standard topology on $\mathbb{R}$ for $n = 1$. Let $|x|$ be the *Euclidean* norm of $x \in \mathbb{R}^n$ given by

$$|x| = \sqrt{x_1^2 + \ldots + x_n^2}$$

We remind the reader that a *norm* on the vector space $\mathbb{R}^n$ is a real function

$$x \mapsto ||x||$$

which satisfies the properties:

No1: $||x|| \geq 0$, and $||x|| = 0$ if and only if $x = 0$.
No2: $||kx|| = |k|||x||$ for $k \in \mathbb{R}$, $x \in \mathbb{R}^n$
No3: $||x + y|| \leq ||x|| + ||y||$ for x,$y \in \mathbb{R}^n$

For an arbitrary point $a \in \mathbb{R}^n$ we define the set

$$U(a;\epsilon) = \{x \in \mathbb{R}^n : |x - a| < \epsilon\}$$

which we call the *Euclidean ϵ-neighborhood* or open ball centered at a.
Let β be the collection of all Euclidean neighborhoods $U = U(a,\epsilon)$ for varying

$a \in \mathbb{R}^n$ and $\epsilon > 0$. We define the standard topology τ_{st} on $\mathbb{R}^n$ such that $O \subseteq \mathbb{R}^n$ belongs to τ_{st} if for every point $x \in O$ there is a U from β so that $x \in U$ and U is entirely contained in O (compare 1.4). This is the same as saying that for every $x \in O$ there is a U from β centered at x and contained in O.

1.11 Notice that the subcollection $\beta \subseteq \tau_{st}$ is not a topology by itself. However, it generates the standard topology in the sense that every non-empty open set may be written as a union of sets from β. For if $x \in O$, then there exists a set $U_x \in \beta$ such that $x \in U_x \subseteq O$. But this means that $O = \cup_x U_x$, $x \in O$.
We say that β is a *basis for the topology* τ_{st}. By a basis for a topology we generally mean a collection of open sets, such that any non-empty open set may be written as a union of these basis-sets. We have a simple criteria for when a collection of subsets of a set actually constitutes a basis for a topology:

1.12 Lemma Let X be a set and β a collection of subsets such that

B1: Every $x \in X$ belongs to at least a set from β.
B2: For every $U, V \in \beta$ and $x \in U \cap V$, there is a $W \in \beta$ such that $x \in W$ and $W \subseteq U \cap V$.

Then β is a basis for a unique topology on X.

Proof. If β is a basis for the topology on X, then the nonempty, open sets for the topology are all unions of sets from β, which shows that the topology is determined by β.
Let τ be the set of all unions of sets from β, and $\emptyset$. Let us show that τ is a topology on X. Take $U, V \in \tau$. We show that $U \cap V$ is in τ, that is, this intersection is a union of sets from β (property **T1**). We may assume $U = \cup_i U_i$ and $V = \cup_j V_j$ where all the U_i and V_j are sets from β, and i,j indices varying over unspecified index sets. This yields $U \cap V = (\cup_i U_i) \cap (\cup_j V_j) = \cup_{i,j} (U_i \cap V_j)$. But then it suffices to show that every $U_i \cap V_j$ is a union of sets from β.
For every $x \in U_i \cap V_j$ there exists $W_x \in \beta$ such that $x \in W_x$ and $W_x \subseteq U_i \cap V_j$ by the assumptions **B1** and **B2**. This gives $U_i \cap V_j = \cup_x W_x$, where the indexing x varies through $U_i \cap V_j$. Thus **T1** is satisfied. **T2** and **T3** are trivially satisfied.
Hence τ is a topology. □

From lemma 1.12 it follows easily that τ_{st} from example 1.10 in fact is a topology.

1.13 Exercise Show the following.
• Every Euclidean neighborhood $U(x; \epsilon)$ in $\mathbb{R}^n$ is a (standard-) open set.

• The collection τ_{st} is a topology on $\mathbb{R}^n$.
• Every set $\overline{U}(x;\epsilon) = \{y \in \mathbb{R}^n : |y-x| \leq \epsilon\}$ is closed with respect to the standard topology.

(Note: For $n = 1, 2$ it's straightforward to verify these claims by elementary geometrical methods. Then $U(a;\epsilon)$ will be, respectively, an interval on the line and a circular disc in the plane. But to provide a proof which covers all cases of n, one must apply the properties of the norm $|\ |$ in $\mathbb{R}^n$.

1.14 We may define the standard topology on $\mathbb{R}^n$ by means of other norms than the Euclidean. For instance, by setting

$$|x|_{\sup} = \max_{1\leq i\leq n} |x_i|$$

for $x \in \mathbb{R}^n$, properties **No1**-**No3** will be satisfied with $||x|| = |x|_{\sup}$. We call $|x|_{\sup}$ the *supremum norm* of x (sup-norm). If we now define the *sup-neighborhoods* $U_{\sup}(a;\epsilon)$ by the requirement $|x-a|_{\sup} < \epsilon$, we see that the system of sup-neighborhoods for varying x and ϵ defines the same collection of open sets in $\mathbb{R}^n$ as the Euclidean neighborhoods. Namely, the inequalities

$$|x|_{\sup} \leq |x| \leq \sqrt{n}|x|_{\sup}$$

yield the following inclusions

$$U_{\sup}(a;\frac{\epsilon}{\sqrt{n}}) \subseteq U(a;\epsilon) \subseteq U_{\sup}(a;\epsilon)$$

for arbitrary $a \in \mathbb{R}^n$ and $\epsilon > 0$. This means that every Euclidean neighborhood of a contains a sup-neighborhood about a, and vice versa. These systems of open neighborhoods are mutually infiltrated. But then they must define the same collections of open sets in $\mathbb{R}^n$. (Explain this in detail).

1.15 Exercise Show that for arbitrary a,$x \in \mathbb{R}^n$ and $\epsilon > 0$ we have
• $|x|_{\sup} \leq |x| \leq \sqrt{n}|x|_{\sup}$
• $U_{\sup}(a;\epsilon/\sqrt{n}) \subseteq U(a;\epsilon) \subseteq U_{\sup}(a;\epsilon)$

1.16 The definition of ϵ-neighborhoods and the related topology can be carried over to general normed vector spaces without further difficulties. If E is a real vector space with norm given by $||x||$, we define the corresponding ϵ-neighborhoods by

$$U(a;\epsilon) = \{x \in E : ||x-a|| < \epsilon\}$$

and call a set $A \subseteq E$ open in the *norm topology* if for every point in A there is an ϵ-neighborhood of this point which is contained entirely in A. Such a collection constitutes a topology on E. The proof of this goes along the same lines as the case for the standard topology on $\mathbb{R}^n$ (see 1.7). We call this topology the *norm topology* on E.
One may ask exactly when two different norms $||||_1$ and $||||_2$ on a vector space V give rise to the same topology, as was the case for the Euclidean norm and the supremum norm on $\mathbb{R}^n$. It is easy to verify that this happens if and only if there exist two positive constants a,b such that

$$a||x||_1 \leq ||x||_2 \leq b||x||_1$$

for all $x \in V$. For these inequalities give appropriate inclusions between ϵ-neighborhoods defined by the two norms. If this property is satisfied, we say that the two norms are *equivalent*.

Could it happen that all norms on a vector space E are equivalent? That is, they all give rise to the same topology on E? Indeed, this is the case if E is of finite dimension, as we shall see. Every norm on $\mathbb{R}^n$ creates the standard topology (as norm topology).

1.17 The norm topology on a vectorspace E with norm $||\ ||$ is generated by the open sets $U(x;\epsilon) = \{y \in E : ||y - x|| < \epsilon\}$ and is therefore determined by the function $d(x, y) = ||y - x||$ of two vector-variables. It follows easily from the norm properties that d satisfies the following

M1: $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$
M2: $d(x, y) = d(y, x)$ for x,$y \in E$
M3: $d(x, z) \leq d(x, y) + d(y, z)$ for x,y,$z \in E$.

A function of this type is called a *metric* (distance function) on E. Generally we define a metric on a set X (not necessarily a vector space) as a function d such that properties **M1**-**M3** are satisfied for all x,y,$z \in X$. The quantity $d(x, y)$ is called the *distance* between x and y. A *metric space* is a set X equipped with a metric.
Every set X may be considered to be a metric space by way of, for instance, the *discrete metric* defined by $d(x, y) = 1$ for $x \neq y$ and $d(x, y) = 0$ for $x = y$. Different metrics may be defined on the same set, but they constitute different metric spaces as such. If X is a subset of a normed vectorspace E, then the metric $d(x, y) = ||y - x||$ on E will give a metric on X if we restrict x,y to vary in X.
In a metric space it makes sense to talk about the distance from a point x to

a subset A, the distance between two subsets A and B, and the diameter of a subset. If d is the metric on X, we set

$$d(x, A) = \inf\{d(x, a) : a \in A\}$$

$$d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$$

A is a *bounded* set if there is an upper bound for the distances $d(x, y)$ as x and y vary in A. In that case the least upper bound exists, which we call the *diameter* of A and write diam A,

$$\text{diam } A = \sup\{d(x, y) : x, y \in A\}$$

If A is unbounded, we write diam $A = \infty$.

1.18 If X is a metric space, we define the ϵ-neighborhood of $a \in X$, similar to the case of normed vector spaces, by

$$U(a; \epsilon) = \{x \in X : d(x, a) < \epsilon\}$$

for $\epsilon > 0$. Not surprisingly, we get a topology on X called the *metric topology*, where a set $A \subseteq X$ is open if for every point in A there is an ϵ-neighborhood of the point, contained in A. The ϵ-neighborhoods in a metric space are of course open sets (with respect to the metric topology). Every metric space is thus a topological space too.

1.19 Let $M(m, n)$ denote the set of all real (m, n)-matrices (m rows, n columns). $M(m, n)$ is a vector space over $\mathbb{R}$ where the vector space operations are the usual matrix addition and multiplication of scalar and matrix. We have a linear isomorphism of $M(m, n)$ onto $\mathbb{R}^{mn}$ defined by assigning the matrix $A = [a_{ij}] \in M(m, n)$ to the point $(a_{11}, \ldots, a_{1n}, \ldots, a_{mn}) \in \mathbb{R}^{mn}$, that is, the matrix is "read row-by-row". By this linear isomorphism we transfer the standard topology of $\mathbb{R}^{mn}$ to our newly introduced space $M(m, n)$.
We note that the examples 1.10 and 1.14 also give rise to two norms on $M(m, n)$. The Euclidean norm, also called the *Hilbert-Schmidt-norm*, may be written

$$|A| = (\sum_{ij} a_{ij}^2)^{\frac{1}{2}} = (Tr(A^t A))^{\frac{1}{2}}$$

where A^t means the transpose matrix of A, and Tr denotes the trace (sum of main diagonal elements). The sup-norm, which we have shown to be equivalent to the Euclidean, then gives

$$|A|_{\sup} = \max_{i,j} |a_{ij}|$$

We may also define the so-called *operator norm* on $M(m,n)$ by

$$||A|| = \sup_{|x|=1} |Ax|$$

where Ax denotes the matrix-vector product of the matrix A and the column vector x, $|x|$ and $|Ax|$ denote the Euclidean norm of x and Ax. It is not hard to verify properties **No1**-**No3**. To show that the operator norm is also equivalent to the Euclidean norm, it is enough to show the inequalities $|A|_{\sup} \leq ||A|| \leq \sqrt{mn}|A|_{\sup}$. This also shows that the supremum value $||A|| = \sup_{|x|=1} |Ax|$ above is finite.

1.20 Example Let $C[a,b]$ be the set of all continuous, real functions defined on a bounded, closed interval $[a,b]$ in $\mathbb{R}$. This function space is a vector space over $\mathbb{R}$, where the vector space operations are, as usual, addition of functions and multiplication of a constant and a function. From real analysis we know that a continuous function defined on a bounded, closed interval in $\mathbb{R}$ is bounded. We may therefore define the *supremum norm* of an element $f \in C[a,b]$ by

$$|f|_{\sup} = \sup_{x\in[a,b]} |f(x)|$$

This makes $C[a,b]$ into a normed vector space, that is, requirements **No1**-**No3** are met. Thus we also have the corresponding *sup-metric* given by

$$d_{\sup}(f,g) = \sup_{x\in[a,b]} |f(x) - g(x)|$$

which makes $C[a,b]$ into a metric space and thus also a topological space. The ϵ-neighborhood of f is defined by the requirement $d_{\sup}(f,g) < \epsilon$, and will therefore consist of all g whose (graphical) images lie within a strip of half-width ϵ surrounding the (graphical) image of f. This topology is usually called the topology for *uniform convergence*. (This name is explained in chapter 2).
In general, if X is an arbitrary set and $B(X)$ is the set of all bounded, real functions defined on X, we can make $B(X)$ into a topological space in a similar way. Then $B(X)$ too will be a normed vectorspace with the sup-norm.

1.21 Example The function space $C[a,b]$ may also be considered as a topological space in the following way. Let $f \in C[a,b]$. For a *finite* set $A \subseteq [a,b]$ and $\epsilon > 0$ we define the set (consisting of functions)

$$U(f;A,\epsilon) = \{g \in C[a,b] : |g(x) - f(x)| < \epsilon \text{ for all } x \in A\}$$

To see that the sets $U(f;A,\epsilon)$ constitute a basis for a topology on $C[a,b]$, it is enough to note that we have

(i) $f \in U(f; A, \epsilon)$ for all A,ϵ.
(ii) $U(f; A_1, \epsilon_1) \cap U(f; A_2, \epsilon_2) \supseteq U(f; A, \epsilon)$ when $A = A_1 \cup A_2$ and $\epsilon = \min(\epsilon_1, \epsilon_2)$
(iii) $U(f; A, \epsilon) \supseteq U(g; A, \delta)$ when $g \in U(f; A, \epsilon)$ and $\delta = \epsilon - \max_{x \in A} |f(x) - g(x)|$.

It is clear, then, that the premises in lemma 1.12 are met. The topology we obtain here is called the topology for *pointwise convergence*. (See chapter 2).
In general we can topologize the vector space $F(X)$ of all real functions defined on an arbitrary set X in a similar fashion.

1.22 The two topologies on $C[a, b]$ defined in examples 1.20 and 1.21 are different. To be more precise, we have that every "pointwise" basis-set $U(f; A, \epsilon)$ contains a "uniform" basis-set $U(f; \epsilon)$, but not vice versa, as is easily seen. It follows that every pointwise open set is a uniform open set, but not vice versa. (Explain). Therefore we have the inclusion $\tau_p \subset \tau_u$ for the corresponding topologies. We say that the topology for pointwise convergence is *strictly coarser* than the topology for uniform convergence, which is said to be *strictly finer* than the first. It is, by the way, remarkable that two topologies on a set are comparable.

1.23 Every subset A in a topological space X inherits the topology from X in the following sense: The open sets in the subspace A are sets $O \cap A$ where O is open in X. If τ is the topology on X, we let τ_A denote the topology induced on A in this manner. Consequently, the closed sets in the subspace A are the sets $F \cap A$ where F is closed in X. This follows from

$$(X - O) \cap A = A - (O \cap A)$$

which applies for arbitrary subsets A and O of X.
A metric d on X gives by restriction a metric d_A on A, and if τ is the topology on X defined by the metric d, then the topology on A defined by the metric d_A is precisely τ_A.

1.24 Example Consider $X = \mathbb{R}$ and $A = [0, 1]$. Sine $\mathbb{R}$ has a basis consisting of all intervals (a, b), the subspace $[0, 1]$ has a basis consisting of all intervals $[0, b)$,$(a, 1]$ and (a, b) where $0 \leq a < b \leq 1$.
Note that the sets $[0, 1]$,$[0, b)$ and $(a, 1]$ which are all open in the subspace $[0, 1]$, are not open in $\mathbb{R}$. On the other hand, all closed subsets of $[0, 1]$ are also closed in $\mathbb{R}$. In general we have the following

1.25 Lemma Let X be a topological space with subspace $Y \subseteq X$. If $A \subseteq Y$ is open in Y and Y is open in X, then A is open in X. (Equivalent argument applies to closed sets).

1.26 We introduce standard descriptions for the following spaces defined by the Euclidean norm,

$$\begin{aligned} D^n &= \{x \in \mathbb{R}^n : |x| \leq 1\} && \text{the closed n-cell.} \\ E^n &= \{x \in \mathbb{R}^n : |x| < 1\} && \text{the open n-cell.} \\ S^{n-1} &= \{x \in \mathbb{R}^n : |x| = 1\} && \text{the (n-1)-sphere.} \end{aligned}$$

As topological spaces these are subspaces of $\mathbb{R}^n$.

1.27 The definition of topological spaces can essentially be traced back to Felix Hausdorff's book "Grundz*ü*ge der Mengenlehre" from 1914. Previous to this work there were some more or less successful attempts at establishing a general notion of space. An important predecessor was the concept of metric space which was introduced by Maurice Frechet in 1906. Most function spaces which occur frequently in functional analysis are viewed as metric spaces in natural ways. The choice of a metric depends on the structure being studied. The idea of considering "spaces" of functions stems from Riemann.
The theory of normed spaces, which combines linear algebra with a notion of distance, was developed by several mathematicians in the period 1907-1932. Of particular importance was Stefan Banach's work "Theorie des operations lineaires" from 1932.

Chapter 2

Open and closed sets

In this chapter we still examine the properties of open and closed sets in topological spaces. We shall also introduce the notion of convergence of a sequence.

2.1 The class of open sets in a topological space allows (per definition) finite intersections and arbitrary unions. The class of closed sets allows the complementary operations; namely finite unions and arbitrary intersections.

Lemma. Let X be a topological space.
L1 Every finite union of closed sets is closed.
L2 Every intersection of closed sets is closed.
L3 The set X itself and the empty set $\emptyset$ are both closed.

Proof. **L3** is trival. **L1** and **L2** follow immediately from the properties of open sets and by using De Morgan's laws

$$X - \bigcup_i A_i = \bigcap_j (X - A_j)$$

$$X - \bigcap_i A_i = \bigcup_j (X - A_j)$$

where the indices i and j are from an arbitrary index set, finite or infinite. Let us show **L1** in detail:
Assume $A_1, \ldots, A_n$ are closed sets. Then $X - A_1, \ldots, X - A_n$ are open and so is $(X - A_1) \cap \cdots \cap (X - A_n)$ by **T1**. But

$$(X - A_1) \cap \cdots \cap (X - A_n) = X - (A_1 \cup \cdots \cup A_n)$$

by De Morgan's first law, so $X - (A_1 \cup \cdots \cup A_n)$ is open and hence $A_1 \cup \cdots \cup A_n$ must be closed. $\square$

2.2 Exercise Prove **L2**, **L3** and De Morgan's Laws.

2.3 By an *open neighborhood* of a point $x \in X$ we shall mean an open set which contains x. A *neighborhood* of x is a set which contains an open neighborhood of x. Note that an open set is a neighborhood of all its points, and vice versa. The system of all such neighborhoods of x is denoted $\eta(x)$ and the subsystem of open neighborhoods $\eta_0(x)$.
Let A be a subset of X. By int A, *the interior of* A, we mean the union of all open sets contained in A, and by cl A, *the closure of* A, we mean the intersection of all closed sets which contain A.
As a union of open sets, int A is open, and as a union of sets contained in A, int A is also contained in A. It follows then, that int A is the largest open set contained in A, since if U is an open set contained in A, then U must be contained in int A. Similarly we see that cl A is closed, contains A and is the smallest closed set which contains A.
It follows that A is open if and only if $A = \text{int } A$, and closed if and only if $A = \text{cl } A$. Generally we have

$$\text{int } A \subseteq A \subseteq \text{cl } A$$

From the definition of interior and closure we also get the formula

$$X - \text{cl } A = \text{int } (X - A)$$

A point $x \in X$ is called an *interior point* if x belongs to int A. This is the same as saying that A is a neigborhood of x. (Why?) The interior points of A make up the interior of A. We say that x is an *exterior point* of A if x is an interior point of $X - A$. The set of all exterior points make up the *exterior of* A, which is simply int $(X - A)$. The remaining points x of X are called *boundary points* to A. They constitute the *boundary* of A, denoted bd A. We then have a disjoint union of interior, boundary and exterior of A, which fills the entire space X,

$$X = \text{int } A \cup \text{bd } A \cup \text{int } (X - A)$$

Furthermore, we have a disjoint union

$$\text{cl } A = \text{int } A \cup \text{bd } A.$$

We note that a set and the complementary set both have the same boundary

$$\text{bd } A = \text{bd } (X - A).$$

It follows from the splitting of X into interior, boundary and exterior of A, that bd A must be a closed set.

2.4 Exercise Show the following
• A set is open if and only if it does not contain any of its boundary points
• A set is closed if and only if it contains all of its boundary points
• The intervals (a, b), $[a, b)$, $[a, b]$, with the standard topology, all have interior (a, b) and a boundary consisting of the endpoints a and b.
• int $\mathbb{Q} = \emptyset$ and bd $\mathbb{Q} = \mathbb{R}$. Thus the boundary of a set can be bigger than the set itself.
For the open n-cell E^n and closed n-cell D^n in $\mathbb{R}^n$ the interior in both cases is E^n and the boundary is the (n-1)-sphere S^{n-1}.

2.5 A point $x \in X$ is called a *closure point* for $A \subseteq X$ if x is an element of cl A. Every element of A is a closure point.

Lemma.
• x is a closure point of A if and only if every neighborhood of x intersects A.
• x is a boundary point of A if and only if every neighborhood of x intersects both A and $X - A$.

Proof. Assume that every neighborhood of x intersects A. The open set $X-$cl A does not intersect A and hence does not contain x. This means that x belongs to the complement, namley cl A.
Assume there exists a neighborhood of x which does not intersect A. Then there exists a (possibly smaller) neighborhood U which also does not intersect A. The complement $X - U$ is closed and contains A, hence it contains cl A. In other words, x belongs to a set U which does not intersect cl A. This means that x does not belong to cl A. □

A subset $A \subseteq X$ is said to be *dense* in X if cl$\breve{a}A = X$. This is equivalent to the fact that every neighborhood in X contains points from A. For example, $\mathbb{Q}$ is dense in $\mathbb{R}$, and moreover $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$.

2.6 Let again X be a topological space, and $(x_n)_{n=1,2,\ldots}$ a sequence of elements in X. We say that (x_n) *converges* to an element $x \in X$ if for every neighborhood V of x there is a natural number N such that $x_n \in V$ for $n > N$. It is enough that this property holds for every neighborhood in a basis of neighborhoods - it will then hold in general. We call x the *limit* for the sequence (x_n) and write $x_n \to x$ as $n \to \infty$. A sequence that does not converge is said to *diverge*.
From this definition it may just as well happen that a sequence in X may converge to several elements. In a space with trivial topology every sequence is convergent and converges to every element, but for a large class of spaces this cannot occur. We call X a *Hausdorff space* if two distinct points of X always have disjoint

neighborhoods. In a Hausdorff space a sequence (x_n) may converge to at most one point since the sequence may not, from a certain index n and on, lie in two disjoint neighborhoods. Hence the limit for the sequence is unique if it exists, and we denote it $\lim x_n$.
It is worth noting that in a Hausdorff space every one-point-set x is closed. For if $x' \neq x$, it follows from the Hausdorff property that $x' \notin \text{cl } x$. Thus $\text{cl } x = x$.

2.7 A metric space X is generally Hausdorff. For if $x \neq x'$, then $d(x, x') = r > 0$, and the neighborhoods $U(x; \frac{r}{2})$ and $U(x'; \frac{r}{2})$ are disjoint.
A sequence (x_n) in a metric space X converges to a point x if and only if for every $\epsilon > 0$ there exists N such that $d(x, x_n) < \epsilon$ for all $n > N$.
For sequences in $\mathbb{R}^n$ this specializes to the usual notion of convergence we are already familiar with from analysis.

2.8 Example In the space $B(X)$ of bounded, real functions on X, equipped with the sup-norm

$$|f|_{\text{sup}} = \sup_{x \in [a,b]} |f(x)|$$

a sequence (f_n) converges to a limit f if and only if $\sup_{x \in X} |f(x) - f_n(x)| \to 0$ when $n \to \infty$. This means that f_n converges uniformly to f in X, which explains the name given to this topology: "topology for uniform convergence". $B(X)$ is of course a Hausdorff space, since it is a normed vector space.

2.9 Example The space $F(X)$ of all real functions on X with the topology of pointwise convergence, i.e. using the open sets

$$U(f; A; \epsilon) = \{g \in C[a, b] : |g(x) - f(x)| < \epsilon \text{ for all } x \in A\} \ (\epsilon > 0, A \text{ finite}),$$

as the basis-sets, makes $F(X)$ a Hausdorff space.
Convergence of a sequence of functions (f_n) to a limit function f in $F(X)$ is in other words convergence of the sequence of numbers $(f_n(x))$ to the number $f(x)$ for every point $x \in X$. Because if f_n converges to f, then, if x is an arbitrary point in X and an arbitrary $\epsilon > 0$ is chosen, then $f_n \in U(f; x, \epsilon)$ from a certain n and onwards. This means precisely that $(f_n(x))$ is converging to $f(x)$. If, conversely, $(f_n(x))$ is converging to $f(x)$ for every $x \in X$, then (f_n) is in fact converging to f in the topology given on $F(X)$. Thus also explaining the name "topology for pointwise convergence".
To explicitly see the difference between *pointwise* and *uniform* convergence, consider the sequence of functions (f_n) on the interval $[0, \infty)$ given by $f_n(x) =$

n^2xe^{-nx}. The sequence (f_n) converges pointwise to the function $f = 0$, but not uniformly.

2.10 We will say that a space X has *countable neighborhood bases* if every point in X has a countable neighborhood basis. This is equivalent to requiring that every point in X has a *descending* countable basis of neighborhoods, that is, a basis of neighborhoods

$$V_1 \supseteq V_2 \supseteq \cdots$$

For if (W_n) is an arbitrary countable basis of neighborhoods of x, we simply set $V_n = W_1 \cap \cdots \cap W_n$. Then (V_n) is a descending countable basis of neighborhoods of x.
A metric space, for instance, has countable neighborhood bases. We may choose the basic neighborhoods $U(x; \frac{1}{n})$ for $n = 1, 2, \ldots$ of every point x.
In particular, the space $B(X)$ with the topology for uniform convergence, has countable neighborhood bases. The next example shows that not all function spaces possess this desirable property.

2.11 Example Consider the function space $F(\mathbb{R})$ of all functions from $\mathbb{R}$ to $\mathbb{R}$, equipped with the topology for pointwise convergence. We shall show that $F(\mathbb{R})$ does not have countable neighborhood bases.
Let $f \in F(\mathbb{R})$ and assume $(V_n)_{n=1,2,\ldots}$ is a neighborhood basis of f. For each n we can find a finite subset $A_n \subseteq \mathbb{R}$ and an $\epsilon_n > 0$ such that $U(f; A_n, \epsilon_n) \subseteq V_n$, since $(U(f; A, \epsilon))_{A,\epsilon}$ is a neighborhood basis of f. Now, $T = A_1 \cup A_2 \cup \cdots$ is a countable set and hence a proper subset of $\mathbb{R}$. Thus we may choose $c \in \mathbb{R} - T$. It then follows that none of the sets $U(f; A_n, \epsilon_n)$ are contained in $U(f; \{c\}, 1)$ since c does not belong to any A_n. This contradicts the fact that $(U(f; A_n, \epsilon_n))_n$ is a basis of neighborhoods and so we must reject the assumption of countable neighborhood bases.
This also shows that $F(X)$ is *non-metrizable*, meaning there is no metric on $F(X)$ which defines the topology we originally started with in this example, namely the topology of pointwise convergence. Hence metric spaces are not sufficient for a treatment of all the interesting types of convergence in the branch of analysis.

2.12 Theorem. Let A be an arbitrary subset of a topological space X. Every element in X which is a limit for a sequence of elements in A, lies in cl A. If X has countable neighborhood bases, then conversely every element in cl A is a limit for a sequence of elements in A.

Proof. Assume that $x_n \in A$ for $n = 1, 2, \ldots$ and that $x_n \to x$ as $n \to \infty$.

Every neighborhood of x contains all the x_n from a certain n onwards, so $x \in \mathrm{cl}\, A$ indeed. Assume now, that X has countable neighborhood bases. Let $x \in \mathrm{cl}\, A$ and choose $(V_n)_{n=1,2,\ldots}$ as a descending basis of neighborhoods of x. Since $V_n \cap A \neq \emptyset$ for all n, we may choose $x_n \in V_n \cap A$. This yields a sequence (x_n) of elements from A such that $x_n \to x$ as $n \to \infty$. □

The next example illustrates the fact that the requirement of countability regarding the neighborhood bases was necessary in the theorem just proved.

2.13 Example. Consider $F(\mathbb{R})$ again. Let $F_0(\mathbb{R})$ denote the subset consisting of functions which take the value 0 except at a finite number of points. Then $F_0(\mathbb{R})$ is dense in $F(\mathbb{R})$. For if $f \in F(\mathbb{R})$ is arbitray and $U(f; A, \epsilon)$ is a basis neighborhood (meaning it is a neighborhood taken from the basis) of f, then we may set

$$g(x) = \begin{cases} f(x) & \text{for } x \in A \\ 0 & \text{otherwise} \end{cases}$$

We then have that $g \in U(f; A, \epsilon) \cup F_0((R)$. This shows that every neighborhood in $F(\mathbb{R})$ intersects $F_0((R)$, so f is a closure point for $F_0(\mathbb{R})$.
On the other hand, it is incorrect to say that every f in $F(\mathbb{R})$ may be written as a limit for a sequence of functions from $F_0((R)$. Let (g_n) be a sequence in $F_0(\mathbb{R})$ and $g = \lim g_n$, that is (g_n) converges pointwise to g. By the assumption, g_n is zero outside of a finite set $A_n \subseteq \mathbb{R}$. Let $A = \bigcup_{n=1}^{\infty} A_n$. Then A is countable. Outside of A, all g_n are identically zero, and the same applies to g. A function f which does not vanish outside a countable set in $\mathbb{R}$ can obviously not be a limit for a sequence from $F_0(\mathbb{R})$.

2.14 Some elementary topological concepts related to point sets on the real line and $\mathbb{R}^n$ were used in mathematical analysis already in the 1800s. Karl Weierstrass, in the last half of the last century, contributed greatly in imposing the analysis a rigorous and formal structure, and introduced set-oriented notions such as "open set" and "accumulation point". In the early 1870s Georg Cantor, by investigating convergence for trigonometrical series, was led to study topological properties of point sets in $\mathbb{R}$. His work was to eventually become the foundation of a more general set-theoretic topology.
The separation requirement regarding two points always having disjoint neighborhoods was originally one of the axioms in Hausdorff's definition of topological spaces. This axiom was later removed due to the discovering of applications where this requirement simply was not met. The requirement of countable neighborhood bases also stems from Hausdorff (first countability axiom), and also the existence of a countable basis for the topology (second countability axiom).

Frechet defined, in 1906, a general notion of space based on the convergence of sequences (limes-spaces). But this was not very fruitful. To improve this shortcoming of countably indexed sequences, E.H. Moore and H.L. Smith introduced, in 1932, so-called generalized sequences, which are sequences indexed by more general partially ordered (or directed) sets, other than the natural numbers.

Chapter 3

Continuity

In this chapter we will deal with the crucial concept of continuity and related issues. Continuity is the most central term of point-set or set-theoretic topology.

3.1 Let $f : X \to Y$ be a map between topological spaces X and Y. We say that f is *continuous* at the point $x \in X$ if for every neighborhood V of $f(x)$ there exists a neighborhood U of x such that $f(U) \subseteq V$, or equivalently, if the inverse image $f^{-1}(V)$ of every neighborhood V of $f(x)$ is a neighborhood of x.
It is enough, then, that this holds for every V in a neighborhood basis for $f(x)$; it then holds in general.
We say that f is *continuous*, altogether, if f is continuous at all points $x \in X$ in the manner explained above.

3.2 Example Let X and Y be metrizable by metrics d and d' respectively, so x and $f(x)$ have neighborhood bases of ϵ-neighborhoods $(U(x;\epsilon))_\epsilon$ and $(U'(f(x);\epsilon))_\epsilon$. Then f is continuous at x if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $f(U(x;\delta)) \subseteq U'(f(x);\epsilon)$. In other words, if for every $\epsilon > 0$ there exists $\delta > 0$ so that $z \in X$ and $d(x,z) < \delta$ implies $d'(f(x),f(z)) < \epsilon$. For $X = \mathbb{R}^n$, $Y = \mathbb{R}$ this specializes to the usual notion of continuity for real functions of several variables.

3.3 Example The fact that we may determine continuity at a point by having the set V, as above, chosen among a neighborhood basis, has a certain practical significance. Let us for instance look at the case $Y = \mathbb{R}^p$. A map $f : X \to \mathbb{R}^p$ can be written $f(x) = (f_1(x), \ldots, f_p(x))$, where $f_1, \ldots, f_p$ are real functions on X. We will show that f is continuous if and only if the component functions f_i are continuous. This is easiest to realize by using the fact that the topology on $\mathbb{R}^p$ can be derived from the supremum metric, the consequence being that f is continuous at the point $x_0 \in X$ if for every $\epsilon > 0$ there exists a neighborhood U

of x_0 such that $x \in U$ implies $|f(x) - f(x_0)|_{\sup} < \epsilon$. Since

$$|f(x) - f(x_0)|_{\sup} < \epsilon$$

is equivalent to

$$|f_i(x) - f_i(x_0)| < \epsilon \text{ for } i = 1, 2, \ldots, p$$

the conclusion follows.

3.4 Example The four arithmetical operations define continuous functions $a, s, m : \mathbb{R}^2 \to \mathbb{R}$ and $d : U \to \mathbb{R}$ by the assignments

- $a(x, y) = x + y$
- $s(x, y) = x - y$
- $m(x, y) = xy$
- $d(x, y) = x/y$

where the last function is only defined on the complement U of the x-aksis. For instance, we have for the addition operation $a(x, y) = x + y$ that $|(x, y) - (x', y')|_{\sup} < \epsilon/2$ implies $|a(x, y) - a(x', y')| < \epsilon$. More generally, we have that a_n and m_n given by

- $a_n(x_1, \ldots, x_n) = x_1 + \ldots + x_n$
- $m_n(x_1, \ldots, x_n) = x_1 x_2 \cdots x_n$

are continuous functions on $\mathbb{R}^n$ for $n = 1, 2, \ldots$. We may here, of course, replace the real numbers $\mathbb{R}$ with the complex numbers $\mathbb{C}$.

3.5 Example Let $A : \mathbb{R}^n \to \mathbb{R}^p$ be a linear map. Then A is continuous. This follows from a general inequality:

$$|Ax - Ax'| \leq c_A |x - x'| \qquad x, x' \in \mathbb{R}^n \tag{3.5.1}$$

where c_A is a constant (depending on A) which we need not know. It is obvious that (3.5.1) yields $d(Ax, Ax') < \epsilon$ for $d(x, x') < \epsilon/c_A$. We may thus parry ϵ with $\delta = \epsilon/c_A$ everywhere in $\mathbb{R}^n$.

Because of the linearity of A, (3.5.1) is equivalent to the simpler inequality

(3.5.2) $\qquad |Ax| \leq c_A|x|, \qquad x \in \mathbb{R}^n$

It is enough to prove the validity of (3.5.2) for all $x \neq 0$ in $\mathbb{R}^n$, the case $x = 0$ poses no difficulty. After dividing by $|x|$ on both sides of the inequality we see that (3.5.2) is equivalent to

(3.5.3) $\qquad |Ay| \leq c_A, \qquad x \in \mathbb{R}^n, |y| = 1$

It's straightforward to verify (3.5.3) for an appropriate c_A. Because it is fairly obvious that the left-hand side $|Ay|$ is dominated by

$$\sup_{|y|=1} |Ay| = ||A||$$

and so it's enough to choose $c_A \geq ||A||$. In particular, $c_A = ||A||$ is the least constant to satisfy (3.5.1)-(3.5.3). (See 1.19 for an estimate of $||A||$).

3.6 Example Assume V to be a normed vector space with norm $||x||$. Set $||x|| = p(x)$. Then we have

$$|p(x) - p(y)| \leq ||x - y||$$

for all x,$y \in V$. It follows that the norm is a continuous function on V, when V is equipped with the norm topology.

3.7 Let $f : X \to Y$ be a map between two sets. A sequence $(x_n)_{n=1,2,\ldots}$ in X is sent by f to the sequence $(f(x_n))_{n=1,2,\ldots}$ in Y. If X and Y are topological spaces, there is a close relationship between continuous functions and convergent sequences, which is even stronger if X has countable neighborhood bases.

Theorem Let $f : X \to Y$ be a map between topological spaces. If f is continuous at the point x, then f sends every sequence which converges to x, to a sequence which converges to $f(x)$. The converse holds if x has a countable neighborhood basis.

Proof. Assume first that f is continuous at x. Let (x_n) be a sequence in X converging to x, and let V be an arbitrary neighborhood of $f(x)$ in Y. Then there is a neighborhood U of x such that $f(U) \subseteq V$ (since f is continuous at x). And we have $x_n \in U$ from a certain n and onwards(since (x_n) is converging to x). This gives $f(x_n) \in V$ from a certain n and onwards. Thus $(f(x_n))$ is converging to $f(x)$.

Now assume that f is discontinuous at x. Then there exists a neighborhood

V of $f(x)$ such that $f(U) \not\subseteq V$ for every neighborhood U of x. If x has a countable neighborhood basis, let $U_1 \supseteq U_2 \supseteq \cdots$ be one such and choose $x_n \in U_n$ such that $f(x_n) \notin V$. Then (x_n) is a sequence in X converging to x, but $f(x_n)$ is a sequence which does not converge to $f(x)$. The theorem is proved. □

In summary, we can say that if X has countable neighborhood bases, then a map $f : X \to Y$ is continuous if and only if f sends convergent sequences in X to convergent sequences in Y.

3.8 Theorem If $f : X \to Y$ is continuous at $x \in X$, and $g : Y \to Z$ is continuous at $f(x) \in Y$, then $g \circ f : X \to Z$ is continuous at x.

Proof. Set $y = f(x)$ and $z = g(y)$. For every neighborhood W of $z \in Z$ there is a neighborhood V of $y \in Y$ such that $g(V) \subseteq W$ (by the continuity of g at y), and a neighborhood U of $x \in X$ such that $f(U) \subseteq V$ (by the continuity of f at x). But this gives $g \circ f(U) \subseteq W$ and shows that $g \circ f$ indeed is continuous at x. □

We make much use of the following

Corollary If f and g are continuous, then so is $g \circ f$.

If $A \subseteq X$ is an arbitrary subset of X, and $i : A \to X$ is the inclusion map $i(x) = x$, then i is a continuous map from the subspace A to X, which is readily seen. If $f : X \to Y$ is a continuous map, then so is $f' = f \circ i : A \to Y$. We call f' *the restriction* of f to A and often write it as $f|_A$.

3.9 Example We write $C(X, \mathbb{R})$ or just $C(X)$ for the space of continuous real functions on a topological space X. $C(X, \mathbb{R})$ is an algebra over $\mathbb{R}$. That is, if f,g are continuous maps, then so are $f + g$, fg and every scalar multiple αf. The quotient f/g is continuous whereever $g(x) \neq 0$.
It is not difficult to verify this directly. But we may rationalize the argument by the following. The function $f + g$, for instance, may be considered to be the composition

$$X \overset{(f,g)}{\to} \mathbb{R}^2 \overset{a}{\to} \mathbb{R}^2$$

where $(f, g) : X \to \mathbb{R}^2$ is given by $(f, g)(x) = (f(x), g(x))$ and a is the usual addition operation on $\mathbb{R}^2$. According to example 3.3 (f, g) is continuous, and by example 3.4 the arithmetical operations are continuous. Hence the composition $a \circ (f, g)$ is continuous.
It now also follows that $f_1 + \ldots + f_n$ and $f_1 \cdots f_n$ are continuous functions on X

whenever $f_1, \ldots, f_n$ are. And the same argument applies to the set $C(X, \mathbb{C})$ of continuous complex functions on X (again 3.3).

3.10 Example We consider the case $X = \mathbb{R}^n$ and the function

$$x \to x_j$$

where x_j is the j-th component of x. This is a linear function on $\mathbb{R}^n$ which we also write as x_j. It follows from 3.5 that x_j is continuous and from 3.9 that the productfunctions

$$x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$$

are continuous, where $i_1, \ldots, i_n$ are integers ≥ 0. If $i_i + i_2 + \ldots + i_n = i$ we have a *monomial* of degree i. A *homogenous polynomial* of degree i or an i-th degree *form* is a linear combination of i-th degree monomials,

$$p(x) = \sum_{i_1 + \ldots + i_n = i} \alpha_{i_1 \cdots i_n} x_1^{i_i} \cdots x_n^{i_n}$$

For $i = 2$ in particular we have the second degree forms or *quadratic forms*. In two variables ($n = 2$) they are simply

$$p(x) = \alpha_{20} x_1^2 + \alpha_{11} x_1 x_2 + \alpha_{02} x_2^2$$

We write $\mathbb{R}[x]$ for the set of all polynomials in the variables $x_1, x_2, \ldots, x_n$. It is clear that $\mathbb{R}[x]$ is a subalgebra of $C(\mathbb{R}^n, \mathbb{R})$.
The quotient of two polynomials $p, q \in \mathbb{R}[x]$ is called a *rational* function. A rational function $r = p/q$ is defined and continuous at all points $x \in \mathbb{R}^n$ where $q(x) \neq 0$ (again 3.9).

3.11 Global continuity may be formulated in quite a simple manner by means of open and closed sets.

Theorem A map $f : X \to Y$ is continuous if and only if the inverse image under f of every open set (in Y) is open (in X), or equivalently, the inverse image of every closed set is closed.

Proof. It follows from the general formula $f^{-1}(Y - B) = X - f^{-1}(B)$ for maps between arbitrary sets that the last two conditions are equivalent. Hence we need only consider the case of open sets.
Assume first that the inverse image under f of an open set always is open. Let $x \in X$ and V be an open neighborhood of $f(x)$. Then $U = f^{-1}(V)$ is an (open)

neighborhood of x. This shows that f is continuous at x and therefore entire X, since x was chosen arbitrarily. Assume, for the converse, that f is continuous and let $W \subseteq Y$ be an open set. Then W is a neighborhood of all its points, so $f^{-1}(W)$ is a neighborhood of all of its points, and thus open. □

It is, however, enough to check continuity of f on open sets from a basis for the topology. (Why?)

3.12 Example Let $M(n) = M(n, n)$ be the space of n-by-n matrices (see example 1.19), and det: $M(n) \to \mathbb{R}$ be the function which to every matrix assigns the determinant of the matrix. If $A = [a_{ij}]$ is a matrix from $M(n)$, then A is a homogenous polynomial of degree n in the matrix elements a_{ij}, which means that $det : M(n) \to \mathbb{R}$ is a continuous function on $M(n)$ (see 3.10). The solution set of the equation

$$\det A = 0$$

is precisely the subset of noninvertible (singular) matrices. It is a closed subset of $M(n)$ which we denote $\Sigma(n)$. The complement, consisting of the invertible (regular or nonsingular) matrices, is then an open subset of $M(n)$. It is called the *general linear group* $GL(n)$. As the name suggests, $GL(n)$ is a group with respect to matrix multiplication. This follows from familiar properties of the determinant function:

(3.13.1) $\det (AB) = \det A \cdot \det B$

(3.13.2) $\det I = 1$ where I is the identity matrix. From the same relations it also follows that the matrices with determinant 1 constitute a subgroup of $GL(n)$, called the *special linear group* and denoted $SL(n)$. Continuity of the determinant shows that $SL(n)$ is closed in $M(n)$ as well as in $GL(n)$.
The matrices with determinant > 0 make a subgroup $GL^+(n)$ of $GL(n)$. We have a disjoint union $GL(n) = GL^+(n) \cup GL^-(n)$, where $GL^-(n)$ are the matrices of determinant < 0 in $GL(n)$. Both $GL^+(n)$ and $GL^-(n)$ are open subsets of $GL(n)$. Consequently $GL(n)$ is splitted in a disjoint union of two open, nonempty subsets. On the other hand, $GL^+(n)$ and $GL^-(n)$ are also closed subsets of $GL(n)$, because they are complements of each other.

3.14 Example A matrix $A \in M(n)$ is called *orthogonal* if A preserves the inner product in $\mathbb{R}^n$, that is, if $Ax \cdot Ay = x \cdot y$ for all x,$y \in \mathbb{R}^n$, or equivalently, if A preserves the norm of the innerproduct, meaning $|Ax| = |x|$ for all $x \in \mathbb{R}^n$. Every orthogonal matrix is invertible, and the orthogonal matrices are a subgroup of $GL(n)$, called the *orthogonal group* and denoted $O(n)$.

It is well-known that orthogonality of $A \in M(n)$ is equivalent to the requirement $A^tA = I$, where A^t is the matrix transpose of A. If we define a map $f : M(n) \to M(n)$ by $f(A) = A^tA$, then the component functions of f are

$$f_{ij}(A) = \sum_k a_{ki}a_{kj}$$

where $A = [a_{ij}]$. These are continuous (again 3.10), hence f is continuous. It follows that $O(n)$ is closed in $M(n)$. In particular, $O(n)$ is a closed subgroup of $GL(n)$.
From the equation $A^tA = I$ it also follows that $(\det A)^2 = 1$, that is, $\det A = \pm 1$. The orthogonal matrices of determinant 1 are a closed subgroup of $O(n)$, called *the special orthogonal group* or *the rotation group*, and denoted $SO(n)$. Since orthogonal matrices have determinant ± 1, it follows that $SO(n)$ is both open and closed in $O(n)$. This gives the splitting $O(n) = O^+(n) \cup O^{-1}(n)$ in openclosed sets, and $O^+(n) = SO(n)$.

3.15 A *homeomorphism* is a map $f : X \to Y$ which is bijective and such that both the map itself and its inverse are continuous. In that case, the inverse map $g : Y \to X$ is also a homeomorphism. Two spaces X and Y are said to be *homeomorphic* if there is a homeomorphism between them.

It is clear that a homeomorphism $f : X \to Y$ gives a unique correspondence between the topologies on X and Y; a set $A \subseteq X$ is open in X if and only if the image set $f(A)$ is open in Y. Alternatively, A is a neighborhood of $x \in X$ if and only if $f(A)$ is a neighborhood of $f(x) \in Y$.
The identity map $id : X \to X$ is a homeomorphism for every topological space X. A composition of homeomorphisms is a homeomorphism as well, a fact which follows from corollary 3.8.

3.16 Continuity is not the only interesting topological condition which a map may be subject to. A map between topological spaces is said to be *open* if it maps open sets to open sets. It is, yet again, enough that it maps open sets from a basis for the topology to open sets (since the image of a union is the union of the images). A map is *closed*if it maps closed sets to closed sets. A homeomorphism is then a bijective, continuous and open (closed) map.
If $f : X \to Y$ maps X homeomorphically onto the subspace $f(X)$ in Y, then f is called a (topological) *embedding* of X in Y. An injective, continuous and open (closed) map is therefore an embedding.
A *local homeomorphism* is a map $f : X \to Y$ such that every point $x \in X$ has an open neighborhood which is sent homeomorphically by f onto an open neighborhood of $f(x)$. It is clear that a local homeomorphism is an open map,

and even an open embedding if it is injective also.
We give some examples on homeomorphisms.

3.17 Example The closed n-cell $D^n = \{x \in \mathbb{R}^n : |x| \leq 1\}$ and the closed, upper half of the sphere, $S^n_+ = \{x \in S^n : x_{n+1} \geq 0\}$ are homeomorphic spaces. We have the projection $p : S^n_+ \to D^n$ defined by $p(x_1, \ldots, x_{n+1}) = (x_1, \ldots, x_n)$ and the map $q : D^n \to S^n_+$ by $q(x_1, \ldots, x_n) = (x_1, \ldots, x_n, (1 - \sum_{i=1}^n x_i^2)^{\frac{1}{2}})$ which are continuous maps (why?) and mutually inverse (check), i.e. they are homeomorphisms.

3.18 Example We will show that the open n-cell $E^n = \{x \in \mathbb{R}^n : |x| < 1\}$ is homeomorphic to $\mathbb{R}^n$. Let $f : E^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to E^n$ be the radial projections defined by

$$f(x) = \frac{x}{1 - |x|}, \qquad x \in E^n$$

$$g(y) = \frac{y}{1 + |y|}, \qquad y \in \mathbb{R}^n$$

It is readily checked that f and g are continuous and the compositions $g \circ f$ and $f \circ g$ are identity maps. So f and g are mutually inverse homeomorphisms.

3.19 Example The map $f : S^1 \to SO(2)$ given by

$$f(e^{i\theta}) = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

is an isomorphism between the multiplicative circle group $S^1 \subset \mathbb{C}$ and the rotation group $SO(2)$. Furthermore, f is a homeomorphism since its inverse
$g : SO(2) \to S^1$ given by $g(A) = Ae_1$ is continuous. (Explain.)
Instead of expressing f in polar coordinates we can express f in Cartesian coordinates. For a point $(x, y) \in S^1$ (such that $x^2 + y^2 = 1$) we have $x = \cos\phi$, $y = \sin\phi$ and thus

$$f(x, y) = \begin{bmatrix} x & -y \\ y & x \end{bmatrix}$$

3.20 Example Every linear isomorphism $A : \mathbb{R}^n \to \mathbb{R}^n$ is a homeomorphism. A is bijective and the inverse A^{-1} is also linear. Then both A and A^{-1} are continuous (example 3.5).
We may exploit this to topologize all finite dimensional real vector spaces V. Every such space is isomorphic to $\mathbb{R}^n$, for appropriate n. We choose an isomorphism $S : \mathbb{R}^n \to V$ and transport the standard topology on $\mathbb{R}^n$ to V by using S. Indeed,

we say that a set $O \subseteq V$ is open if $S^{-1}(O)$ is open in $\mathbb{R}^n$. It is clear that this provides a topology τ on V. But to what extent does τ depend on the choice of the map S?
Let $\tau = \tau(S)$, and let $\tau(T)$ be the topology defined in the same manner, by using the map $T : \mathbb{R}^n \to V$. Now let $O \subseteq V$ be an arbitrary set from $\tau(S)$. This means that $S^{-1}(O)$ is an open set in $\mathbb{R}^n$. What about $T^{-1}(O)$? We have $T^{-1}(O) = T^{-1}(S(S^{-1}(O))) = A(S^{-1}(O))$, where $A = T^{-1}S$. But A is an isomorphism of $\mathbb{R}^n$ to itself, and thus a homeomorphism on $\mathbb{R}^n$. So A is sending open sets to open sets. Consequently, $T^{-1}(O)$ is an open set in $\mathbb{R}^n$, which means that O belongs to the topology $\tau(T)$. This shows that $\tau(S) \subseteq \tau(T)$. By symmetrical reasoning we also get $\tau(T) \subseteq \tau(S)$, which now ultimately gives us $\tau(T) = \tau(S)$.
We are led to conclude that all isomorphisms $\mathbb{R}^n \to V$ transport the topology on $\mathbb{R}^n$ to the same topology on V, which we name the *standard topology* on V.

3.21 Example We will show that the sphere S^n minus one point is homeomorphic to the Euclidean space $\mathbb{R}^n$. Because of the rotation symmetry of the sphere, the spaces $S^n - \{a\}$ for $a \in S^n$ are all homeomorphic. More precisely, the transformations in $O(n+1)$ are homeomorphisms of S^n onto itself, and by orthogonal transformations we may map every point of S^n to any other point. Therefore, we may choose $a = (0, \ldots, 1) \in S^n$. We define the map $\pi : S^n \to \mathbb{R}^n$, *stereographic projection* from a, as following: for $x \in S^n$, $x \neq a$, we set $\pi(x)$ to be the point of intersection of the line through a and x, with the subspace $\mathbb{R}^n$ of $\mathbb{R}^{n+1}$.
We have here identified $\mathbb{R}^n$ with a subspace of $\mathbb{R}^{n+1}$ by the injection $y \to (y, 0)$. Geometrically, it is clear that π is a homeomorphism of $S^n - \{a\}$ onto $\mathbb{R}^n$. The inverse homeomorphism $\rho : \mathbb{R}^n \to S^n - \{a\}$ is acquired by a similar geometrical construction. For $y \in \mathbb{R}^n$, $\rho(y)$ is the point of intersection of the line through y and a with the sphere S^n.
The maps π and ρ may also be defined analytically. If $p : \mathbb{R}^{n+1} \to \mathbb{R}^n$ denotes the projection $p(x_1, \ldots, x_{n+1}) = (x_1, \ldots, x_n)$, we set

$$\pi(x) = \frac{p(x)}{1 - x_{n+1}}, \qquad x \in S^n, x \neq a$$

and

$$\rho(y) = \frac{(2y, |y|^2 - 1)}{|y|^2 + 1} \qquad y \in \mathbb{R}^n$$

It is straightforward to verify that π and ρ are mutually inverse maps. Since both are continuous, they are homeomorphisms between $S^n - \{a\}$ and $\mathbb{R}^n$.

3.22 Example The set $(\mathbb{R}^n)^* = \mathbb{R}^n \cup \{\omega\}$, where ω is an element which doesn't belong to $\mathbb{R}^n$, can be topologized such that it is homeomorphic to the sphere S^n, and such that the subspace $\mathbb{R}^n$ becomes identical to the standard space $\mathbb{R}^n$. (In other words, the subspace topology on $\mathbb{R}^n$ inherited from $(R^n)^*$is identical with the standard topology).
Let U be an arbitrary (standard) open set in $\mathbb{R}^n$ and V be an open set in $\mathbb{R}^n$ with bounded complement, that is, $\mathbb{R}^n - V$ is a bounded, closed set in $\mathbb{R}^n$. Set $V^* = V \cup \omega$. Let τ be the class of all sets U and V^* in $(\mathbb{R}^n)^*$. Then τ is a topology on $(\mathbb{R}^n)^*$ which induces the standard topology on the subset $\mathbb{R}^n$ (check) and such that $(\mathbb{R}^n)^*$ with τ becomes homeomorphic to the sphere S^n. For we may extend the stereographic projection π (see 3.21) to a continuous map $\pi^* : S^n \to (\mathbb{R}^n)^*$ by setting $\pi^*(a) = \omega$. By definition of the topology on $(\mathbb{R}^n)^*$ it follows that π^* is also continuous at the point a. (Check.) The inverse map ρ may be extended to a map $\rho^* : (\mathbb{R}^n)^* \to S^n$ by $\rho^*(\omega) = a$. Thus ρ^* is also continuous. This shows that $(\mathbb{R}^n)^*$ is homeomorphic to S^n.

3.23 Homeomorphisms are the structure preserving maps between topological spaces. The term homeomorphism plays a similar role in topology as the term congruence does in geometry. We write $X \approx Y$ to express that X and Y are homeomorphic. This defines an equivalence relation between topological spaces. The equivalence classes are called *homeomorphism types*. A central task in topology is to determine or classify topological spaces according to homeomorphism type. Like we mentioned earlier, this task, to its full extent, is far too comprehensive to be of particular interest. It becomes more interesting when we restrict ourselves to some reasonable class of spaces. From a geometrical point of view, the locally Euclidean spaces are among the most interesting. A topological space is *locally Euclidean* of dimension n if every point has an open neighborhood homeomorphic to an open set in $\mathbb{R}^n$. If the space in addition is Hausdorff and has a countable basis for the topology, we call it an *n-manifold*. A manifold X may be covered by *local parametrizations* $\phi : U \to X$ where U is open in $\mathbb{R}^n$, $\phi(U)$ is open in X and ϕ sends U omeomorphically onto $\phi(U)$. Alternatively, X may be covered by open sets W, *charts*, homeomorphic to open sets in $\mathbb{R}^n$ by maps $\phi : W \to \mathbb{R}^n$. $\mathbb{R}^n$ is itself a manifold, charted by the identity map. It follows from 3.21 that S^n is an n-manifold, charted by stereographic projections, and that two maps suffice to cover S^n. Why can't one chart do the job?
In the following we shall study manifolds mainly through examples.

3.24 We end this chapter by making a few observations regarding the concept of continuity. The first observation is that continuity is a *local* concept. Whether or not $f : X \to Y$ is continuous at a point $x \in X$ depends only on the topology on X in an arbitrary neighborhood V of x: If $f|_V$ is continuous at x, then f is continuous

at x. If, therefore, $(U)_{i\in J}$ is an open covering of X and $f|_{U_i}$ is continuous for every i, then f is continuous.
The same is not generally true for closed coverings. Consider, for instance, a function $f : \mathbb{R} \to R$ and let $(F_x)_{x\in\mathbb{R}}$ be the closed covering of $\mathbb{R}$ where $F_x = \{x\}$. Then $f|_{F_x}$ is continuous for every x no matter what f is. However, we have the following weaker, yet somewhat useful, version for closed coverings.

Theorem If $(F_k)_{k=1,\dots,n}$ is a finite closed covering of X, and $f : X \to Y$ such that $f|F_i$ is continuous for all i, then f is continuous.

Proof. Let B be an arbitrary closed subset of Y. We show that $f^{-1}(B)$ is closed in X. We have

$$f^{-1}(B) = \bigcup_{k=1}^{m}(f^{-1}(B) \cap F_k) = \bigcup_{k=1}^{m} f_k^{-1}(B)$$

where $f_k = f|F_k$, $k = 1, \dots, n$. Since we are dealing with a finite union, it's enough to show that $f^{-1}(B)$ is closed in X for every k. And since $f^{-1}(B)$ is contained in F_k, and F_k is closed in X, it is also enough to show that $f_k^{-1}(B)$ is closed in F_k. But that follows from the fact that $f_k : F_k \to Y$ is continuous. □

3.25 The term continuity was first given a precise definition by Bernhard Bolzano in an article from 1817, in which there was given a proof of the intermediate value theorem for real functions of one real variable. However, Bolzano's article was not duely noted. Continuity entered the spotlight with Cauchy's "Cours d'Analyse" from 1821. But even here the definition was vague and imprecise according to our rigorous standards. The modern $\epsilon - \delta$-formulation of the definition of limit was first discussed in an article from 1872, by H.E. Heine, but was supposedly used earlier by Weierstrass in his unpublished lecture series from 1861.

Chapter 4

Products

Product spaces are important constructions in topology. The intention is, as with the introduction of $\mathbb{R}^n$, to be able to handle functions of several variables, among other things.

4.1 If X and Y are topological spaces, the product set $X \times Y$ (which consists of all pairs (x, y), where $x \in X$ and $y \in Y$) can be topologized in the following way. We give $X \times Y$ the topology which has as basis the system of product sets $U \times V$, where U and V are open sets in X and Y. This topology is called *the product topology*, and the space $X \times Y$ *the product space*. An open set in the product space is thus a set which is a union of (open) product sets $U \times V$. Note that the class of product sets $U \times V$ allows finite intersections, which follows from the general formula

$$(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2).$$

That is more than we require from the criteria regarding a basis (see 1.12). Furthermore, it is enough that U and V are sets from each of their bases of open sets in X and Y. Then the corresponding sets $U \times V$ still make a basis for the product topology. (Explain.)
The construction of the product space is easily generalized from the product of two spaces, to the product of a finite number of spaces $X_1, \ldots, X_n$. We give the product set $X = X_1 \times \cdots \times X_n = \Pi_{i=1}^n X_i$ the product topology consisting of all unions of product sets $U = U_1 \times \cdots U_n$ where U_i is open in X_i for every i. The product topology on X has the class of sets $U = U_1 \times \cdots U_n$ as basis. Also here it is enough to use open sets U_i from a basis for the topologies on the sets X_i. It is also worth noticing that the product sets

$$\hat{U}_i = X_1 \times \cdots X_{i-1} \times U_i \times X_{i+1} \times \cdots X_n \subseteq X,$$

(for varying i and U_i open in X_i) are generator sets for the product topology, i.e. the class of all finite intersections of sets $\hat{U}_i$ is a basis. This follows from the

formula

$$\hat{U}_1 \cap \cdots \cap \hat{U}_n = U_1 \times \cdots \times U_n$$

(Explain.)

The product of topological spaces is associative in the sense that the spaces $(X \times Y) \times Z$ and $X \times (Y \times Z)$ are canonically homeomorphic. For they are homeomorphic to the space $X \times Y \times Z$. (Explain.)

4.2 The standard topology on $\mathbb{R}$ has a basis consisting of intervals (a, b). In the product space $\mathbb{R}^n = \mathbb{R} \times \cdots \times \mathbb{R}$ we therefore get a basis consisting of "boxes" $(a_1, b_1) \times \cdots \times (a_n, b_n)$. This basis yields the same topology as the basis of supremum neighborhoods $U_{\sup}(x; \epsilon)$ for varying x and ϵ. (Why?) The product topology on $\mathbb{R}^n$ therefore coincides with the topology defined by the supremum metric, that is, with the standard topology on $\mathbb{R}^n$.
Generally, the product topology on $\mathbb{R}^n = \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_r}$ where $n = n_1 + \ldots + n_r$ coincide with the standard topology on $\mathbb{R}^n$. As basis neighborhoods we may, in both these topologies, use the open boxes in $\mathbb{R}^n$.

4.3 We can now reformulate the Hausdorff requirement introduced earlier in section 2.6.

Theorem A topological space X is a Hausdorff space if and only if the diagonal Δ is closed in $X \times X$.

Proof. The diagonal Δ in $X \times X$ consists of all points (x, y) where $y = x$. X being Hausdorff means that for all points x,$y \in X$ where $y \neq x$, there exist disjoint open neighborhoods U and V of x and y. This is equivalent to every point $(x, y) \in X \times Y$ outside of Δ having an open neighborhood $U \times V$ disjoint with Δ. The complement of this neighborhood is a closed set which contains Δ, and the intersection of all such is Δ. (Why?) This means that Δ is closed in $X \times X$. We leave it to the reader to reverse these conclusions. □

4.4 Assume that A and B are subspaces of X and Y. Then the product topology on $A \times B$ will be identical to the subspace topology from $X \times Y$. This follows from the formula

$$(U \times V) \cap (A \times B) = (U \cap A) \times (V \cap B).$$

If we choose U and V amongst all open sets in X and Y, we get on the left-hand side all sets in a basis for the subspace topology on $A \times B$, and on the right-hand

side all sets in a basis for the product topology on $A \times B$. The two topologies are therefore identical.

4.5 Let M and N be manifolds of dimensions m and n. Every point $x \in M$ has an open neighborhood U homeomorphic to an open set U' in $\mathbb{R}^m$, and every point $y \in N$ has an open neighborhood V homeomorphic to an open set V' in $\mathbb{R}^n$. It follows that $U \times V$ is an open neighborhood of $(x, y) \in M \times N$ and that $U \times V$ is homeomorphic to the open set $U' \times V'$ in $\mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}$. Therefore, $M \times N$ is a manifold of dimension $m + n$. Hence the class of manifolds allows construction of products. Some simple examples on product manifolds are the cylinder surface $S^1 \times \mathbb{R}$ and the torus surface $S^1 \times S^1$. The former appears naturally as a subspace of $\mathbb{R}^2 \times \mathbb{R} = \mathbb{C} \times \mathbb{R}$. The latter is a subspace of $\mathbb{R}^2 \times \mathbb{R}^2 = \mathbb{C} \times \mathbb{C}$, but can be embedded topologically in $\mathbb{R}^3$, that is, it has a homeomorphic image in $\mathbb{R}^3$. One such embedding in $\mathbb{R}^3$ may be defined as

$$(e^{i\theta}, e^{i\psi}) \to ((R + r\cos\psi)\cos\phi, (R + r\cos\psi)\sin\phi, r\sin\psi)$$

where $0 < r < R$.

4.6 The projections $pr_1 : X \times Y \to X$ and $pr_2 : X \times Y \to Y$ are continuous. For if U is an open set in X, then by definition the product topology $U \times Y = pr_1^{-1}(U)$ is an open set in $X \times Y$. This shows that pr_1 is continuous, and analogously for pr_2.
For arbitrary $U \subseteq X$ and $V \subseteq Y$ we have $pr_1^{-1}(U) = U \times Y = \hat{U}$ and $pr_2^{-1}(V) = X \times V = \hat{V}$ using our earlier notation (4.1). This means that the product topology on $X \times Y$ is generated by the inverse images $pr_1^{-1}(U)$ and $pr_2^{-1}(V)$ for open sets U and V in X and Y. From this, it follows that the product topology is the coarsest topology on $X \times Y$ which makes the projections pr_1 and pr_2 continuous. For a topology τ on $X \times Y$ which makes the projections continuous must have all inverse images $pr_1^{-1}(U)$ and $pr_2^{-1}(V)$ as open sets. Since the product topology is generated by these inverse images, it must be coarser than the topology τ.
The same applies to products of several spaces. The product topology on $X = \Pi_{i=1}^n X_i$ is generated by the inverse images $\hat{U}_i = pr_i^{-1}(U_i)$ where U_i is open in X_i, and therefore the coarsest topology on X making the projections $pr_i : X \to X_i$ continuous.
This suggests a natural definition of a topology on infinite product spaces. If $X = \Pi_{\alpha \in J} X_\alpha$ is the product of a family of topological spaces X_α indexed over an arbitrary index set J, finite or infinite, then we give X the topology generated by the sets $\hat{U}_\alpha = pr_\alpha^{-1}(U_\alpha)$, where $U_\alpha \subseteq X_\alpha$ is open and $\alpha \in J$. This is then the coarsest topology on X making the projections $pr_\alpha : X \to X_\alpha$ continuous. We note that the product topology has a basis consisting of all product sets $\Pi_\alpha U_\alpha$ where $U_\alpha \subseteq X_\alpha$ is open and *equals* X_α except for finitely many indices α.

The projections of a product space (finite or infinite) are always open maps since the open basis sets $\Pi_\alpha U_\alpha$ in the product space project onto the open sets U_α for each of the factors. But on the other hand, the projections are not necessarily closed. For instance, the closed set $A = \{(x, y) \in \mathbb{R}^2 : xy = 1\}$ will, by projection to the first factor, be mapped to $pr_1(A) = \{x \in \mathbb{R} : x \neq 0\}$, which is not closed in $\mathbb{R}$.

4.7 If $f : X \times Y \to Z$ is a continuous map from the product space of two spaces into a topological space, then the map $x \to f(x, y)$ of X to Z is continuous for all $y \in Y$. This is a consequence of the fact that it is the composition of the continuous maps $x \to (x, y)$ and $(x, y) \to f(x, y)$. Analogously we see that the map $y \to f(x, y)$ of Y to Z is continuous for all $x \in X$. We may express this shortly by saying that a continuous function of two variables is *partially continuous* in each of the variables. The same is the case for functions defined on arbitrary product spaces.
However, the converse does not apply. A counterexample would be the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{for } (x, y) \neq (0, 0) \\ 0 & \text{for } (x, y) = (0, 0) \end{cases}$$

f is continuous in each variable separately, but discontinuous at the origin, since $f(x, x) = \frac{1}{2}$ for $x \neq 0$ and $f(0, 0) = 0$.

4.8 Let $f : Z \to X$ be a map of a space Z to a product space $X = \Pi_\alpha X_\alpha$. The component maps of f are given by $f_\alpha = pr_\alpha \circ f$. In the finite case we have $f(z) = (f_1(z), \ldots, f_n(z))$ for all $z \in Z$, and we write $f = (f_1, \ldots, f_n)$.
If f is continuous, then naturally each f_α is continuous, since the composition of two continuous maps is continuous. The interesting is, however, that the converse also holds.

Theorem A map to a product space is continuous if and only if the component maps are continuous.

Proof. Assume $f_\alpha = pr_\alpha \circ f$ is continuous for each α. If U_α is an open set in X_α, then the set

$$f^{-1}(pr_\alpha^{-1}(U_\alpha)) = (pr_\alpha \circ f)^{-1}(U_\alpha) = f_\alpha^{-1}(U_\alpha)$$

is open in Z. Since the sets $pr_\alpha^{-1}(U_\alpha)$ generate the product topology on X, it follows that the inverse image under f of *every* open set in X is an open set in Z (We make use of the fact that the inverse image of a map respects intersections and unions). Therefore f is continuous. Hence the theorem follows. □

Corollary Assume that $f,g : Z \longrightarrow X$ are continuous maps of a topological space Z to a Hausdorff space X. Then the set $\{z \in Z : f(z) = g(z)\}$ is closed in Z.

Proof. $(f, g) : Z \longrightarrow X \times X$ is continuous by the theorem and the diagonal Δ is closed in $X \times X$ by 4.3. From 3.10 it then follows that

$$\{z \in Z : f(z) = g(z)\} = (f, g)^{-1}(\Delta)$$

is closed in Z. □

4.9 Example In the ring $M(n)$ of real (n, n)-matrices the ring operations addition and multiplication give rise to two maps $\alpha,\mu : M(n) \times M(n) \longrightarrow M(n)$, i.e. $\alpha(A, B) = A + B$ and $\mu(A, B) = AB$. We defined the topology on $M(n)$ by identifying $M(n)$ with $\mathbb{R}^{n^2}$. For $A = [a_{ij}]$ and $B = [b_{ij}]$, the component maps of α and μ are given by

$$\alpha_{ij}(A, B) = a_{ij} + b_{ij}$$

$$\mu_{ij}(A, B) = \sum_k a_{ik} b_{kj}$$

As polynomials, α_{ij} and μ_{ij} are continuous (3.10). But then α and μ are continuous, too, according to theorem 4.8.
The operation of matrices in $M(n)$ on vectors in $\mathbb{R}^n$ gives a map $\mu : M(n)\times\mathbb{R}^n \longrightarrow \mathbb{R}^n$, $\mu(A, x) = Ax$. We get the component maps

$$\mu_i(A, x) = \sum_j a_{ij} x_j$$

This shows that μ is continuous.

4.10 Example For the general linear group $GL(n)$ of invertible matrices in $M(n)$ the multiplication $\mu : GL(n) \times GL(n) \longrightarrow GL(n)$ is continuous, as a restriction of the continuous map $\mu : M(n) \times M(n) \longrightarrow M(n)$ from the previous example. The matrix inversion map $\iota : GL(n) \longrightarrow GL(n)$ defined by $\iota(A) = A^{-1}$ is continuous. This we see from the fact that ι has component maps given by

$$\iota_{ij}(A) = \frac{(-1)^{i+j} det A^{ji}}{det A}$$

where A^{ji} is the cofactor matrix obtained from A by removing the j-th row and i-th column. As a rational function of the matrix elements, this function depends continuously on A (3.9). Therefore ι is continuous. $GL(n)$ is therefore an example of a topological group, that is, a group with a topology which makes the group operations, multiplication and inversion. The same automatically applies to subgroups of $GL(n)$, such as $SL(n)$ and $O(n)$.

4.11 Example Let $f : SL(n) \times R_+ \longrightarrow GL^+(n)$ be the group isomorphism defined by $f(A, \lambda) = \lambda A$ with inverse $g : GL^+(n) \longrightarrow SL(n) \times R_+$ given by $g(B) = ((\det B)^{-1/n}B, (\det B)^{1/n})$. We see, componentwise, that f and g are continuous. Therefore f and g are mutually inverse homeomorphisms. Thus $SL(n) \times R_+$ and $GL^+(n)$ are homeomorphic spaces.

4.12 Example Denote by $PT(2)$ the set of all real $(2,2)$-matrices on the form

$$C = \begin{bmatrix} c_{11} & c_{12} \\ 0 & c_{22} \end{bmatrix}$$

with positive diagonal elements (i.e. the set of positive uppertriangular $(2,2)$-matrices). It is straightforward to verify that $PT(2)$ is a subgroup of $GL(2)$.
We will show that every matrix $A \in GL(2)$ may be written uniquely on the form $A = BC$ where $B \in O(2)$ and $C \in PT(2)$. Let $e_1, e_2 \in \mathbb{R}^2$ be the standard vectors. Then $v_1 = Ae_1$, $v_2 = Ae_2$ is a basis in $\mathbb{R}^2$ which by Gram-Schmidt's construction yields an orthonormal basis f_1, f_2 in $\mathbb{R}^2$. Explicitly we have

$$f_1 = \frac{v_1}{|v_1|} \qquad f_2 = \frac{v_2 - (v_2 \cdot f_1)f_1}{|v_2 - (v_2 \cdot f_1)f_1|}$$

In this construction $v_1 \cdot f_2 = 0$ and $v_i \cdot f_i > 0$ for $i = 1, 2$.

We now let $B \in O(2)$ be defined by $Be_i = f_i$ for $i = 1, 2$ and set $C = B^{-1}A$. Then we have, for $C = (c_{ij})$

$$c_{ij} = e_i \cdot Ce_j = e_i \cdot B^{-1}v_j = e_i \cdot B^{-1}((v_j \cdot f_1)f_1 + (v_j \cdot f_2)f_2) = v_j \cdot f_i$$

Therefore $c_{21} = 0$ and $c_{ii} > 0$ for $i = 1, 2$, that is $C \in PT(2)$. This gives $A = BC$ with $B \in O(2)$ and $C \in PT(2)$ as proposed. The uniqueness of this decomposition follows from the fact that $O(2)$ and $PT(2)$ are subgroups of $GL(2)$ with $O(2) \cap PT(2) = \{I\}$. For if $B_1C_1 = B_2C_2$ with $B_1, B_2 \in O(2)$ and $C_1, C_2 \in PT(2)$, then it follows that $B_2^{-1}B_1 = C_2C_1^{-1} = I$, that is $B_1 = B_2$ and $C_1 = C_2$.
The map $\mu : O(2) \times PT(2) \longrightarrow GL(2)$ defined by $\mu(B, C) = BC$ is bijective. The inverse map $\mu^{-1} : GL(2) \longrightarrow O(2) \times PT(2)$ is given by $\mu^{-1}(A) = (B, C)$ where B and C are defined according to the previous decomposition. From 4.9 we know that μ is continuous. Also, the basis vectors f_1, f_2 constructed by using the Gram-Schmidt process, depend continuously on the initial basis vectors v_1, v_2. This is clear from the above formulae. Therefore B depends continuously on A. The same applies to $C = B^{-1}A$ by 4.9 and 4.10. Hence μ^{-1} is continuous by 4.8. This means that μ is a homeomorphism. (But note that μ is not an isomorphism between groups, since $B \in O(2)$ and $C \in PT(2)$ don't always commute).

The space $PT(2)$ is obviously homeomorphic to $\mathbb{R}^2_+ \times \mathbb{R}$ and therefore with $\mathbb{R}^3$. So we have a homeomorphism

$$GL(2) \approx O(2) \times \mathbb{R}^3$$

In the decomposition $A = BC$ we have $\det C > 0$. Therefore $A \in GL^+(2)$ if and only if $B \in SO(2)$. This gives a homeomorphism

$$GL^+(2) \approx SO(2) \times \mathbb{R}^3 \approx S^1 \times \mathbb{R}^3$$

We get that $A \in SL(2)$ if and only if $B \in SO(2)$ and $\det C = 1$. Since $\{C \in PT(2) : \det C = 1\} \approx \mathbb{R}_+ \times \mathbb{R} \approx \mathbb{R}^2$ we also get a homeomorphism

$$SL(2) \approx SO(2) \times \mathbb{R}^2 \approx S^1 \times \mathbb{R}^2$$

This method is easily generalized to $\mathbb{R}^n$ and then yields a homeomorphism $\mu : O(n) \times PT(n) \longrightarrow GL(n)$ where $PT(n)$ denotes the space of positive uppertriangular (n, n)-matrices. Then we get homeomorphisms $GL(n) \approx O(n) \times \mathbb{R}^{n(n+1)/2}$ and $GL^+(n) \approx SO(n) \times \mathbb{R}^{n(n+1)/2}$. We also get a homeomorphism $SL(n) \approx SO(n) \times \mathbb{R}^{n(n+1)/2}$.
These results mean that we transfer the topological "appearance" of the spaces $GL(n)$ and $SL(n)$ back to the space $SO(n)$. Topologically speaking it's the factor $SO(n)$ which is the interesting part in the products; the Euclidean spaces don't do anything but increase dimensions. We shall study the structure of the space $SO(n)$ in greater detail. For now, we will remark that the homeomorphism between $SO(2)$ and S^1 is most particular for dimension $n = 2$. For $n > 2$ the spaces $SO(n)$ are not homeomorphic to spheres.

4.13 We include a useful result regarding convergence in product spaces. Let $X = \Pi_\alpha X_\alpha$ be a product space and $(x_n)_{n=1,2,\ldots}$ a sequence in X. We then get a component sequence $(x_{n\alpha})_{n=1,2,\ldots}$ in X_α for every α, where $x_{n\alpha} = pr_\alpha(x_n)$.

Theorem A sequence in a product space converges if and only if each of the component sequence converge. More precisely, $x_n \to a$ when $n \to \infty$ if and only if $x_{n\alpha} \to a_\alpha$ when $n \to \infty$ for every α.

Proof. Assume $x_n \to a$ in X. Continuity of the projection $pr_\alpha : X \longrightarrow X_\alpha$ gives $x_{n\alpha} \to a_\alpha$. Assume conversely that $x_{n\alpha} \to a_\alpha$ in X_α for every α. Let $U = \Pi_\alpha U_\alpha$ be a basis neighborhood of a, that is, U_α is a neighborhood of a_α and $U_\alpha = X_\alpha$ except for finitely many α. By the assumption of convergence, there exists N_α such that $x_{n\alpha} \in U_\alpha$ for $n > N_\alpha$, and we may choose $N_\alpha = 0$ except for finitely many α. Let $N = \max_\alpha N_\alpha$. Then obviously $n > N$ implies that $x_{n\alpha} \in U_\alpha$ for all α, that is $x_n \in U$. This means that $x_n \to a$.

4.14 The product space X is Hausdorff if and only if each factor X_α is Hausdorff (we assume all $X_\alpha \neq \emptyset$). This follows from the fact that two distinct points in X may already be separated by disjoint generator sets $\hat{U}_\alpha$ and $\hat{V}_\alpha$ for appropriate α. In that case, the limit for a sequence is unique if it exists. We may then use limes-notation and write theorems more briefly: $\lim_{n\to\infty} x_n = a$ if and only if $\lim_{n\to\infty} x_{n\alpha} = a_\alpha$ for every α.

4.15 The definition of product topology for infinite product spaces stems from Tychonov 1930. Tychonov's work brought new insight especially regarding function spaces. A real function f on set X may alternatively be regarded as a family $(f(x))_{x\in X}$ indexed over X. The function set $F(X,\mathbb{R})$ is then identical to the set of family of real numbers indexed over X, which we also write $\Pi_{x\in X}\mathbb{R}_x$ ($\mathbb{R}_x = \mathbb{R}$) or simply R^X. It is readily seen that under this identification a "pointwise" basis neighborhood $U(f;A,\epsilon)$ corresponds precisely to the standard basis neighborhood $\Pi_{x\in X}U_x$ in the product topology, where $U_x = U(f(x);\epsilon)$ for $x \in A$ and $U_x = \mathbb{R}$ for $x \in X - A$. This means that the topology for pointwise convergence on $F(X,\mathbb{R}) = \mathbb{R}^X$ is identical to the product topology.

Chapter 5

Quotient spaces

In this chapter we deal with yet another general topological construction with many important applications, the forming of a *quotient space*. A quotient space is constructed by way of an identification of elements in a given space. That is, points which are considered equivalent for some purpose, are merged into a single point. We begin by an example where such an identification of points is natural.

5.1 Example A function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is called *periodic* with the number $\omega > 0$ being its *period* if $f(x+\omega) = f(x)$ for all $x \in \mathbb{R}$. In which case also $f(x+n\omega) = f(x)$ for all $x \in \mathbb{R}$, n being an arbitrary integer. In this situation we may define all points of the form $x + n\omega$ for a given $x \in \mathbb{R}$ and arbitrary $n \in \mathbb{Z}$ to be equivalent. We let the map $q : \mathbb{R} \longrightarrow S^1$ from the real line to the unit circle in the complex plane be defined by

$$q(x) = e^{2\pi i x/\omega}$$

The map q is surjective and identifies precisely the equivalent points in $\mathbb{R}$, namely $q(x) = q(x')$ if and only if $x' = x + n\omega$ for some $n \in \mathbb{Z}$. Thus the periodic function f gives rise to a new function $\hat{f} : S^1 \longrightarrow \mathbb{R}$ defined by

$$\hat{f}(q(x)) = f(x) \tag{5.1.1}$$

for all $x \in \mathbb{R}$. Conversely, every function $\hat{f} : S^1 \longrightarrow \mathbb{R}$ will define a periodic function $f : \mathbb{R} \longrightarrow \mathbb{R}$ with period ω by equation (5.1.1). The assignment $f \to \hat{f}$ is therefore a bijection on the periodic functions with period ω on $\mathbb{R}$ and the functions on S^1.
The function $\hat{f}$ is continuous on S^1 if and only if f is continuous on $\mathbb{R}$. It is enough to compare the graphs of the functions f and $\hat{f}$ to realize that this is true, but we may formally verify this as follows.
One direction is obvious: if $\hat{f}$ is continuous, then f is continuous since we have

$f = \hat{f} \circ q$ and the map q is continuous. For the other direction, assume that f is continuous and let W be an open set in $\mathbb{R}$. We must show that $V = \hat{f}^{-1}(W)$ is open in S^1. We have $q^{-1}(V) = q^{-1}(\hat{f}^{-1}(W)) = (\hat{f} \circ q)^{-1}(W) = f^{-1}(W)$, so $q^{-1}(V)$ is open in $\mathbb{R}$ by the continuity of f. It is enough to show that V is open in S^1 when $q^{-1}(V)$ is open in $\mathbb{R}$. But this is true since q is a surjective open map. Hence $\hat{f}$ is continuous.
Thus we also have a bijection $f \to \hat{f}$ between the continuous ω-periodic functions on $\mathbb{R}$ and the continuous functions on S^1.
The main ingredient in the above argument is the surjective continuous map $q : \mathbb{R} \longrightarrow S^1$, which has the property that $V \subseteq S^1$ is open if $q^{-1}(V) \subseteq \mathbb{R}$ is open. We shall use these properties for construction of topological quotient spaces in general.

5.2 We may identify elements of a set X by introducing an equivalence relation R in X and obtain the quotient set X/R, where the elements, per definition, are the equivalence classes of R. Elements in each equivalence class in X are thus "identified", that is, they give rise to one element in X/R.
If X is a topological space, we wish to define a topology on the quotient set X/R. There is a natural way of doing this. Let $p : X \longrightarrow X/R$ be the canonical map which to every element in X assigns the corresponding equivalence class. A set V in X/R is defined to be open if the inverse image $p^{-1}(V)$ is an open set in X. The class of open sets in X/R will then meet the requirements for a topology, since forming the inverse image preserves intersection and union. The topological space X/R we obtain this way is called the *quotient space* of X with respect to R.
The canonical map $p : X \longrightarrow X/R$ is now continuous, since inverse images under p of open sets are open. The quotient topology may therefore be characterized as the finest topology on X/R which makes p continuous.
Furthermore, we have that the open sets in X/R are images under p of the open, *saturated* sets in X, that is, the open sets in X which are unions of equivalence classes. For if V is an open set in X/R, then $p^{-1}(V)$ is an open, saturated set in X, and we have $V = p(p^{-1}(V))$. If on the other hand U is an open, saturated set in X, then $p^{-1}(p(U)) = U$ is open in X, and therefore $p(U)$ is open in X/R.
Complementarily, we have that a set B is closed in X/R if and only if $p^{-1}(B)$ is closed in X. The closed sets in X/R are therefore the images under p of the closed, saturated sets in X.

5.3 An important property regarding quotient construction is the following condition for continuity of a map *from* a quotient space. As before, $p : X \longrightarrow X/R$ is the canonical map.

Theorem A map $g : X/R \longrightarrow Y$ is continuous if and only if the composition $g \circ p : X \longrightarrow Y$ is continuous.

Proof. One part of the theorem follows from the fact that the composition of two continuous maps is continuous. For the other part, assume that $g \circ p$ is continuous. If W is an open set in Y, then $p^{-1}(g^{-1}(W)) = (g \circ p)^{-1}(W)$ is open in X, that is, $g^{-1}(W)$ is open in X/R. This means that g is continuous. □

5.4 We say that a map $f : X \longrightarrow Y$ is *compatible* with an equivalence relation R in X if f is constant on each equivalence class in X. In that case we get an *induced* map $\hat{f} : X/R \longrightarrow Y$ given by $\hat{f} \circ p = f$. This means that f may be factorized via X/R.
By 5.3 $\hat{f}$ is continuous if and only if f is continuous. There is a bijective correspondence between continuous maps from X to Y compatible with R, and continuous maps from X/R to Y.
In particular, the map f itself gives rise to an equivalence relation in X by defining two points x,$x' \in X$ to be equivalent if $f(x) = f(x')$. We write X/f for the quotient space in this case. Then we have an injective induced map $\hat{f} : X/f \longrightarrow Y$. We shall give a necessary and sufficient condition for $\hat{f}$ being a homeomorphism.

5.5 A map $f : X \longrightarrow Y$ between topological spaces is called an *identification* if f is surjective and has the property that a set W is open in Y if and only if $f^{-1}(W)$ is open in X. (We may replace "open" with "closed" in this definition). An identification is continuous.

It is clear that the composition $g \circ f : X \longrightarrow Z$ of two identifications $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$, is an identification. The canonical map of a quotient space is obviously an identification.

Theorem A map $f : X \longrightarrow Y$ between topological spaces induces a homeomorphism $\hat{f} : X/f \longrightarrow Y$ if and only if f is an identification.

Proof. Assume that $\hat{f}$ is a homeomorphism, thus $\hat{f}$ is an identification. Since p is an identification too, it follows that the composition f is an identification. Assume conversely that f is an identification. Then firstly, $\hat{f}$ is bijective. Secondly, a set W is open in Y if and only if $p^{-1}(\hat{f}^{-1}(W)) = f^{-1}(W)$ is open in X, in other words if and only if $\hat{f}^{-1}(W)$ is open in X/f. This means that $\hat{f}$ is a homeomorphism. □.

When we are given an identification $f : X \longrightarrow Y$ we may topologically consider Y as a quotient space of X. (In 5.1 this was the setting we worked in).

We say that Y has the *quotient topology* from X defined by f. This is the finest topology on Y which makes f continuous.

Corollary An identification which is bijective is a homeomorphism.

5.6 For applications the following criteria for a map being an identification is especially important.

Theorem A surjective, continuous and either open or closed map is an identification.

Proof. If $f : X \longrightarrow Y$ is surjective, then $W = f(f^{-1}(W))$ for every set W in Y. This yields the theorem since we can characterize an identification by considering inverse images of open or closed sets. □

In general, an identification need not be open nor closed, and the same applies to the canonical map $p : X \longrightarrow X/R$ for the quotient construction. An equivalence relation R in a topological space X is called *open (closed)* if $p : X \longrightarrow X/R$ is an open (closed) map. This is equivalent to requiring that the saturation of every open (closed) set in X with respect to R is open (closed) in X.

5.7 Example We may define an equivalence relation in $\mathbb{R}$ by proclaiming x,$x' \in \mathbb{R}$ to be equivalent if $x - x' \in \mathbb{Z}$. This is the situation from example 5.1, where we for simplicity have chosen the period $\omega = 1$. The quotient space is then denoted $\mathbb{R}/\mathbb{Z}$. The map $q : \mathbb{R} \longrightarrow S^1$ given by $q(x) = e^{2\pi ix}$ is surjective, continuous and open, i.e. an identification by 5.6. Furthermore, $q(x) = q(x')$ if and only if $x - x' \in \mathbb{Z}$. Thus the induced map $\hat{q} : \mathbb{R}/\mathbb{Z} \longrightarrow S^1$ is a homeomorphism by 5.5. Topologically speaking, we may consider S^1 to be the quotient space of $\mathbb{R}$ with respect the equivalence relation defined by $\mathbb{Z}$. The equivalence relation in this case is open, but not closed in $\mathbb{R}$ since the map q is not closed. Note the geometric interpretation of q. The map q bends the line $\mathbb{R}$ by a factor 2π and winds it around S^1 in the positive direction such that $0 \in \mathbb{R}$ falls on $1 \in S^1$.
The forming of quotient spaces allow us to shape some geometrically interesting spaces. We will look at a few simple examples.

5.8 Example Consider a rectangle in the plane with two parallell sides chosen. By reflecting across the middle axis and identifying one side, point by point, with the opposite side, we get a cylinder band. Thereafter, identify each point on one of the boundary circles of the cylinder, with the point on the same generatorline on the opposite boundary circle, and we get the surface of a torus. Formally speaking,

we have realized the torus as a quotient of the rectangle $X = [0,1] \times [0,1]$ with respect to the equivalence relation $\sim$ given by $(x,0) \sim (x,1)$ and $(0,y) \sim (1,y)$ for x,$y \in [0,1]$.

5.9 Example If in the previous example we instead identify each point on the lower boundary circle of the cylinder band with the point on the upper boundary circle obtained by reflecting about a fixed diameter centred in the cylinder, the resulting surface is what is called *Klein's bottle*. Such a merging of boundary circle points is impossible, however, to realize concretely in 3−space without letting the surface intersect itself.
Klein's bottle may therefore not be realized in 3−space. But we may perceive the surface abstractly as a rectangle with boundary points identified, in this case as a quotient of the rectangle $X = [0,1] \times [0,1]$ under the equivalence relation $\sim$ defined by $(x,0) \sim (1-x,1)$ and $(0,y) \sim (1,y)$ for x,$y \in [0,1]$.

5.10 Example An important space in many respects in topology and geometry, is the *real projective n-space* P^n. The elements of P^n are the lines through the origin in $\mathbb{R}^{n+1}$ (directions in $\mathbb{R}^{n+1}$). We then have a natural surjection $p : \mathbb{R}^{n+1} - \{0\} \longrightarrow P^n$ by defining $p(x)$ to be the line through the origin and x in $\mathbb{R}^{n+1}$, that is, $p(x) = \mathbb{R}x$. We give P^n the quotient topology defined by p, i.e. the topology making p an identification. By 5.5 we then have a homeomorphism $P^n \approx (\mathbb{R}^{n+1} - \{0\})/\sim$, where $\sim$ denotes the equivalence relation which identifies proportional vectors in $\mathbb{R}^{n+1} - \{0\}$. We set $x \sim x'$ if there is some scalar λ such that $x' = \lambda x$, when x,$x' \in \mathbb{R}^{n+1} - \{0\}$.
We shall show that the projective n-space P^n is an n-manifold, that is, a Hausdorff space which is locally Euclidean of dimension n. For $x = (x_1, \ldots, x_{n+1})$ in $\mathbb{R}^{n+1} - \{0\}$ we introduce the notation $p(x) = [x_1, \ldots, x_{n+1}]$. We then have per definition $[x_1, \ldots, x_{n+1}] = [x'_1, \ldots, x'_{n+1}]$ if and only if $x'_i = \lambda x_i$ for $i = 1, \ldots, n+1$ and an appropriate scalar λ. The quantities $x_1, \ldots, x_{n+1}$ are called *homogenous coordinates* for the point $p(x)$ in P^n.
Let $U_i \subseteq P^n$ for $i = 1, \ldots, n+1$ be defined by the condition $x_i \neq 0$. It is clear that U_i is an open set in P^n and that $U_1 \cup \cdots \cup U_{n+1} = P^n$. We now define the map $\phi_i : U_i \longrightarrow \mathbb{R}^n$ by

$$\phi_i[x_1, \ldots, x_{n+1}] = (\tfrac{x_1}{x_i}, \ldots, \tfrac{x_{i-1}}{x_i}, \tfrac{x_{i+1}}{x_i}, \ldots, \tfrac{x_{n+1}}{x_i}) \tag{5.10.1}$$

and $\psi_i : \mathbb{R}^n \longrightarrow U_i$ by

$$\psi_i(y_i, \ldots, y_n) = [y_1, \ldots, y_{i-1}, 1, y_i, \ldots, y_n]. \tag{5.10.2}$$

It follows that ϕ_i is well-defined since (5.10.1) remains unchanged if all x_i are

multiplied by some scalar $\lambda \neq 0$. Furthermore, ϕ_i is continuous by theorem 5.3, since the composition $\phi_i \circ p$ is continuous. ψ_i is clearly continuous. It is also easy to verify that the compositions $\phi_i \circ \psi_i$ and $\psi_i \circ \phi_i$ are identity maps. So ϕ_i is a homeomorphism of the open set $U_i \subseteq P^n$ onto R^n. The maps $\phi_i, \ldots, \phi_{n+1}$ are therefore local charts covering P^n. This shows that P^n is a locally Euclidean space.
If $p(x) \neq p(x')$ are distinct points in P^n we may separate them by disjoint open neighborhoods $p(V)$ and $p(V')$ in P^n. We assume $||x|| = ||x'|| = 1$ and set $V = \{y \in \mathbb{R}^{n+1} - \{0\} : ||y \cdot x'|| \leq ||y \cdot x||\}$ and $V' = \{y \in \mathbb{R}^{n+1} - \{0\} : ||y \cdot x|| \leq ||y \cdot x'||\}$. Then V,V' are dense, saturated sets in $\mathbb{R}^{n+1} - \{0\}$. This shows that the topology defined on P^n is Hausdorff.
All in all, we have shown that P^n is an n-manifold.

5.11 Example We give an alternative construction of the real projective n-space P^n. Let $q : S^n \longrightarrow P^n$ be the surjection defined by $q(x) = \mathbb{R}x$ for $x \in S^n$. It is straightforward to see that q is an open, surjective map and therefore an identification by 5.6. We get, by 5.5, a homeomorphism of $S^n/\sim$ onto P^n, where $\sim$ is the equivalence relation identifying antipodal points x and $-x$ of S^n. Hence, the projective n-space is the n-sphere with antipodal points identified. Since the map $q : S^n \longrightarrow P^n$ is a local homeomorphism (every open half sphere on S^n is mapped homeomorphically onto an open set in P^n) we again see that P^n is an n-manifold, since S^n is.

5.12 Example Let X and Y be topological spaces. We give the disjoint union $X + Y$ a topology by defining a set $U \subseteq X + Y$ to be open if $U \cap X$ and $U \cap Y$ are open sets in X and Y respectively. The space $X + Y$ is called (the topological) *sum* of X and Y. Let $i : X \longrightarrow X + Y$ and $j : Y \longrightarrow X + Y$ be the natural embeddings.
Assume $f : A \longrightarrow Y$ to be a continuous map defined on a subspace A of X. Let $\sim$ be the equivalence relation in $X + Y$ given by $x \sim f(x)$ for all $x \in A$. The quotient space $(X + Y)/\sim$ is then denoted $X \cup_f Y$ and is called the *adjunction space* defined by X, Y and f.
We have natural maps $\bar{i} : X \longrightarrow X \cup_f Y$ and $\bar{j} : Y \longrightarrow X \cup_f Y$ given by $\bar{i} = p \circ i$ and $\bar{j} = p \circ j$, where $p : X + Y \longrightarrow X \cup_f Y$ is the quotient map. It's easily seen that $\bar{j}$ is an embedding. By this, Y is identified with a subspace of $X \cup_f Y$. If we assume A to be closed in X, then Y is a closed subspace of $X \cup_f Y$ and $\bar{i}$ defines a homeomorphism of $X - A$ onto the complement $X \cup_f Y - Y$.
An important special case is choosing X to be the n-cell D^n and A to be the boundary sphere S^{n-1}. The space $D^n \cup_f^Y$ is said to be constructed by adjoining an n-cell to Y. A topological space built by successively adjoining cells of growing dimension, is called a *cell complex* . For instance, the torus surface may be

constructed as a cell complex by starting with a 0-cell, that is, a point, adjoining two 1-cells to this, and finally adjoining a 2-cell to this so-called 1-sceleton.

5.13 Example We shall show that the projective n-space P^n may be realized as a cell complex with one cell in each dimension from 0 to n. It is enough to show that we have a homeomorphism $D^n \cup_q P^{n-1} \approx P^n$ where $q : S^{n-1} \longrightarrow P^{n-1}$ is the map identifying antipodal points defined in 5.11. For this shows that P^n arises from P^{n-1} by attaching an n-cell. By successively attaching cells of increasing dimension we can build a space homeomorphic to P^n. Let $h : D^n \approx S^n_+$ be the homeomorphism from the closed n-cell D^n onto the closed upper half sphere $S^n_+ = \{x \in S^n : x_{n+1} \geq 0\}$, which is inverse to orthogonal projection. The composed map $f : D^n \longrightarrow S^n_+ \longrightarrow P^n$ is a closed surjection which identifies antipodal points on S^{n-1}. Furthermore, we have a natural inclusion map $g : P^{n-1} \longrightarrow P^n$ which is also a closed map. Hence we get an induced map $f + g : D^n + P^{n-1}$, which is closed and surjective, thus an identification by 5.6, and which exactly identifies the points $x \in S^{n-1} \subseteq D^n$ and $q(x) \in P^{n-1}$. By 5.5 we then have a homeomorphism $D^n \cup_q P^{n-1} \approx P^n$, as was to be shown.
Since $f : D^n \longrightarrow P^n$ defined above is an identification, we see that $P^n \approx D^n/\sim$, where $\sim$ identifies antipodal points on S^{n-1}. This gives a characterization of the projective n-space.

5.14 We end this chapter by discussing a special type of quotient construction, namely the collapsing of a subspace. For a subspace $A \neq \emptyset$ in a topological space X we may consider the equivalence relation in X which identifies all points in A. The quotient space in this case is denoted X/A and is said to be made of X by *collapsing* of A. We note that X/A can only be a Hausdorff space when A is closed in X.
As an example, we consider the quotient space D^n/S^{n-1}, the closed n-cell where the points on the boundary sphere are identified. We have a map $f : D^n \longrightarrow (\mathbb{R}^n)^*$ (see 1.8) given by

$$f(x) = \begin{cases} \frac{x}{1-|x|} & \text{for } |x| < 1 \\ \infty \end{cases}$$

This map is continuous on all of D^n, also at boundary points by our definition of the neighborhoods of ∞ in $(\mathbb{R}^n)^*$. The map f is also closed, which follows from theorem 6.10, therefore an identification by 5.6. It is clear that f is surjective and identifies the points on the boundary S^{n-1} in D^n. We then get, by 5.5, an induced homeomorphism $\hat{f} : D^n/S^{n-1} \longrightarrow (\mathbb{R}^n)^*$. According to 3.22 then, D^n/S^{n-1} is homeomorphic to S^n.

5.15 The consept of quotient space was first explicitly defined in N. Bourbaki's book "Topologie générale" from 1940. Constructions of spaces by identification of points were known earlier, however. In 1863 A.F. Moebius published a topological classification of surfaces in $\mathbb{R}^3$ in the form of spheres with handles and by cutting and pasting of surfaces. Nonorientable surfaces were first defined by F. Klein in 1876 and topologically classified by W. Dyck in 1888. The realization of the projective plane as the sphere with diametrically opposite points identified, is due to F. Klein and L. Schläfli, 1874. Klein's bottle was introduced by Klein in 1882. The study of spaces by way of cell division stems from H. Poincaré, who founded combinatorical topology, 1895-1904.

Chapter 6

Compactness

Among the most interesting topological spaces we find those whose topology is "finitely reducable". A bounded closed interval J on the real line is an example of such a space. This is illustrated by, for instance, the extremal value theorem, which states that every continuous function $f : J \longrightarrow \mathbb{R}$ achieves its maximum (minimum). Here J behaves like a finite point set. Even more relevant is the Heine-Borel theorem below, which shows that this sense of finiteness is tied to the topology. The Heine-Borel theorem has brought about the notion *compact space*, which we shall study in this chapter.

6.1 Bounded, closed intervals on the real line posess a topological property which is stated in a well-known result from real analysis.

Theorem. (Heine-Borel) If a class of open sets in $\mathbb{R}$ covers a bounded, closed interval $[a, b]$, then finitely many of the sets will cover $[a, b]$.

Proof. We will prove this using the upper bound principle for the real numbers. Let $\{U_i\}_{j\in J}$ be an open cover of $[a, b]$ by open sets from $\mathbb{R}$, and let A be the set of points $x \in [a, b]$ such that $[a, x]$ can be covered by finitely many U_i. Then $s = \sup A$ exists, and we have $s \leq b$. Now observe that s belongs to A. For s belongs to a set U_j, which also contains an interval $[x, s]$ for an $x \in A$ sufficiently close to s. Since $[a, x]$ is finitely covered, so is $[a, s]$, hence $s \in A$. On the other hand, U_j cannot contain any interval $[s, y]$ with $s < y \leq b$. For in that case, also y must belong to A, contrary to the definition of s. This shows that $s = b$, and thus $b \in A$. But this means that $[a, b]$ is finitely covered. □

The requirement that the interval be bounded and closed is necessary. If, for instance, we consider the interval $[0, 1)$, then the open sets $U_n = (-1, 1 - \frac{1}{n})$ for $n = 1, 2, \ldots$ together cover the interval, but no finite selection will.

Upon generalizing, the covering property in Heine-Borel's theorem propagates an important type of space.

6.2 A topological space is called *compact* if every class of open sets which together cover the space, contains a finite selection which already covers the space. A subset is called compact if it is compact as subspace, or equivalently, if every cover of the subset by open sets from the space may be reduced to a finite cover.
From the definition it then follows that compactness is a topological property; a property which is preserved under homeomorphisms. It is also a consequence that a union of finitely many compact sets is compact.
Bounded, closed intervals are compact subsets of $\mathbb{R}$ by Heine-Borel's theorem, even though $\mathbb{R}$ itself is obviously not compact. A discrete set is compact if and only if it is finite. However, it quickly becomes tedious to construct interesting examples by brute force of hand. We shall therefore first establish some general properties concerning compactness.

6.3 Theorem Closed subsets of compact spaces are compact.

Proof. Assume that X is a compact space, and $A \subset X$ a closed set. Let $\{U_i\}_{i\in I}$ be a class of open sets in X which together cover A. Then $\{X - A, U_i\}_{i\in I}$ is an open cover of X. This may be reduced to a finite cover since X is compact,

$$X = (X - A) \cup U_{i_1} \cup \ldots \cup U_{i_n}$$

But then $\{U_{i_1}, \ldots, U_{i_n}\}$ is a finite cover of A. □

6.4 Theorem If X and Y are compact spaces, then $X \times Y$ is compact.

Proof. Let $\{W_i\}_{i\in J}$ be an open cover of $X \times Y$. We show that it's possible to extract a finite cover. Since every set W_i in $X \times Y$ is a union of open product sets $U \times V$, we may without loss of generality assume that all the W_i are themselves of this form. For if finitely many such $U \times V$ cover, then also finitely many W_i will. Assume therefore $W_i = U_i \times V_i$, $i \in J$.
Let $x \in X$ be arbitrary. Then $\{x\} \times Y$ is a compact subset in $X \times Y$ and may be covered by finitely many W_i,

$$\{x\} \times Y \subseteq W_{i_1} \cup \ldots \cup W_{i_n}$$

Set $U_{i_1} \cap \cdots \cap U_{i_n} = U_x$. Then U_x is an open neighborhood of x, and $U_x \times Y$ an open column in $X \times Y$ which contains $\{x\} \times Y$ and is covered by finitely many W_i,

$$\{x\} \times Y \subseteq U_x \times Y \subseteq W_{i_1} \cup \ldots \cup W_{i_n}$$

It is therefore enough to show that finitely many such columns $\{U_x \times Y\}_{x\in X}$ will cover $X \times Y$. But this is straightforward: For $\{U_x\}_{x\in X}$ makes an open cover of X and may therefore be reduced to a finite cover (since X is compact),

$$X = U_{x_1} \cup \ldots \cup U_{x_m}$$

which gives $X \times Y = (U_{x_1} \times Y) \cup \ldots \cup (U_{x_m} \times Y)$. □

Corollary If A and B are compact sets in X and Y, then $A \times B$ is compact in $X \times Y$.

Both the theorem and the corollary extend immediately to products of a finite number of spaces. The fact is that they apply for products in general (Tychonov's theorem).
One may wonder whether a nonempty product space $X \times Y$ can be compact without the factor spaces X and Y being compact. The answer is no. In section 6.8 we show that the image of a compact space under a continuous map is compact. From this it follows that $X = pr_1(X \times Y)$ and $Y = pr_2(X \times Y)$ are compact.

6.5 It so happens that the compactness property is most fruitful when combined with Hausdorff's separation axiom. We shall further study this combination.

Theorem In a Hausdorff space two disjoint, compact sets may be separated by disjoint neighborhoods.

Proof. Let X be a Hausdorff space and A, $B \subseteq X$ two disjoint, compact sets. We first consider the case where B consists of a single point, $B = \{y\}$. For every $x \in A$ we can find neighborhoods U_x and V_x of x and y, respectively, such that $U_x \cap V_x = \emptyset$ (since X is Hausdorff). The class $\{U_x\}_{x\in A}$ is an open cover of A and can be reduced to a finite cover $\{U_x\}_{i=1,\ldots,n}$ (since A is compact). Set $U_y = U_{x_1} \cup \ldots \cup U_{x_n}$ and $V_y = V_{x_1} \cap \cdots \cap V_{x_n}$. Then U_y is an open neighborhood of A and V_y is an open neighborhood of y such that $U_y \cap V_y = \emptyset$.
Then the general case: For every $y \in B$ we choose neighborhoods U_y of A and V_y of y such that $U_y \cap V_y = \emptyset$ as above. The class $\{V_y\}_{y\in B}$ is an open cover of B, which is reducable to a finite cover $\{V_{y_j}\}_{j=1,\ldots,p}$. Set $U = U_{y_1} \cap \cdots \cap U_{y_p}$ and $V = V_{y_1} \cup \ldots \cup V_{y_p}$. Then U is an open neighborhood of A and V is an open neighborhood of B such that $U \cap V = \emptyset$. □

Corollary 1 In a Hausdorff space every compact set is closed.

Therefore, in a compact Hausdorff space only the closed sets are compact.

Corollary 2 In a compact Hausdorff space every point has a basis of closed (and therefore compact) neighborhoods.

6.6 We may now determine the compact subsets of $\mathbb{R}^n$. First we apply our knowledge of products of compact sets to bounded, closed intervals in $\mathbb{R}$, and deduce that the cubes $[a,b]^n$ are compact. By theorem 6.3 we get that every closed subset of a cube is compact. But this means that every bounded, closed subset of $\mathbb{R}^n$ is compact; for a set in $\mathbb{R}^n$ is bounded if and only if it is contained in some cube $[-a,a]^n$, for appropriate $a > 0$.
The converse does also apply. If $K \subset \mathbb{R}^n$ is compact, then K is closed by 6.5, corollary 1. Furthermore, K is bounded, since K can be covered by a finite number of metric (Euclidean) neighborhoods $U(x;1)$, $x \in K$. All in all, we have:

Theorem A set in $\mathbb{R}^n$ is compact if and only if it is bounded and closed.

6.7 Example The orthogonal group $O(n)$ is compact: We recall that $O(n)$ has the subspace topology from the matrix space $M(n)$, and that $M(n)$ with the Hilbert-Schmidt norm is identifiable with R^{n^2} with Euclidean norm, under a norm preserving isomorphism $M(n) \cong R^{n^2}$ (1.19). Therefore it is enough to show that $O(n)$ is a bounded, closed set in $M(n)$. The fact that $O(n)$ is closed we already know from (3.14). It is straightforward to see that $O(n)$ is bounded. If $A \in O(n)$, then we have

$$|A|^2 = tr(A^tA) = tr(I) = n$$

which shows that A lies on the (Hilbert-Schmidt) sphere centred at the origin with radius $\sqrt{n}$. Since $SO(n)$ is a closed subgroup of $O(n)$, it also follows that $SO(n)$ is compact. On the other hand, $GL(n)$ and $SL(n)$ are not compact (from $n \geq 2$ for $SL(n)$). For instance, $SL(2)$ contains the matrices

$$D_k = \begin{bmatrix} k & 0 \\ 0 & \frac{1}{k} \end{bmatrix}$$

for all $k \neq 0$. This shows that $SL(2)$ and $GL(2)$ are unbounded.

6.8 We have already noted that compactness is a property which is preserved under homeomorphisms. The fact is that it's even preserved under continuous maps.

Theorem The image of a compact space under a continuous map is compact.

Proof. Assume X to be compact, and let $f : X \to Y$ be a continuous map to a space Y. We must show that $f(X)$ is compact. Without loss of generality we assume $f(X) = Y$.
Let $\{O_i\}_{i\in J}$ be a class of open sets in Y which cover Y. Then $\{f^{-1}(O_i)\}_{i\in J}$ is a class of open sets in X which cover X. Since X is compact, there is a finite selection which already covers, $X = f^{-1}(O_{i_1}) \cup \ldots \cup f^{-1}(O_{i_p})$. But then it follows that $Y = f(X) = O_{i_1} \cup \ldots \cup O_{i_p}$. This shows that $\{O_i\}_{i\in J}$ contains a finite cover of Y, hence Y is compact. □

Corollary A continuous map sends compact sets to compact sets.

From the theorem it follows, for instance, that quotients of compact spaces are compact.

6.9 Example The sphere S^n and the projective space P^n are compact. For S^n is a bounded, closed subset of $\mathbb{R}^{n+1}$ and P^n is a quotient space of S^n (5.11).
The next result yields some of the most important and applicable consequences of compactness.

6.10 Theorem A continuous map from a compact space to a Hausdorff space is closed.

Proof. Let $f : X \longrightarrow Y$ be the map, and let $F \subseteq X$ be a closed set. Then F is compact (6.3) and $f(F)$ must be compact (6.8). But then $f(F)$ is closed (6.5, corollary 1). □

Corollary A bijective, continuous map from a compact space to a Hausdorff space is a homeomorphism.

The requirement of compactness cannot be removed. For instance, the map $p : [0, 2\pi) \longrightarrow S^1$ given by $p(t) = e^{it}$ is continuous and bijective, but not a homeomorphism, since the interval $[0, 2\pi)$ is not compact and thus not homeomorphic to a compact space.

6.11 A standard result from analysis states that every continuous function on a bounded, closed interval has a maximum and minimum. This result holds more generally:

Theorem (Extremal value theorem) Let X be a compact space and f a continuous, real function on X. Then f is bounded and takes the value $\sup f(X)$ ($\inf f(X)$) at least at one point in X.

Proof. The image $f(X) \subseteq \mathbb{R}$ is compact and therefore bounded and closed. This means that $\sup f(X)$ ($\inf f(X)$) exists and belongs to $f(X)$. □

We give an application of the extremal value theorem.

6.12 Example All norms in $\mathbb{R}^n$ are equivalent (see 1.16):
Let $||\ ||$ be an arbitrary norm on $\mathbb{R}^n$. Then we have (see 3.6)

$$\Big|\,||x|| - ||y||\,\Big| \leq ||x - y||$$

for all x, y in $\mathbb{R}^n$. Furthermore, for arbitrary $z \in \mathbb{R}^n$

$$||z|| \leq |z_1|\ ||e_1|| + \ldots + |z_n|\ ||e_n||$$

where $e_1,\ldots,e_n$ is the standard basis in $\mathbb{R}^n$. Since generally $|z_i| \leq |z|$ (absolute value of z_i, Euclidean norm of z), we obtain the estimate

$$||z|| \leq Mn|z|$$

where M is the maximum of $||e_1||,\ldots,||e_n||$, a constant independent of z. If we substitute $z = x - y$ we get

$$||x - y|| \leq Mn|x - y|$$

which combined with the first inequality above yields

$$\Big|\,||x|| - ||y||\,\Big| \leq Mn|x - y|$$

This shows that the general norm function $x \mapsto ||x||$ is continuous on $\mathbb{R}^n$.
Now consider two arbitrary norms $||\ ||_1$ and $||\ ||_2$ on $\mathbb{R}^n$. As positive, continuous functions on S^{n-1} they have positive maxima and minima (extremal value theorem.) Hence there exist positive numbers c_1 and c_2 such that $||x||_1 \leq c_1$ and $||x||_2 \geq c_2$ for all $x \in S^{n-1}$. This gives $||x||_1/||x||_2 \leq c_1/c_2$ for $x \in S^{n-1}$ and thus for all $x \in \mathbb{R}^n$. A similar reasoning applies to $||x||_2/||x||_1$. Then it follows that the norms are equivalent.

6.13 The theory of compact spaces does also interplay with spaces which are not compact themselves, but have "a sufficient amount" of compact subsets.
A topological space is called *locally compact* if every point has a basis of compact

neighborhoods.
The coordinate spaces $\mathbb{R}^n$ are locally compact. As neighborhood basis of $x \in \mathbb{R}^n$ we may take $\{\overline{U}(x; r)\}_{r>0}$. In general, all manifolds are locally compact.
Every compact Hausdorff space is locally compact. This fact follows from 6.5, corollary 2. It is also a consequence of the same corollary, that a Hausdorff space is locally compact if every point has one compact neighborhood; then it has a basis of compact neighborhoods.
It is clear that every open subset of a locally compact space is locally compact. A topological space may therefore not be embedded as an open subspace of a locally compact space, unless it is locally compact itself. In particular, it is impossible to realize other than locally compact Hausdorff spaces as open subspaces of compact Hausdorff spaces. On the other hand, the following is always possible: Let X be a locally compact Hausdorff space. Let ω be a point which does not belong to X and set $X \cup \{\omega\} = X^*$. We topologize X^* as follows:
Let U be an arbitrary open set in X, and V be an open set with compact complement, that is, $X - V$ is compact (and thus closed.) Set $V^* = V \cup \{\omega\}$. Let τ be the class of all sets U and V^* in X^*. Then τ is a topology on X^* which makes X^* into a Hausdorff space with X as open subspace. A special case of this construction is the space $(\mathbb{R}^n)^*$ from (3.22) which we showed to be homeomorphic to S^n.
The product rule for compact spaces has a locally compact analogue:

Theorem If X and Y are locally compact spaces, then $X \times Y$ is locally compact.

Proof. If U and V are compact neighborhoods of x and y in $X \times Y$, then $U \times V$ is a compact neighborhood of $(x, y) \in X \times Y$. (And if U and V are from neighborhood bases, then so is $U \times V$). □

The theorem may immediately be extended to the product of a finite number of sets. When it comes to infinite products, however, the case is somewhat different. If $X = \Pi X_\alpha$ is locally compact, then a point $x = (x_\alpha)$ in X has a compact neighborhood K. The projections $pr_\alpha : X \longrightarrow X_\alpha$ send K to X_α entirely, except for finitely many indices, by the definition of the product topology. This gives that, except for finitely many indices, the sets X_α are compact. An infinite product of locally compact spaces can therefore not be locally compact unless all except for a finite number of the factors are compact.

6.15 Example Let F be the vector space of all real sequences $(x_n)_{n=1,2,\ldots}$ (with componentwise operations), and let $H \subset F$ be the subset of sequences (x_n) such that the series $x_1^2 + x_2^2 + \ldots$ converges. Then H is a linear subspace of F:
First we note that if (x_n), $(y_n) \in H$, then the series $x_1y_1 + x_2y_2 + \ldots$ is convergent.

For by Schwarz inequality in $\mathbb{R}^N$ we get

$$(\sum_1^N |x_n||y_n|)^2 \leq \sum_1^N |x_n|^2 \sum_1^N |y_n|^2$$

that is,

$$(\sum_1^N |x_n y_n|) \leq (\sum_1^N |x_n|^2)^{\frac{1}{2}} (\sum_1^N |y_n|^2)^{\frac{1}{2}})$$

Letting $N \to \infty$ it follows that the series $\sum x_n y_n$ is absolutely convergent, and that

$$(\sum_1^\infty |x_n y_n|) \leq (\sum_1^\infty |x_n|^2)^{\frac{1}{2}} (\sum_1^\infty |y_n|^2)^{\frac{1}{2}})$$

From the identity $(x_n + y_n)^2 = x_n^2 + y_n^2 + 2x_n y_n$ it follows in a similar manner (by summing and going to the limit), that the series $\sum (x_n + y_n)^2$ is convergent. It follows that the sum sequence $(x_n) + (y_n) := (x_n + y_n)$ belongs to H. Finally, it is easy to see that the product $\alpha(x_n) := (\alpha x_n)$ belongs to H for every real scalar α. Thus H is a vector space with respect to component wise operations.
We define a scalar product on H analogously to the Euclidean product in $\mathbb{R}^n$,

$$(x_n) \cdot (y_n) = \sum_1^\infty x_n y_n$$

It is clear that the definition satisfies the requirements of a scalar product and therefore induces a norm in H,

$$|(x_n)| = (\sum_1^\infty x_n^2)^{\frac{1}{2}}$$

Hence H is a normed vector space, with a well-defined norm topology. We may consider H to be a natural infinite dimensional culmination of the Euclidean spaces $\mathbb{R}^1 \subseteq \mathbb{R}^2 \subseteq \ldots$.
H has a lot in common with the Euclidean spaces $\mathbb{R}^n$, but also some notable differences. For instance, H is not locally compact:
Let e_i be the standard vector $(0, \ldots, 0, 1, 0, \ldots)$, with 1 at the i-th position. Then $e_i \in H$ and we have $|e_i - e_j| = \sqrt{2}$ for $i \neq j$. Therefore the open cover $\{U(x; \frac{1}{2}\sqrt{2})\}_{x \in S(H)}$ of the unit sphere $S(H)$ cannot be reduced to a finite cover. (In that case at least one $\{U(x; \frac{1}{2}\sqrt{2})\}_x$ would contain two standard vectors.) This shows that $S(H)$ is not compact (even though being both closed and bounded in H). The same applies to the closed unit cell $D(H) = \overline{U}(0; 1)$ in H. Since every

closed ϵ-neighborhood $\overline{U}(0;\epsilon)$ is homeomorphic to $D(H)$, neither of these spaces are compact. It follows that H does not have any compact neighborhood of the origin 0 and thus is not a locally compact space. (In fact, H does not have any compact neighborhood of any point. Why?)
We call H the *Hilbert space* of square summable sequences (square convergent sequences).

6.16 The definition of compact spaces by way of Heine-Borel's covering property stems from a work of P. Alexandrov and P. Urysohn from 1929. Earlier, the concept of compactness was defined in terms of convergence of point sequences. The importance of the covering property was first made clear by Alexandrov's and Urysohn's work. Spaces with the covering property were called "bicompact" to separate the notion from the earlier one. But eventually it was made clear that the definition of compactness by open coverings, and not by countable point sequences, was the right notion for general spaces, and the terminology was changed from "bicompact" to "compact". Also locally compact spaces were introduced by Alexandrov and Urysohn in 1929. One-point compactification of a locally compact space was defined by Alexandrov in 1924.
A. Tychonov proved his famous theorem regarding compactness of general product spaces in 1930. He also studied compactifications of spaces, that is, the possibility to embed a given space as a subspace of a compact space.
The extremal value theorem for functions defined on a compact interval was known from analysis in the last century. Cauchy assumed this result without sufficient proof. Weierstrass gave a proof later on, regarding the existence of an extremal point.

Chapter 7

Connectedness

An elementary but important result from real analysis is the intermediate value theorem. It states that if f is a continuous, real function on an interval $[a, b]$, and z a real number between $f(a)$ and $f(b)$, then $z = f(x)$ for at least one $x \in [a, b]$. This means that f takes *all* values between $f(a)$ and $f(b)$ on the interval $[a, b]$. If we replace the domain of definition $[a, b]$ with a "disconnected" set, by for instance removing one or several interior points in $[a, b]$, then the result obviously no longer applies. (Consider $f(x) = x$.) Clearly the intermediate value theorem depends on the domain of definition being "connected" in some sense. In this chapter we shall study the concept of connectedness, in particular in light of the spaces we have already seen.

7.1 A topological space is called *disconnected* if it can be separated in two disjoint, nonempty, open subsets; in short: if it can be *splitted*. Otherwise we say the space is *connected*. A subset of a space is (dis-)connected if it is (dis-)connected as a subspace.
For example, $[0, 1) \cup (1, 2]$ is a disconnected subset of the real line $\mathbb{R}$. The same applies to $[0, 1] \cup [2, 3] \cup [4, 5] \cup \ldots$. Discrete spaces with more than one element are naturally disconnected.
It is indifferent if we in the above definition replace the word "open" with "closed" (or "openclosed"). For if two complementary sets in a space are open, then they are also closed, and vice versa. It then follows that a space X is disconnected if and only if it contains an openclosed subset different from $\emptyset$ and X.
The following characterization is often useful.

7.2 Theorem A space X is connected if (and only if) every pair of points in X belongs to a connected subset.

Proof. It suffices to show that if X is disconnected, then there exist two points

x, y in X which do not belong to any connected subset.
Let $X = U \cup V$ be a splitting of X, and choose $x \in U$ and $y \in V$. Every set K which contains x and y is splitted similarly, $K = (K \cap U) \cup (K \cap V)$, and is therefore disconnected. □

It is clear from the definition that connectedness is a topological notion, i.e. invariant under homeomorphisms: Homeomorphic spaces X and Y are either both connected or both disconnected.

7.3 Example We will show that the interval $[a, b]$ is connected. Assume, for the sake of contradiction, there is a splitting $[a, b] = U \cup V$. Let $b \in V$ and set $m = \sup U$. Then m is a boundary point for U and thus also for V, since U and V are complementary. But then m must belong to both U and V, since both are closed. This contradiction shows that $[a, b]$ cannot be splitted, hence it is connected.
By theorem 7.2 we now deduce that all convex subsets of $\mathbb{R}$ are connected. That $K \subset \mathbb{R}$ is convex means that $[a, b] \subseteq K$ whenever $a, b \in K$. Conversely, it is clear that a connected subset of $\mathbb{R}$ is convex.
The convex subsets of $\mathbb{R}$ are easily determined. These are exactly the intervals, when we among these also include infinite intervals, one point sets and the empty set. We have thus determined the connected subsets of $\mathbb{R}$.
We may now formulate the intermediate value theorem as follows: A continuous, real function on an interval has a connected image. From this point of view, the result has a powerful generalization.

7.4 Theorem The image of a connected space under a continuous map is connected.

Proof. Assume X to be connected and $f : X \longrightarrow Y$ a continuous map to a space Y. We must show that $f(X) \subseteq Y$ is connected. Without loss of generality, assume $f(X) = Y$, i.e. f is surjective.
If Y was not connected, then we would have a splitting $Y = U \cup V$. But this would give a splitting $X = f^{-1}(U) \cup f^{-1}(V)$, in contradiction to the assumption. □

Corollary A continuous map sends connected sets to connected sets.

7.5 Example Let K be a convex subset of $\mathbb{R}^n$ or more generally of an arbitrary normed vector space V. K is *convex* if the segment $[x, y] = \{z = (1 - t)x + ty : 0 \leq t \leq 1\}$ lies in K whenever x, y belong to K. The segment $[x, y]$

is connected, as image of the unit interval $[0,1] \subseteq \mathbb{R}$ under the continuous map $\sigma(t) = (1-t)x+ty$ into K. It then follows from theorem 7.2 that K is connected. In particular, the cells E^n and D^n in $\mathbb{R}^n$, as well as $\mathbb{R}^n$ itself, are connected spaces. It follows from corollary 7.4 that topological spaces with continuous, real functions having disconnected images, are themselves disconnected.

7.6 Example The general linear group $GL(n)$ has a splitting $GL(n) = GL^+(n) \cup GL^-(n)$ and is thus disconnected. This can also be seen by way of the determinant function which is > 0 on $GL^+(n)$ and < 0 on $GL^-(n)$, does not take the value 0 on $GL(n)$.

7.7 It is reasonable to expect, intuitively speaking, that the union of a class of connected subsets with nonempty intersection, to be connected. By theorem 7.2, it's sufficient to show

Theorem Let A and B be connected subsets with nonempty intersection in a space X. Then $A \cup B$ is connected.

Proof. If not, then there should be a splitting of $A \cup B$, that is $A \cup B = U \cup V$. By connectedness, A must be entirely contained in one of the sets U or V, and B must be contained in the other. But this is impossible since $A \cap B \neq \emptyset$. □

The theorem provides great flexibility with respect to construction of connected sets.

7.8 Example The sphere S^n and the projective space P^n are connected for $n \geq 1$. It is enough to point this out for the sphere, since P^n is a continuous image of S^n (example 5.11). The connectedness of the sphere follows from theorem 7.7 when we write S^n as a union of two closed half-spheres, each homeomorphic to the unit cell D^n (example 3.15). For D^n is obviously connected (7.5).
S^0 is a discrete space consisting of two points, so it is disconnected. P^0 consists of only one point, and is naturally connected.

7.9 Let X be a topological space. We introduce a relation C between the points in X, writing xCy whenever x and y belong to a mutual connected subset. It is clear that the relation C is an equivalence relation; transitivity follows by an application of theorem 7.7. The space X is therefore partitioned in equivalence classes called *components*. Two points in X belong to the same component if and only if they can be joined by some connected set.
From theorem 7.2 it then follows that every component is connected, and by

definition that components are *maximal* as connected sets, that is, a component can not be subset of a larger connected set.
Since a point $x \in X$ belongs to exactly one component, we may speak of *the component* of x, which we denote C_x. It is the largest connected subset of X containing x. Naturally we have $C_x = C_y$ if x and y can be joined by some connected set.
If $A \subseteq X$ is an openclosed set and $x \in A$, then it follows that $C_x \subseteq A$, or else $C_x \cap A$ would be a proper, openclosed subset of C_x. From this we get

$$A = \bigcup_{x \in A} C_x$$

In other words, every nonempty openclosed set in X is a union of components.
One may wonder whether components themselves are openclosed. The fact is that they are closed, but not always open. To realize this, it suffices to show that the closure of a connected set is connected. If C is a component, we get
$C = \mathrm{cl}\ C$ because of the maximality of C. Hence C is closed.

7.10 Theorem If A is a connected subset of a space X, and B a subset such that $A \subseteq B \subseteq clA$, then B is connected. In particular, cl A is connected.

Proof. The condition $A \subseteq B \subseteq \mathrm{cl}\ A$ implies that A is dense in B. (Check.) We may therefore without loss of generality assume that $B = X$ and A dense in X.
Let $F \subseteq X$ be a nonempty, openclosed subset of X. Then F must intersect A (because of density) in a nonempty, openclosed subset of A. But since A is connected, we must have $F \cap A = A$, that is, $F \supseteq A$. Since A is dense and F is closed, this means that $F = X$. Hence there are no openclosed subsets of X other than $\emptyset$ and X, i.e. X is connected. □

7.11 Example We consider $\mathbb{Q}$, the set of rational numbers. If q is rational and i irrational and positive, then

$$\overline{U}(q; i) \cap \mathbb{Q} = U(q; i) \cap \mathbb{Q}$$

is an openclosed neighborhood of q in $\mathbb{Q}$, which we denote $V(q; i)$. The system

$$\{V(q; i)\}_{i \text{ irrational}}$$

is clearly a neighborhood basis for q in $\mathbb{Q}$. Since the component C_q of q in $\mathbb{Q}$ is contained in all $V(q; i)$ and $\cap_i V(q; i) = \{q\}$, we get that $C_q = \{q\}$. In other words the components of $\mathbb{Q}$ are all sets consisting of a single point! It can't get any worse. We say that $\mathbb{Q}$ is *totally disconnected*. Notice that the components here are open sets.

7.12 We say that a space X is *locally connected* if every point has a basis of connected neighborhoods. If X is *locally connected*, then the components of X are open subsets.

Theorem If X is locally connected, then the components of X are open subsets.

Proof. Let $C \subseteq X$ be a component. For every $x \in C$ we can find a connected neighborhood V_x of x. But then $V_x \subseteq C$, since C is the largest connected set containing x. C is therefore neighborhood of all its points, that is, open. □

Thus in a locally connected space the components are openclosed sets (since they are always closed). It follows from 7.9, last part, that components are *minimal* (nonempty) openclosed subsets in the space.
For instance, the set of rational numbers $\mathbb{Q}$ is not locally connected; we have already established that the components of $\mathbb{Q}$ are not open.
Local connectedness is obviously a local concept: If every point in X has a neighborhood which is locally connected (as a subspace), then X itself is locally connected. Since all Euclidean spaces clearly are locally connected, therefore all manifolds too, are locally connected. This applies in particular to S^n, P^n and $GL(n)$ for all $n \geq 0$.

7.13 There is a more intuitive notion of connectedness than the one we have considered till now. A space X is called *path connected* if for every pair of points $x, y \in X$ there is a path from x to y. By a *path* from x to y we mean a continuous map $\gamma : [0,1] \longrightarrow X$ such that $\gamma(0) = x$ and $\gamma(1) = y$. The image $\gamma[0,1]$ is then a connected subset of X containing x and y. It therefore follows from theorem 7.2 that a path connected space is connected. On the other hand, the converse does not generally hold. There are examples of spaces which are connected without being path connected.
We introduce a relation P between the points in X, writing xPy if there exists a path from x to y. It is easy to see that the relation P is an equivalence relation. The space X is therefore partitioned in equivalence classes called *path components*. Two points lie in the same path component if and only if they can be joined by a path. It is clear that the path components of X are the maximal path connected subsets of X.
Since a path component in particular is connected, it is contained in a unique component. Every component is thus a union of path components.
A topological space X is called *locally path connected* if every point in X has a basis of path connected neighborhoods. All manifolds, for example, are locally path connected. It is clear that local path connectedness implies local connectedness.

7.14 Theorem If X is locally path connected, then the path components of X are open subsets.

The proof is completely analogous to the proof of theorem 7.12.

Corollary In a locally path connected space components and path components coincide.

So in particular, a locally path connected space is connected if and only if it's path connected.

Finally we shall take a quick peek at products and quotients of connected spaces.

7.15 Theorem If X and Y are connected, then the product $X \times Y$ is connected.

Proof. It is enough to show that two arbitrary points (x, y) and (x', y') in $X \times Y$ belong to a connected subset; we may take $(X \times \{y'\}) \cup (\{x\} \times Y)$. □

The converse of the theorem is also valid: If $X \times Y$ is connected (and nonempty), then $X = pr_1(X \times Y)$ and $Y = pr_2(X \times Y)$ are also connected (theorem 7.4).
The theorem is readily generalized to any finite number of factors. In fact, the theorem holds for product spaces $X = \Pi X_\alpha$ in general.
The corresponding result regarding quotient spaces ("the quotient of a connected space is connected") is trivial by theorem 7.4. This is true since X/R is the continuous image of X under the canonical map, and it's therefore connected if X is connected. Even more interestingly, there is a converse statement.

7.16 Theorem Let X be a space and R an equivalence relation in X. If X/R and every equivalence class in X is connected, then X is connected.

Proof. Consider an openclosed set $A \subseteq X$. Let $p : X \longrightarrow X/R$ be the canonical map. For every $x \in A$, the equivalence class $p^{-1}\{p(x)\}$ lies entirely in A. (Or else $p^{-1}\{p(x)\}$ wouldn't be connected.) Thus it follows that A equals $p^{-1}(p(A))$, which is then openclosed. But then $p(A)$ is openclosed in X/R. Since X/R is connected we must have $p(A) = X/R$ or $p(A) = \emptyset$. But that means $A = X$ or $A = \emptyset$. It follows that X is connected. □

Theorem 7.16 covers the product theorem 7.15. For $pr_1 : X \times Y \longrightarrow X$ is an identification where the equivalence classes $pr_1^{-1}\{x\}$ are all homeomorphic to Y.

7.17 The definition of a connected space was given by C. Jordan. In Jordan's book "Cours d'Analyse" from 1887 we find formulated the statement that every simple closed curve in the plane divides the plane in two connected regions. This result, known as Jordan's curve theorem, would come to play an important role in the history of topology. (A correct proof of the theorem was first given in 1905 by O. Veblen). A systematic study of the concept of connectedness in general was first established with Hausdorff's book in 1914.

Chapter 8

Complete metric spaces

In this chapter we again deal with metric spaces, focusing mainly on the important notion of *completeness* associated with the Cauchy convergence criteria, and the interplay between completeness and compactness.

8.1 An important result in analysis is the so-called Cauchy's convergence principle for real sequences. This result provides an intrinsic characterization of convergent sequences, meaning, a characterization which refers only to the sequence itself and not to any eventual limit.

Theorem (Cauchy). A real sequence (x_n) is convergent if and only if for every real number $\epsilon > 0$ there is a natural number N such that $|x_m - x_n| < \epsilon$ when $m,n \geq N$.

A real sequence (x_n) with the property specified in the last part of the theorem, namely $|x_m - x_n| \to 0$ when $m,n \to \infty$, is called a *Cauchy sequence*. So the theorem states that a real sequence is convergent if and only if it is a Cauchy sequence.

Proof. One direction of the proof is a simple application of the triangle inequality. Assume that (x_n) converges to x. Then $|x_n - x| \to 0$ when $n \to \infty$. From this we get $|x_m - x_n| \leq |x_n - x| + |x_m - x| \to 0$ when $|x_n - x| \to 0$. Hence (x_n) is a Cauchy sequence.
The most important part of the theorem is the converse, namely the fact that every Cauchy sequence of real numbers is convergent. To prove this we make use of the principle of existence of supremum (infimum) in the real number system.
Let (x_n) be a Cauchy sequence. Then (x_n) is bounded. (Check this.) Therefore $s_m = \sup_{n \geq m} x_n$ exists for all m. The sequence (s_m) is also bounded, and furthermore it is decreasing. Thus it converges to a number $x = \inf_m s_m$. We show that

the sequence x_n converges to x. For given $\epsilon > 0$ we may choose N such that $|s_n - x| < \epsilon/3$ and $|x_n - x_m| < \epsilon/3$ for m, $n \geq N$. Next, we may choose $m \geq N$ such that $|x_m - s_N| < \epsilon/3$. By adding the inequalities we get $|x_n - x| < \epsilon$ for $n \geq N$. This means that (x_n) converges to x. The theorem is proved. □

In the set of rational numbers $\mathbb{Q}$, the Cauchy convergence principle does not hold. We can choose a rational sequence (r_n) which converges to the irrational number $\sqrt{2}$. Then (r_n) is a Cauchy sequence in $\mathbb{Q}$, but not convergent in $\mathbb{Q}$. The limit point $\sqrt{2}$ is missing in the rational number system.

8.2 A point sequence (x_n) in a metric space X is called a *Cauchy sequence* if $d(x_m, x_n) \to 0$ when $m,n \to \infty$. Every convergent sequence in X is a Cauchy sequence. Again, this is easily realized by applying the triangle inequality. The metric space X is said to be *complete* if every Cauchy sequence in X is convergent. According to 8.1, $\mathbb{R}$ equipped with the usual metric $d(x, y) = |x - y|$ is complete, while $\mathbb{Q}$ is not complete.
It is important to be aware of the fact that completeness is not a topological notion. For instance, all open intervals in $\mathbb{R}$ are homeomorphic, but only $\mathbb{R}$ itself amongst these is complete with respect to the usual metric. In the open interval $(0, \infty)$, for example, the sequence $1, \frac{1}{2}, \ldots, \frac{1}{n}, \ldots$ is a Cauchy sequence which is not convergent.

8.3 Theorem A subspace A of a complete metric space X is complete if and only if A is closed in X.

Proof. Assume A is complete. Every sequence in A which converges in X is a Cauchy sequence and therefore must have its limit point in A. By 2.14 A is closed in X. Now assume A is closed in X. Any Cauchy sequence in A converges in X. By 2.14 the limit point will belong to A. Hence the sequence is convergent in A. This means that A is complete.
We shall study a few important examples of complete metric spaces.

8.4 Example The space $\mathbb{R}^k$ with the Euclidean metric is complete. This is a simple consequence of the completeness of $\mathbb{R}$. It suffices to note that a point sequence (x_n) in $\mathbb{R}^k$ is a Cauchy sequence if and only if every component sequence in $\mathbb{R}$ is a Cauchy sequence, and that (x_n) is convergent if and only if every component sequence in $\mathbb{R}$ is convergent (see 4.11).
As shown in 6.13, all norms on $\mathbb{R}^k$ are equivalent. Two equivalent norms define the same Cauchy sequences and naturally the same convergent sequences (the convergence depends only on the topology). It then follows that $\mathbb{R}^k$ is complete also with respect to the metric induced by an arbitrary norm. Thus every finite

dimensional normed vector space over $\mathbb{R}$ is complete.
As for infinite dimensional normed vector spaces, completeness is not automatic, but something, if it is the case anyway, which requires proof.

8.5 Example We will show that the Hilbert space H of square convergent real sequences (square summable sequences), see 6.15 for the definition, is complete with respect to the metric induced by the inner product in H.
Let (x_n) be a Cauchy sequence in H. So we have for each n, $x_n = (x_{nk})_{k=1,2,\ldots}$ where $|x_n|^2 = \sum_k x_{nk}^2$. By the inequality

$$|x_{mk} - x_{nk}| \leq |x_m - x_n|$$

each of the sequences $(x_{nk})_{n=1,2,\ldots}$ is a Cauchy sequence in $\mathbb{R}$ and hence converges to a limit $a_k \in \mathbb{R}$. This yields a sequence $a = (a_k)$. We shall show that $a \in H$ and that (x_n) converges to a. For a given $\epsilon > 0$ we choose N such that

$$\sum_k (x_{mk} - x_{nk})^2 = |x_m - x_n|^2 \leq \epsilon$$

when $m,n \geq N$. For every natural number M we then have

$$\sum_{k \leq M} (x_{mk} - x_{nk})^2 \leq \epsilon$$

and by letting $n \to \infty$ we get

$$\sum_{k \leq M} (x_{mk} - a_k)^2 \leq \epsilon$$

Thus also

$$\sum_k (x_{mk} - a_k)^2 \leq \epsilon$$

for $m \geq N$. This shows that $x_m - a \in H$, hence $a \in H$. In particular we have $|x_m - a|^2 \leq \epsilon$ for $m \geq N$. Therefore (x_n) must converge to the limit a. This means H is complete, as was claimed.

Among the infinite dimensional normed vector spaces, the Hilbert space H has the geometry which most closely resembles the usual Euclidean geometry in $\mathbb{R}^n$. This plays a role, first and foremost, in analysis, where it is used for instance in the theory of Fourier series. Generally speaking, a complete inner product space is called a *Hilbert space*.

8.6 Example Let $B(X)$, as in 1.14, denote the vector space of bounded, real functions on a set X, with the supremum norm. Assume that (f_n) is a Cauchy sequence in $B(X)$. Then for every $\epsilon > 0$ there is an N such that $|f_m - f_n|_{\sup} \leq \epsilon$ when $m,\ n \geq N$. This implies $|f_m(x) - f_n(x)| \leq \epsilon$ when $m,\ n \geq N$ for all $x \in X$. Therefore $(f_n(x))$, for all $x \in X$, is a Cauchy sequence in $\mathbb{R}$, and the limit $f(x) = \lim_{n\to\infty} f_n(x)$ exists. By letting $n \to \infty$ in the preceding we get $|f(x) \leq |f_m|_{\sup} + \epsilon$ for all $x \in X$, so f is bounded. Furthermore, $|f_m - f|_{\sup} \leq \epsilon$ for $m \geq N$. This means (f_n) converges to f in $B(X)$. The space $B(X)$ is therefore complete.

8.7 Example Let (f_n) be a sequence of continuous, real functions defined on a topological space X, and assume (f_n) converges to f uniformly on X. Then the limit f is also continuous. This is a well-known and fundamental result from analysis in a general form. The usual $\epsilon/3$-proof goes without any changes: If $x \in X$ and $\epsilon > 0$ is given, we may choose N such that $|f - f_n|_{\sup} \leq \epsilon/3$ for $n \geq N$, and a neighborhood U of x such that $|f_N(x) - f_N(y)| \leq \epsilon/3$ for $y \in U$. Then we have

$$|f(x)-f(y)| \leq |f(x)-f_N(x)|+|f_N(x)-f_N(y)|+|f_N(y)-f(y)| \leq \frac{\epsilon}{3}+\frac{\epsilon}{3}+\frac{\epsilon}{3} = \epsilon$$

for all $y \in U$. This shows that f is continuous at x.

We let $BC(X) = B(X) \cap C(X)$ denote the vector space of all continuous, bounded, real functions defined on X. By previous descussions we have already shown $BC(X)$ to be closed in $B(X)$. Therefore $BC(X)$ is complete with respect to the supremum norm. In particular, if X is a compact space, then by the extremal value theorem (6.12), $C(X) \subseteq B(X)$. In this case $C(X)$ is complete with respect to the supremum norm.
Complete normed vector spaces are called *Banach spaces*. They play a central role in analysis.

8.8 In a metric space compactness can be characterized by the behaviour of sequences. We shall take a brief look at this aspect of compactness.

Let $(x_n)_{n=1,2,\ldots}$ be a sequence in a topological space X. By an *accumulation point* for (x_n) we mean a point $a \in X$ such that every neighborhood U of a contains terms x_n of the sequence for infinitely many n. ("The sequence is frequently near a".) For instance, the sequence $(\sin\frac{n\pi}{2})$ has three accumulation points: 0,1 and -1, which are the only values the sequence takes on. On the other hand, the sequence $(\sin(\frac{n\pi}{2} + \frac{1}{n}))$ has the same accumulation points, even though this sequence never takes on these values.

A limit point for a convergent sequence is clearly also an accumulation point for the sequence. More generally, a limit point for a subsequence of a sequence in X will always be an accumulation point for the sequence; the converse holds if X has countable neighborhood bases. (Check.)
A well-known result from analysis states that every bounded real sequence has an accumulation point (Bolzano-Weierstrass). This is a special case of the following:

Theorem In a compact space every sequence has an accumulation point.

Proof. Assume there is a sequence (x_n) in X without an accumulation point. Then every point $x \in X$ has an open neighborhood U_x which contains x_n for only finitely many n. This gives an open cover $\{U_x\}_{x\in X}$ of X which obviously cannot be reduced to a finite cover. □

8.9 We shall show that the converse of the theorem holds in metric spaces. Let again X be a metric space.
We observe that if a Cauchy sequence has an accumulation point, then it is convergent to the point. Assume that (x_n) is a Cauchy sequence in X with x as accumulation point. For given $\epsilon > 0$ we can find an N such that $d(x_m, x_n) < \epsilon/2$ for $m,n \geq N$, and choose $m \geq N$ such that $d(x_m, x) < \epsilon/2$. It then follows that $d(x_n, x) \leq d(x_m, x_n) + d(x_m, x) < \epsilon$ for $n \geq N$. Hence (x_n) is convergent to x.

Corollary Every compact metric space is complete.

A compact metric space is also bounded (see 6.6). Regarding subsets of $\mathbb{R}^n$, we know that they are compact if and only if they are closed and bounded, that is, complete and bounded as metric spaces (see 6.6). However, this does not apply to metric spaces in general. For instance, the unit sphere in the Hilbert space H, as we have seen, is not compact. This in spite of being closed in H and thus complete, and of course bounded.
A metric space X is called *totally bounded* if for every $\epsilon > 0$ it is possible to cover X with a finite number of open ϵ-neighborhoods. Another way to express this is by saying that for every $\epsilon > 0$, X can be divided into a finite number of sets of diameter $\leq \epsilon$. It is clear that a totally bounded metric space is bounded. On the other hand, the unit sphere in H illustrates the fact that the converse is not always valid. Here a subset of diameter $< \sqrt{2}$ can at most contain one of the basis vectors $e_1, e_2, \ldots$. A finite number of such sets can therefore not cover the whole unit sphere. We now have an important theorem.

Theorem For a metric space the following statements are equivalent.

a) X is compact

b) Every sequence in X has an accumulation point

c) X is complete and totally bounded

d) Every sequence in X has a convergent subsequence.

Proof.
(a) $\Rightarrow$ (b) follows by theorem 8.8.
(b) $\Rightarrow$ (c). Assume (b). Every Cauchy sequence in X is then convergent, so X is complete. If X is not totally bounded, there is an $\epsilon > 0$ such that X cannot be covered by a finite number of ϵ-neighborhoods. Then we may successively choose points x_1, x_2, ... in X such that every point has distance $\geq \epsilon$ from all the previous points. The sequence (x_n) does not have an accumulation point in X, in contradiction to the assumption, hence X is totally bounded.
(c) $\Rightarrow$ (a). Assume (c). Let $\{U_i\}_{i\in I}$ be an open cover of X. Suppose X is not finitely covered by sets from the collection $\{U_i\}$. Since X is totally bounded, we may divide X into a finite number of sets of diameter ≤ 1. Then there must be one of these, A_0, which is not finitely covered either. We may divide A_0 into a finite number of sets of diameter $\leq \frac{1}{2}$. One of these, A_1, is not finitely covered, etc. Continuing this way we inductively get a decreasing sequence of sets $A_0 \supseteq A_1 \supseteq \ldots$ where diam $A_n \leq 2^{-n}$, and where A_n is not finitely covered for any n. We now choose $x_n \in A_n$. Then (x_n) is a Cauchy sequence in X. Since X is complete, $x = \lim_{n\to\infty} x_n$ exists. We then have $x \in U_j$ for a $j \in I$. It then follows that $U_j \supseteq A_n$ for n sufficiently large. But this is in contradiction to A_n not being finitely covered. Thus $\{U_i\}_{i\in I}$ can be reduced to a finite cover, so X is compact.
(b) $\Leftarrow\Rightarrow$ (d) is trivial. The theorem is proved. $\square$

It is clear that in $\mathbb{R}^n$ every bounded set is also totally bounded. We have thus again proved that sets in $\mathbb{R}^n$ are compact if and only if they are bounded and closed (theorem 6.6).

8.10 A map $f : X \longrightarrow X$ of a metric space into itself is called a *contraction* if there exists a real constant $\lambda < 1$ such that

$$d(f(x), f(x')) \leq \lambda d(x, x')$$

for all x, $x' \in X$. This of course implies that f is continuous. A point $x \in X$ is called a *fixed point* for f if $f(x) = x$.

Theorem (The contraction principle). A contraction of a complete metric space has a unique fixed point.

Proof. Assume $X \neq \emptyset$ is a complete metric space and $f : X \longrightarrow X$ a contraction with constant λ. Let $x_0 \in X$ be an arbitrary point. We define $x_n = f^n(x_0)$ for $n \geq 1$ (the map f applied n times), and let $r = d(x_0, x_1)$. It follows that $d(x_m, x_{m+1}) \leq \lambda^m r$ and thus

$$d(x_n, x_{n+k}) \leq d(x_n, x_{n+1}) + \ldots + d(x_{n+k-1}, x_{n+k}) \leq \lambda^n r + \ldots + \lambda^{n+k-1} r \leq \frac{\lambda^n r}{1-\lambda} \to 0$$

Therefore (x_n) is a Cauchy sequence in X. Let $a = \lim_{n\to\infty} x_n$. Since f is continuous, we have (see 3.5)

$$f(a) = f(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = a$$

which shows that a is a fixed point for f.
Assume a, $b \in X$ are fixed points for f. Then

$$d(a, b) = d(f(a), f(b)) \leq \lambda d(a, b)$$

which gives $d(a, b) = 0$ and therefore $a = b$. This shows the uniqueness of the fixed point.
The theorem is thus proved. □
We may consider x_n to be an n-th approximation to the fixed point a. The inequality

$$d(x_n, a) = \lim_{k\to\infty} d(x_n, x_{n+k}) \leq \frac{\lambda^n r}{1-\lambda}$$

is a measure of the effectiveness of this approximation.
The contraction principle has a number of interesting and important applications. We shall give only one example of this.

8.11 Example We consider a differential equation of the form

$$y' = f(t, y)$$

$$y(t_0) = y_0$$

where $f : [t_0, t_1] \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous, $[t_0, t_1] \subseteq \mathbb{R}$ and $y_0 \in \mathbb{R}$. With a *solution* we mean a differentiable function $y : [t_0, t_1] \longrightarrow \mathbb{R}$ which satisfies the differential equation and meets the initial requirement.
We shall apply the contraction principle to prove the existence and uniqueness of a solution, under the assumption that f satisfies an additional condition.

It is clear that finding a solution is in fact the same as finding a continuous function $y : [t_0, t_1] \longrightarrow \mathbb{R}$ which satisfies the integral equation

$$y(u) = y_0 + \int_{t_0}^{u} f(t, y(t))dt$$

We may therefore work in the space $C[t_0, t_1]$ of continuous, real functions on $[t_0, t_1]$, which is a complete metric space with the supremum norm d of 8.7. Let $F : C[t_0, t_1] \longrightarrow C[t_0, t_1]$ be defined by

$$F(y)(u) = y_0 + \int_{t_0}^{u} f(t, y(t))dt$$

A solution of the integral equation is then a fixed point for F, and vice versa. Let us assume that f satisfies a *Lipschitz*-requirement with respect to the second variable, that is,

$$|f(t, y_1) - f(t, y_2)| \leq L|y_1 - y_2|$$

for all $t \in [t_0, t_1]$ and $y_1, y_2 \in \mathbb{R}$, where L is a constant. We then get $d(F(\phi), F(\psi)) \leq (t_1 - t_0)Ld(\phi, \psi)$ for all $\phi, \psi \in C[t_0, t_1]$ since

$$\begin{aligned} d(F(\phi), F(\psi)) &= \sup_{t_0 \leq u \leq t_1} |F(\phi)(u) - F(\phi)(u)| \\ &= \sup_{t_0 \leq u \leq t_1} |\int_{t_0}^{u} (f(t, \phi(t)) - f(t, \psi(t)))dt| \\ &\leq (t_1 - t_0) \sup_{t_0 \leq t \leq t_1} |f(t, \phi(t)) - f(t, \psi(t))| \\ &\leq (t_1 - t_0) \sup_{t_0 \leq t \leq t_1} L|\phi(t) - \psi(t)| = (t_1 - t_0)Ld(\phi, \psi) \end{aligned}$$

Therefore F is a contraction if $\lambda = (t_1 - t_0)L < 1$, that is, if $L < \frac{1}{t_1 - t_0}$. By the contraction principle we arrive at the conclusion: The differential equation in question has a unique solution on the interval $[t_0, t_1]$ if f satsifies the Lipschitz-condition with Lipschitz-constant $L < \frac{1}{t_1 - t_0}$.

8.12 In arbitrary topological spaces the intersection of a finite number of open, dense sets is dense. For if $O_1, O_2, \ldots, O_n$ are open, dense sets in a space X and U is a nonempty, open set, then $U \cap O_1$ is nonempty and open since O_1 is dense and open. By the same reasoning $(U \cap O_1) \cap O_2$ is nonempty and open, and so forth. It follows that $(\cdots((U \cap O_1) \cap O_2) \cdots \cap O_{n-1}) \cap O_n = U \cap (\cap O_i)$ is nonempty.

If we replace the finite set $\{O_i\}_{i=1}^{n}$ with an infinite set, then in general the result no longer holds. But it still holds for *countable* sets if X is a complete, metric space, and the result is then called Baire's density theorem. The proof is conducted in a similar manner as the argument above, except that we must take greater care of the repeated intersections to ensure that the infinite intersection doesn't vanish.

Theorem (Baire) In a complete metric space every countable intersection of open, dense sets is dense.

Proof. Assume X to be a complete, metric space and let O_1, O_2, ... be open, dense sets in X. Let U be an arbitrary, nonempty, open set. Then $U \cap O_1$ is nonempty and open; choose $x_1 \in U \cap O_1$ and let ϵ_1 be a positive number such that $\overline{U}(x_1; \epsilon_1) \subseteq U \cap O_1$. Then $U(x_1; \epsilon_1) \cap O_2$ is nonempty and open. Choose $x_2 \in U(x_1; \epsilon_1) \cap O_2$ and let $\epsilon_2 < \epsilon_1/2$ be such that $\overline{U}(x_2; \epsilon_2) \subseteq U(x_1; \epsilon_1) \cap O_2$. By induction we choose x_n and $\epsilon_n < \epsilon_{n-1}/2$ such that $\overline{U}(x_n; \epsilon_n) \subseteq U(x_{n-1}; \epsilon_{n-1}) \cap O_n$ for $n = 1, 2, \ldots$.

It follows that x_n, x_{n+1}, ... all belong to $U(x_n, \epsilon_n)$. Since $\epsilon_n \to 0$, the sequence (x_n) is Cauchy. Now set $x = \lim x_n$. We get $x \in \overline{U}(x_n; \epsilon_n)$ for all n. But then

$$x \in \bigcap_n \overline{U}(x_n; \epsilon_n) \subseteq U \cap (\bigcap_n O_n)$$

It shows that $\cap_n O_n$ intersects every nonempty, open set U, and thus is a dense set in X. □

Corollary In a complete metric space every countable union of closed sets without interior points is itself without interior points.

This is a reformulation of the theorem by complements. For instance, $\mathbb{R}$ may not be written as a countable union of closed sets without interior points.

8.13 In conjunction with Baire's theorem the following notion is in order. A subset of a topological space X is called *residual* if it contains a countable intersection of open, dense sets in X. On the other hand, a subset of X is called *meager* if it is contained in a countable union of closed sets without interior points in X. This is equivalent to the complement set being residual. It is clear that every countable intersection of residual sets in X is a residual set, and every countable union of meager sets is meager. In $\mathbb{R}$, for instance, the irrational numbers constitute a residual set, while the rational numbers make a meager set.
A topological space X is called a *Baire space* if every residual set in X is dense in X, or equivalently every meager set in X is without interior points in X. We thus have

Theorem A complete metric space is a Baire space.

8.14 The formulation of Cauchy's convergence principle in the real number system goes back to Bolzano, who in 1817 gave a rigorous proof of the intermediate

value theorem which was based on this principle. Cauchy later took the principle as a starting point in order to give the theory of infinite series a firm standing. An important task which was undertaken in the last half of the last century, was to find a fully satisfying construction of the real numbers. This task was completed by several mathematicians. A method which was proposed by Charles Méray in 1869 and Georg Cantor in 1872, consisted of perceiving a real number as an equivalence class of Cauchy sequences of rational numbers. This means that the real number field $\mathbb{R}$ is constructed as a completion of the rational number field $\mathbb{Q}$.
The notion "completeness" for metric spaces was defined by Maurice Fréchet in 1906. By a direct generalization of Méray's and Cantor's method, Hausdorff showed in 1914 that every metric space can be embedded in a complete metric space. The completeness of the Hilbert space was however shown by M. Fréchet and E. Schmidt as early as in 1908.
The fact that the limit of a sequence of continuous functions need not be continuous, has not always been clear. Cauchy made a mistake regarding this. It was corrected with Abel's work on series, and later led Weierstrass to define the notion "uniform convergence". The supremum metric for functions was first studied by Fréchet in 1906.
Theorem 8.12 was proved by W. Osgood in 1898 for $\mathbb{R}$ and independently by R. Baire in 1899 for $\mathbb{R}^n$. Baire introduced the concept "meager set" (in his terminology: "a set of first category") in order to characterize functions which are limits for continuous functions.

Chapter 9

Exercises

1.1 Show that in a metric space the following inequality applies

$$d(x, y) \geq |d(x, z) - d(z, y)|$$

for all points x,y,z.

1.2 Verify that in a metric space the set $U(x; \epsilon) = \{y : d(x, y) < \epsilon\}$ is open and the set $\overline{U}(x; \epsilon) = \{y : d(x, y) \leq \epsilon\}$ is closed.

1.3 Show that diam $U(x; r) \leq 2r$ in a metric space.

1.4 Show that a set A in a metric space X is bounded if and only if for all $x \in X$ there is an $r > 0$ such that $A \subseteq U(x; r)$.

1.5 Let X be a metric space with metric d. Assume $f : [0, \infty) \to \mathbb{R}$ is an increasing function such that $f(0) = 0$, $f(u) > 0$ for $u > 0$, and $f(u + v) \leq f(u) + f(v)$.
(a) Show that $d' = f \circ d$ is a metric on X.
(b) Find a condition on f which guarantees that d and d' are equivalent metrics.
(c) Show that the functions $log(1 + u)$, $u/(1 + u)$ and $min(1, u)$ define metrics equivalent to d.

1.6 Let $\rho(x, y)$ denote the length of the shortest great circle arc which connects two points x and y on the surface of the sphere $S^2 = \{x \in \mathbb{R}^3 : |x| = 1\}$ (geodesic distance).
(a) Show that ρ is a metric on S^2.
(b) Show that the topology defined by ρ coincides with the usual topology on S^2.

1.7 Let X be an arbitrary set. A subset V of X is called a neighborhood of an element $x \in X$ if V contains x and is *cofinite*, that is, $X - V$ is finite. Show that this also defines a neighborhood structure on X. Determine the open and closed sets in X.

1.8 Show that $\mathbb{R}^n$ is a normed vector space with norm defined by

$$|x|_1 = |x_1| + \cdots + |x_n|$$

for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Sketch the unit ball $B = \{x \in \mathbb{R}^n : |x|_1 \leq 1\}$ with respect to this norm for $n = 2$. Show that this norm is equivalent to the Euclidean norm in $\mathbb{R}^n$.

1.9 Show that $C[a, b]$ is a normed vector space with norm

$$|f|_1 = \int_a^b |f(x)|dx.$$

Is this norm equivalent to the supremum norm?

1.10 Show that in an inner product space V the Parallelogram Law applies:

$$|x + y|^2 + |x - y|^2 = 2|x|^2 + 2|y|^2$$

for all x,$y \in V$. What is the geometrical interpretation of this law?

1.11 Let V be a vectorspace over the complex numbers $\mathbb{C}$. A *norm* in V is defined analogously to the real case by axioms **No1**-**No3** in 1.8, where instead $\alpha \in \mathbb{C}$. An *inner product* in V is a function $(x, y) \to (x|y)$ from $V \times V$ to $\mathbb{C}$, such that
(1) $(x + y|z) = (x|z) + (y|z)$
(2) $(\alpha x|y) = \overline{\alpha}(x|y)$
(3) $(x|y) = \overline{(y|x)}$
(4) $(x|x) > 0$ when $x \neq 0$.
for x,y,$z \in \mathbb{C}$. Assume these properties to hold. Let $|x| = (x|x)^{\frac{1}{2}}$.
(a) Prove the Cauchy-Schwarz inequality

$$|(x|y)| \leq |x||y|$$

for x, $y \in V$.
(b) Show that $|\ |$ is a norm in V. (c) Find a formula for the inner product expressed by the norm in a complex innerproduct space.

1.12 A set A in a vectorspace V is called *convex* if for all points $x, y \in A$ the segment $[x, y] = \{(1-t)x + ty : 0 \leq t \leq 1\}$ is also contained in A.
(a) Show that A is convex if and only if $x_0, \ldots, x_n \in A$, $\lambda_0, \ldots, \lambda_n \geq 0$ and $\sum \lambda_i = 1$ implies $\sum \lambda_i x_i \in A$.
(b) Show that there is a smallest convex set in V which contains A, *the convex hull* of A. Describe the points of this set.

1.13 Determine the convex sets in $\mathbb{R}$.

1.14 A function $f : I \to \mathbb{R}$ defined on an interval I in $\mathbb{R}$ is called *convex* if

$$A(f) = \{(x, y) \in \mathbb{R}^2 : x \in I, f(x) \leq y\}$$

(the epigraph of f) is a convex set.
(a) Show that f is convex if and only if

$$f((1-t)x + ty) \leq (1-t)f(x) + tf(y)$$

for all x,$y \in I$ and $0 \leq t \leq 1$. What does this mean geometrically?
(b) Assume f to be once differentiable. Show that f is convex if and only if f' is increasing.
(c) Assume f to be twice differentiable. Show that f is convex if and only if $f'' > 0$.

1.15 Define the concept of convexity for functions $f : D \to \mathbb{R}$ where D is a convex set in a vector space V. Show that
(a) A composition $f \circ A$ of a convex function f and a linear map A is a convex function.
(b) A linear combination $f = \sum \lambda_i f_i$ with coefficients $\lambda_i \geq 0$, of convex functions f_i, is convex.
(c) If $f : V \to \mathbb{R}$ is a convex function, then the set $B = \{x \in V : f(x) \leq 1\}$ is convex.

1.16 Let V be a vector space and $p : V \to \mathbb{R}$ be a function such that $p(x) > 0$ for $x \neq 0$, and $p(\alpha x) = |\alpha| p(x)$ for all $x \in V$, $\alpha \in \mathbb{R}$. Show that p is a norm on V if and only if $B = \{x \in V : p(x) \leq 1\}$ is convex.

1.17 We try to define a norm in $\mathbb{R}^n$ by

$$|x|_p = \left(\sum_i |x_i|^p \right)^{\frac{1}{p}}$$

p being a positive real number.
(a) Check for which values of p this is a norm. Sketch the unit ball $B = \{x \in \mathbb{R}^n : |x|_p \leq 1\}$ for $n = 2$, $p = 2/3, 1, 2, 4$.
(b) Show that the norms $|\ |_p$ in $\mathbb{R}^n$ are equivalent, and that $\lim_{p\to\infty} |x|_p = |x|_{\sup}$.

1.18 Show that $C[a, b]$ is a normed vector space with norm

$$|f|_p = \left(\int_a^b |f(x)|^p dx\right)^{\frac{1}{p}}$$

for $1 \leq p < \infty$. Find $\lim_{p\to\infty} |f|_p$.

1.19 La V and W be normed vectorspaces.
(a) Show that a linear map $A : V \to W$ is continuous if and only if $\sup_{|x|\leq 1} |Ax| < \infty$.
(b) Show that the set $L(V, W)$ of continuous, linear maps $A : V \to W$ is a normed vector space with usual vector space operations, and *operatornorm* defined by $||A|| = sup_{|x|\leq 1}|Ax|$. Show the relation $|Ax| \leq ||A||x|$ for all $x \in V$.

1.20 Verify that a family $(\beta(x))_{x\in X}$ of systems of subsets in a set X constitute bases of open sets for a neighborhood structure on X if and only if the following three conditions are met:
• **NB1**. Every $\beta(x)$ is nonempty, and if $V \in \beta(x)$, then $x \in V$.
• **NB2**. If $V_1, V_2 \in \beta(x)$, then there is a $V_3 \in \beta(x)$ such that $V_3 \subseteq V_1 \cap V_2$.
• **NB3**. If $V \in \beta(x)$ and $y \in V$, there is a $W \in \beta(y)$ such that $W \subseteq V$.
(F. Hausdorff 1914).

2.1 In $\mathbb{R}^2$, consider the subset $A = \{(x, y) : y = \sin \frac{1}{x}, 0 < x < 1\}$. Determine int A, cl A, int $(\mathbb{R}^2 - A)$ and cl $(\mathbb{R}^2 - A)$.

2.2 Prove that

$$\text{cl } (A \cup B) = \text{cl } A \cup \text{cl } B$$

$$\text{int } (A \cap B) = \text{int } A \cap \text{int } B$$

where A and B are subsets of a topological space. See if similar rules apply for the closure of an intersection, and the interior of a union of two sets.

2.3 Show that for a subset A in a topological space we have bd cl $A \subseteq$ bd A and bd int $A \subseteq$ bd A. Give an example in $\mathbb{R}$ where the three sets are different.

2.4 Show by example that cl $U(x; r)$ doesn't always equal $\overline{U}(x; r)$ in a metric space.

2.5
(a) Show that $\mathbb{R}^n$ has a countable basis for the topology.
(b) A space is called *separable* if it has a countable, dense subset. Show that a metric space has a countable basis for the topology if and only if the metric space is separable.

2.6 Let $F(\mathbb{R})$ denote the set of real functions defined on $\mathbb{R}$. Show that the sets

$$N(f; \epsilon) = \{g \in F(\mathbb{R}) : \sup_{x \in \mathbb{R}} |f(x) - g(x)| < \epsilon\}$$

for $\epsilon > 0$ and $f \in F(\mathbb{R})$ constitute neighborhood bases for a topology on $F(\mathbb{R})$. What does it mean that a sequence in $F(\mathbb{R})$ converges in this topology? Can this topology be derived from a metric on $F(\mathbb{R})$?

2.7 Which point sequences converge in $\mathbb{R}$ with the cofinite topology?

2.8 Let X be a metric space with metric d and assume that A is a nonempty subset of X.
(a) Show that the distance $d(x, A)$ depends continuously on $x \in X$.
(b) Show that cl $A = \{x \in X : d(x, A) = 0\}$.

2.9 Show that two disjoint closed sets in a metric space always have disjoint open neighborhoods.

2.10 Let $C(\mathbb{R})$ denote the set of continuous real functions on $\mathbb{R}$. For $f \in C(\mathbb{R})$ and U an open neighborhood of

$$graf(f) = \{(x, y) \in \mathbb{R}^2 : y = f(x)\}$$

we define

$$N(f; U) = \{g \in C(\mathbb{R}) : graf(g) \subseteq U\}$$

(a) Show that the sets $N(f; U)$ constitute neighborhood bases for a topology on $C(\mathbb{R})$, *the graph topology*.
(b) Show that the graph topology on $C(\mathbb{R}$ is strictly finer than the topology for uniform convergence.
(c) What does it mean that a sequence in $C(\mathbb{R})$ converges with respect to the graph topology?

2.11 Let $F = F(X, Y)$ be the set of maps from a set X to a topological space Y. Let τ be the topology on F generated by the sets of the form $U(x, O) = \{f \in F : f(x) \in O\}$ where $x \in X$ and $O \subseteq Y$ is open.
(a) Describe a neighborhood basis of each element $f \in F$ in the topology τ.
(b) Show that $f_n \to f$ with respect to the topology τ if and only if $f_n(x) \to f(x)$ for all $x \in X$.
(τ generalizes the topology from example 1.15 and is called *the topology for pointwise convergence* on F.

2.12 Let X be a subspace of a topological space Y. Assume $A \subseteq X$. Show that for closure and interior relative to X and Y we have
(1) $\mathrm{cl}_X\, A = X \cap \mathrm{cl}_Y A$
(2) $\mathrm{int}_X A \supseteq \mathrm{int}_Y A$

Show by example that there need not be equality in (2) in general.

2.13 Let $C_0 = [0, 1]$. By removing the open, middle third of C_0 we obtain $C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. By again removing the open, middle third of C_1 we obtain $C_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{3}{9}] \cup [\frac{6}{9}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$ etc. C_n is a disjoint union of 2^n closed intervals and C_{n+1} is obtained from C_n by removing the open, middle thirds of these intervals. *The Cantor set* is defined as the intersection $C = \bigcap_{n=0}^{\infty} C_n$. Show that
(a) C is closed in $\mathbb{R}$ and without interior points.
(b) C is without isolated points.
(c) C is uncountable. Show that C consists of all numbers of the form $\sum_{n=1}^{\infty} \frac{c_n}{3^n}$ where $c_n = 0$ or $c_n = 2$.
(A point is called *isolated* in a set if it has a neighborhood which intersects the set in only the point itself.)

2.14

(a) Verify that for the closure $\overline{A} = \text{cl } A$ in a topological space, the following properties apply:

(1) $\overline{\emptyset} = \emptyset$

(2) $A \subseteq \overline{A}$

(3) $\overline{A \cup B} = \overline{A} \cup \overline{B}$

(4) $\overline{\overline{A}} = \overline{A}$

(b) Let X be a set and assume we have at our disposal an operation which to each subset A of X assigns a subset $\overline{A}$ of X such that the relations (1)-(4) above are satisfied. Show then, that X in a unique way is a topological space with the given operation as the closure operation (K. Kuratowski 1922).

3.1 Let $f, g : X \to \mathbb{R}$ be continuous functions defined on a topological space X. Show that
(a) If $f(x) = g(x)$ for all x in a dense subset of X, then $f = g$ (principle of extension of equality). Generalize.
(b) If $f(x) \leq g(x)$ for all x in a dense subset of X, then $f \leq g$ (principle of extension of inequality).
(c) Explain that if $h : X_0 \to \mathbb{R}$ is a function defined on a dense subset $X_0 \subseteq X$ which has a continuous extension $f : X \to \mathbb{R}$, then f is uniquely determined.
(d) Show by example that not every continuous function $h : X_0 \to \mathbb{R}$ may be continuously extended to X, for arbitrary X and X_0.

3.2 Assume f and g to be continuous real functions on a topological space. Show that $h = \max(f, g)$ is continuous.

3.3 Show that a map $f : X \to Y$ between topological spaces is continuous if and only if $f(\text{cl } A) \subseteq \text{cl } f(A)$ for every $A \subseteq X$.

3.4 Consider the vector space $C[a, b]$ with the supremum norm.
(a) Let $C^1[a, b]$ denote the linear subspace of functions with continuous derivative. Is this a closed space?
(b) Define the linear maps $D : C^1[a, b] \to C[a, b]$ and $I : C[a, b] \to C^1[a, b]$ by

$$D(f)(x) = f'(x)$$

$$I(f)(x) = \int_a^x f(t)dt$$

Are these continuous?

3.5 Let $f : V \longrightarrow W$ be a continuous map between normed vector spaces over $\mathbb{R}$. Assume f to be additive, i.e. $f(x + y) = f(x) + f(y)$ for all $x, y \in V$. Show that f is linear.

3.6 Show that a norm on a real vector space may be derived from an inner product if and only if it satisfies the Parallelogram Law (see exercise 1.10). Hint: Define the inner product by using the norm, and show that it's bi-additive, that is, additive in both variables, by using the Parallellogram Law on the pairs of vectors $u = x_1 + y$ and $v = x_2$, $u = x_1 - y$ and $v = x_2$, $u = x_2 + y$ and $v = x_1$, $u = x_2 - y$ and $v = x_1$, and exhibit the alternating sum. (P. Jordan, J. Von Neumann, 1935).

3.7 Let X be an infinite set with the cofinite topology, and let $\mathbb{R}$ have the usual topology. Characterize the continuous functions from X to $\mathbb{R}$.

3.8 Denote by $M(n,\mathbb{C})$ the vector space of all complex (n,n)-matrices with the usual topology. Clarify that the set $GL(n,\mathbb{C})$ of invertible matrices in $M(n,\mathbb{C})$ is open and a group under multiplication. Explain why the set $SL(n,\mathbb{C})$ of matrices with determinant 1 is a closed subgroup.

3.9 A matrix $A \in M(n,\mathbb{C})$ is called *unitary* if A preserves the inner product $(z|w) = \sum_i z_i\overline{w_i}$ in $\mathbb{C}^n$, that is if $(Az|Aw) = (z|w)$ for all z,$w \in \mathbb{C}^n$. Show that this is equivalent to $|Az| = |z|$ for all $z \in \mathbb{C}^n$, and also to $A^*A = I$ where A^* denotes the transpose conjugate matrix of A ($[a_{ij}]^* = [\overline{a}_{ji}]$). Show that $|\det A| = 1$ if A is unitary.

3.10 Show that the unitary matrices $GL(n,X)$ constitute a closed subgroup, *the unitary group* $U(n)$. Also show that the unitary matrices with determinant 1 is a closed subgroup, *the special unitary group* $SU(n)$. Explain how $U(n)$ can be identified with a subgroup of $SO(2n)$.

3.11 Show that the map $f : S^3 \to SU(2)$ defined by

$$f(x,y,z,w) = \begin{bmatrix} x+iy & -z+iw \\ z+iw & x-iy \end{bmatrix}$$

is a homeomorphism.

3.12 Show that the diagonalizable matrices in $M(n,\mathbb{C})$ are a dense subset. Does this also apply in $M(n)$?

3.13
(a) Let $M_k(n)$ be the set of matrices of rank k in $M(n)$, and $\overline{M}_k(n)$ be the set of matrices of rank $\leq k$. Show that $\overline{M}_k(n)$ is closed in $M(n)$ and that $M_k(n)$ is open and dense in $\overline{M}_k(n)$.
(b) Show that the matrices $M \in M(n)$ on the form

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where $A \in M_k$ is invertible, make an open set $U \subseteq M(n)$. Show that a matrix $M \in U$ is of rank k if and only if $D = CA^{-1}B$. Hint: Multiply M from the left by the invertible matrix

$$\begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix}$$

(c) Let $F : GL(k) \times M(k, n-k) \times M(n-k, k) \to M(n)$ be given by

$$f(A, B, C) = \begin{bmatrix} A & B \\ C & CA^{-1}B \end{bmatrix}$$

Show that f is a homeomorphism onto $U \cap M_k(n)$.
(d) Show that $M_k(n)$ is a manifold and determine the dimension.

3.14 Let $C = C(X)$ be the set of continuous real functions on a topological space X, and let C have the topology for pointwise convergence.
(a) Show that the vector space operations $(f, g) \to f + g$ from $C \times C$ to C and $(\alpha, f) \to \alpha f$ from $\mathbb{R} \times C$ to C are continuous.
(b) Let V_1 be the linear subspace of C spanned by an element $f \neq 0$ in C. Show that the map $\alpha \to \alpha f$ is a homeomorphism of $\mathbb{R}$ to V_1.
(c) Let V_2 be the linear subspace of C spanned by two linearly independent elements f_1 and f_2 in C. Show that the map $(\alpha_1, \alpha_2) \to \alpha_1 f_1 + \alpha_2 f_2$ is a homeomorphism of $\mathbb{R}^2$ onto V_2.

3.15 Show that the open set $U = \{(x, y) : x > 0 \text{ or } y \neq 0\}$ in $\mathbb{R}^2$ is homeomorphic to $\mathbb{R}^2$.

3.16 Show that the sets $\mathbb{R}^n - D^n$ and $\mathbb{R}^n - \{0\}$ are homeomorphic.

3.17 Let U be an open neighborhood of 0 in $\mathbb{R}^n$. Define $p : \mathbb{R}^n \to \mathbb{R}$ by $p(x) = \inf\{\lambda > 0 : x \in \lambda U\}$ for $x \in \mathbb{R}^n$ (the Minkowski functional). Find geometric conditions on U such that p is a norm on $\mathbb{R}^n$.

3.18 Show that every nonempty, convex open set in $\mathbb{R}^n$ is homeomorphic to $\mathbb{R}^n$.

3.19 A map $f : X \to Y$ between metric spaces is called *uniformly continuous* if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $d(f(x), f(x')) < \epsilon$ when x, $x' \in X$ and $d(x, x') < \delta$. Show that a uniformly continuous map is continuous, and provide an example that the converse may not always hold.

3.20 A function $f : X \to \mathbb{R}$ on a space X is called *lower semicontinuous* at a point $x \in X$ if for every $\epsilon > 0$ there is a neighborhood U of x such that for all $x' \in U$ we have $f(x') > f(x) - \epsilon$. Exhibit a neighborhoodstructure on $\mathbb{R}$ which gives rise to this concept of continuity. Describe the open and closed sets for this topology on $\mathbb{R}$.

3.21 Let (f_α) be a family of real functions on a space X. Assume $f(x) = \sup f_\alpha(x) < \infty$ for all $x \in X$. Show that if all f_α are lower semicontinuous, then so is f. Show by example that f need not be continuous even if all the f_α are continuous.

3.22 Show that a map $f : X \to Y$ is open if and only if $f(\text{int } A) \subseteq \text{int } f(A)$ for every $A \subseteq X$, and closed if and only if $\text{cl } f(A) \subseteq f(\text{cl } A)$ for every $A \subseteq X$.

3.23 Let V be a normed vector space.
(a) Show that the vector space operations $a : V \times V \to V$ and $m : \mathbb{R} \times V \to V$ defined by $a(x, y) = x + y$ and $m(\alpha, x) = \alpha x$ are continuous.
(b) Show that every translation $t_y : x \to x + y$ with a vector $y \in V$ and every scaling $s_\alpha : x \to \alpha x$ with a factor $\alpha \neq 0$ in $\mathbb{R}$, is a homeomorphism of V to itself.
(c) Show that all closed balls $\overline{U}(x; r)$ for $x \in V$ and $r > 0$ are homeomorphic. Show that every open ball $U(x; r)$ is homeomorphic to V.

4.1 Let X_1 and X_2 be metric spaces with metrics d_1 and d_2. For two points $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $X = X_1 \times X_2$ we define $d(x, y) = \max(d_1(x_1, y_1), d_2(x_2, y_2))$. Show that d is a metric which defines the product topology on X.

4.2 Let A and B be subsets of topological spaces X and Y. Show that in the product space $X \times Y$ the following applies
(a) int $(A \times B) =$ int $A \times$ int B
(b) cl $(A \times B) =$ cl $A \times$ cl B
(c) bd $(A \times B) = (\text{bd } A \times \text{cl } B) \cup (\text{cl } A \times \text{bd } B)$

Show that the product of two open (closed) sets is open (closed).

4.3 Show that $X_1 \approx X_2$ and $Y_1 \approx Y_2$ implies $X_1 \times Y_1 \approx X_2 \times Y_2$ ($\approx$ denotes homeomorphism between topological spaces).

4.4 Show that the cylinder surface $S^1 \times \mathbb{R}$ is homeomorphic to the punctured plane $\mathbb{R}^2 - \{0\}$. Generalize.

4.5 Show that the open, solid torus $S^1 \times E^2$ is homeomorphic to $\mathbb{R}^3 - (z\text{-axis})$.

4.6 Let $H = \{x \in \mathbb{R} : x \geq 0\}$. Show that there exists a homeomorphism $H \times H \approx \mathbb{R} \times H$ ("straigthening of the angle"). Show that $H^n \approx \mathbb{R}^{n-1} \times H$.

4.7 Establish homeomorphisms

$$GL(n, \mathbb{C}) \approx U(n) \times \mathbb{R}^{n^2}$$

$$SL(n, \mathbb{C}) \approx SU(n) \times \mathbb{R}^{n^2-1}$$

4.8 Let X be a metric space with metric d. Show that $d : X \times X \longrightarrow \mathbb{R}$ is continuous.

4.9 Identify the set $F(X, Y)$ of maps from X to Y with the product $Y^X = \Pi_{x \in X} Y_x$ where $Y_x = Y$ for all $x \in X$, and show that the topology for pointwise convergence on $F(X, Y)$ is the product topology.

4.10 For which sets J is the product space $\mathbb{R}^J$:
(a) a space of countable basis for the topology?
(b) a metrizable space?

5.1 Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be continuous maps between topological spaces. Show that if $g \circ f$ is an identification, then g is an identification.

5.2 Let $p : X \longrightarrow Y$ and $s : Y \longrightarrow X$ be continuous maps such that $p \circ s = id$. Show that p is an identification and s an embedding (s is called a *cross section* for p).

5.3 We define an equivalence relation $\sim$ in $\mathbb{C}^{n+1} \{0\}$ by setting $z \sim z'$ if there exists a $\lambda \in \mathbb{C}$ such that $z' = \lambda z$. The quotient space is then called *the complex projective n-space* and denoted $\mathbb{C}P^n$. Show that:
(a) $\mathbb{C}P^1 \approx S^2$
(b) $\mathbb{C}P^n \approx S^{2n+1}/\sim$, where the equivalence classes are great circles
(c) $\mathbb{C}P^n$ is a $2n$-manifold.

5.4 Characterize the quotient topology on the group $\mathbb{R}/\mathbb{Q}$.

5.5 Let R and S be open equivalence relations in X and Y respectively. Show that $(X \times Y)/(R \times S)$ is naturally homeomorphic to $(X/R) \times (Y/S)$.

6.1 Let (x_n) be a convergent sequence with limit x in a topological space X. Show that $C = \{x_1, \ldots, x\}$ is a compact set in X.

6.2 Let $A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \supseteq \cdots$ be a descending sequence of nonempty, closed sets in a compact space X. Show that the intersection $\cap_{n=1}^{\infty} A_n$ is nonempty. Give counter examples when A_n are not closed sets, or X is not compact.

6.3 Let A and B be nonempty sets in a metric space. Show that
(a) If A is compact, then there exists a point $x \in A$ such that $d(x, B) = d(A, B)$.
(b) If A and B are compact, then there exist points $x \in A$ and $y \in B$ such that $d(x, y) = d(A, B)$.

6.4 Find two disjoint sets in $\mathbb{R}^2$ with distance 0. Show that neither of these sets can be compact.

6.5 Let $\{U_i\}_{i \in I}$ be an open cover of a compact metric space X. Show that there exists $\epsilon > 0$ such that every set of diameter $\leq \epsilon$ in X is contained in one of the sets U_i. (Lebesgue's covering lemma).

6.6 By a convex body in $\mathbb{R}^n$ we mean a convex set with an interior point in $\mathbb{R}^n$.
(a) Show that every nonempty convex set in a Euclidean space may be embedded as a convex body in $\mathbb{R}^n$ for appropriate n.
(b) Show that two compact, convex bodies in $\mathbb{R}^n$ are homeomorphic.

6.7 Decide whether or not $\mathbb{R}$ with the cofinite topology is compact.

6.8
(a) Show that the functions $f_{ij}(x) = x_i x_j / |x|^2$ on $\mathbb{R}^3 - \{0\}$ define continuous functions $\hat{f}_{ij}$ on the projective plane P^2.
(b) Show that $F = (\hat{f}_{ij})_{i \leq j}$ is an embedding of P^2 in $\mathbb{R}^6$.

6.9 Let $f : X \longrightarrow Y$ be a continuous map between locally compact Hausdorff-space. We extend f to a map $f^* : X^* \longrightarrow Y^*$ between the one-point compactifications by setting $f^*(\infty) = \infty$. When is f^* continuous?

6.10 Let X be a topological space and $\{X_\alpha\}$ a class of subspaces. We say that X has the topology *defined by* $\{X_\alpha\}$ if the following condition is satisfied: a subset U of X is open in X if and only if $U \cap X_\alpha$ is open in X_α for all α.
(a) Show that if $\{X_\alpha\}$ is an open cover or a finite closed cover, then X has the

topology defined by $\{X_\alpha\}$.
(b) Assume that a space X has the topology defined by a class $\{X_\alpha\}$ of subspaces. Show that a map $f : X \longrightarrow Y$ is continuous if and only if $f|_{X_\alpha}$ is continuous for all α.

6.11 A Hausdorff space X is said to be *compactly generated* if X has the topology defined by the class of compact subspaces. Show that a locally compact Hausdorff space is compactly generated.

6.12 Show that a topological space with a countable neighborhood basis of every point is compactly generated.

6.13 A continuous map $f : X \longrightarrow Y$ between topological spaces is called *proper* if f is closed and $f^{-1}(y)$ is compact for all $y \in Y$.
(a) Show that if f is proper, then $f^{-1}(K)$ is compact for every compact $K \subseteq Y$.
(b) Assume that X and Y are locally compact Hausdorff spaces. Show that f is proper if and only if the inverse image under f of every compact set is compact.

6.14 Assume that X is a compact Hausdorff space and $f : X \longrightarrow Y$ an identification. Show that the following conditions are equivalent.
(a) f is closed
(b) f is proper
(b) Y is Hausdorff.

6.15 Let X and Y be topological spaces and $C(X, Y)$ the set of continuous maps from X to Y. Let τ be the topology on $C(X, Y)$ generated by the sets $U(C, O) = \{f \in C(X, Y) : f(C) \subseteq O\}$ where $C \subseteq X$ is compact, and $O \subseteq Y$ is open (the compact-open topology, R.H. Fox 1945).
(a) Show that τ is finer than the topology for pointwise convergence on $C(X, Y)$.
(b) What does it mean that a sequence in $C(\mathbb{R})$ converges with respect to the topology τ?

6.16 Let W,X,Y be topological spaces and let the map-spaces in the following have the compact-open topology.
(a) Assume X to be locally compact. Show that the natural bijection $(Y^X)^W \approx Y^{W\times X}$ defines a bijection $h : C(W, C(X, Y)) \longrightarrow C(W \times X, Y)$.
(b) Find conditions which ensure that h is a homeomorphism.

6.17 Assume that X is a compact topological space and $f_\alpha : X \longrightarrow \mathbb{R}$, $\alpha = 1, \ldots, N$, a set of continuous functions which separate points in X. That is, for all x,$y \in X$ where $x \neq y$, there is some α such that $f_\alpha(x) \neq f_\alpha(y)$. Show that $F = (f_\alpha)_{1 \leq \alpha \leq N}$ is an embedding of X in $\mathbb{R}^N$.

6.18 Let X be a Hausdorff space. Let $f_\alpha : X \longrightarrow \mathbb{R}$ for $\alpha \in J$ be a family of continuous functions which separates points and closed sets in X, that is, for every $x \in X$ and closed set $A \subseteq X$ such that $x \notin A$ there is some $\alpha \in J$ such that $f_\alpha(x) > 0$ and $f_\alpha|_A = 0$. Show that $F = (f_\alpha)_{\alpha \in J}$ is an embedding of X in $\mathbb{R}^J$.

6.19 Let $C(X, Y)$ have the compact-open topology. Show that the evaluation map $E : C(X, Y) \times X \longrightarrow Y$ defined by $E(f, x) = f(x)$ is continuous if X is locally compact.

6.20 Show that a continuous map $f : X \longrightarrow Y$ from a compact metric space X to a metric space Y is uniformly continuous.

6.21 Let $\mathbb{R}$ have the topology where the open sets are all intervals on the form (a, ∞) for $a \in \mathbb{R}$. Which sets in T are compact with respect to this topology?

6.22 Show that a lower semicontinuous real function on a compact space achieves its minimum.

6.23 Show that every compact metric space is separable.

6.24 Let X be a locally compact Hausdorff space and X^* the one-point compactification. X is called *countable at infinity* if X^* has a countable neighborhood basis of ∞. Show that the following criteria are equivalent.
(a) X is a countable union of compact subsets.
(b) $X = \cup_n K_n$ where K_n is compact and $K_n \subseteq \text{int } K_{n+1}$ for all n.
(c) X is countable at infinity.

6.25
(a) Assume that A and B are nonempty sets in $\mathbb{R}^n$, where A is compact and B is closed. Show that there exist points $x \in A$ and $y \in B$ such that $d(x, y) = d(A, B)$.
(b) Show that the foregoing does not apply if $\mathbb{R}^n$ is replaced by the Hilbert space H.

6.26 Let V be a normed vector space. Prove:
(a) If F is a linear subspace of finite dimension in V and $x \in V$, then there exist $y \in F$ such that $|x - y| = d(x, F)$.
(b) If $F \neq V$, then there exists $u \in V$ such that $|u| = 1$ and $d(u, F) = 1$.
(c) If V is of infinite dimension, then there is a sequence (u_n) of unit vectors in V such that $|u_n - u_m| \leq 1$ for $n \neq m$.
(d) V is locally compact if and only if V is of finite dimension (F. Riesz, 1918).

6.27 Let $f : X \longrightarrow Y$ be a map between topological spaces, where Y is compact Hausdorff. Show that f is continuous if and only if the graph of f is closed in $X \times Y$.

7.1 Let A be a bounded subset of $\mathbb{R}^2$. Show that one of the components of $\mathbb{R}^2 - A$ is unbounded, and the others bounded. Generalize to $\mathbb{R}^n$ for $n \geq 2$.

7.2 Let

$$Z = \{(x, y) \in \mathbb{R}^2 : x \neq 0 \text{ and } y = \sin \frac{1}{x}, \text{ or } x = 0 \text{ and } |y| \leq 1\}$$

Show that Z is connected, but not path connected.

7.3 Let $f : \mathbb{R} \longrightarrow \mathbb{R}^3$ be the map

$$f(t) = (\cos t, \sin t \cos t, \sin^2 t)$$

Set $C = f(\mathbb{R})$.
(a) Show that C is compact and connected.
(b) Show that the projection $p : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ defined by $p(x, y, z) = (x, y)$ maps C homeomorphically onto its image.
(c) Show that the set $F \subseteq \mathbb{R}^3$ defined by the equations

$$x^2 z - y^2 = 0$$

$$x^2 + y^2 + z^2 = 1$$

has C as component. Is F connected?

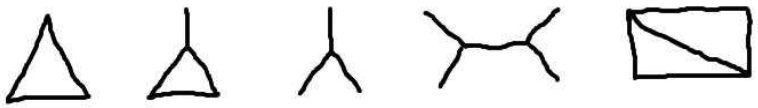
7.4 Let U and V be homeomorphic subsets of $\mathbb{R}$. Show that U is open if and only if V is open.

7.5 Classify the open subsets of $\mathbb{R}$ topologically.

7.6 Show that if $f : X \longrightarrow Y$ is continuous, then f maps every component in X to a component in Y. Show that the number of components in a space is a topological invariant.

7.7 A *cut point* of order n in a space X is a point $c \in X$ such that $X - \{c\}$ has precisely n components. Show that the number of cut points of order n in a space is a topological invariant.

7.8 Show that the following sets, consisting of finite unions of line segments in the plane, are topologically different. Hint: consider cut points.



7.9
(a) Show that $\mathbb{R}$ and $\mathbb{R}^n$ for $n > 1$ are not homeomorphic.
(b) Show that S^1 and S^n for $n > 1$ are not homeomorphic.

7.10 Assume $f : S^1 \longrightarrow \mathbb{R}$ is continuous. Show that there is an $x \in S^1$ such that $f(x) = f(-x)$.

7.11 Let $p : S^1 \longrightarrow S^1$ be defined by $p(z) = z^2$. Show that there does not exist any continuous cross section for p, that is, a continuous map $s : S^1 \longrightarrow S^1$ such that $p \circ s = id$. Generalize to $p(z) = z^n$ for $n \geq 2$.

7.12 Let $X = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\} \times [0,1] \cup \{(0,0)\} \cup \{(0,1)\} \subseteq \mathbb{R}^2$. Give the components of X. Determine the intersection of all openclosed sets in X which contains $(0,0)$.

7.13 Show that a metric space consisting of a countable number of points is totally disconnected.

7.14 Let A and B be subsets of a space X, where B is connected. Assume $A \cap B \neq \emptyset$ and $(X - A) \cap B \neq \emptyset$. Show that bd $A \cap B \neq \emptyset$.

7.15 Let $C_1 \supset C_2 \supset \ldots$ be a decreasing sequence of compact, connected sets in a Hausdorff space. Show that $\bigcap_n C_n$ is connected. Show by example that the assumption of compactness is not redundant.

7.16
(a) Show that a connected metric space X satisfies *Cantor's criteria of connectedness*: for all $\epsilon > 0$ and all pairs of points $a,b \in X$ there is a finite sequence of points $x_1, \ldots, x_n \in X$ such that $x_1 = a, x_n = b$ and $d(x_i, x_{i+1}) \leq \epsilon$ for $1 \leq i \leq n$.
(b) Show that a compact metric space is connected if and only if it satisfies Cantor's criteria for connectedness.

7.17 Characterize the connected sets in $\mathbb{R}$ with the cofinite topology.

7.18 Show that we get an equivalence relation in a space X by setting $x \sim y$ if there do not exist disjoint, open sets A, B in X such that $X = A \cup B$, where $x \in A$ and $y \in B$. The equivalence classes are called *quasi components* of X. Show that every component is contained in a quasi component, and the two notions coincide if X is locally connected.

8.1 Show that a finite product of complete metric spaces is complete (with respect to the product metric $d = \max d_i$).

8.2 Investigate whether the vector space $C[a, b]$ with the norm $|f|_1 = \int_a^b |f(x)|dx$ is complete.

8.3 Let J be the set of all sequences (x_n) from the Hilbert space H such that $0 \leq x_n \leq 1/n$ for all n. Show that J is compact. (*The Hilbert cube*).

8.4 Let X be a metric space. By a *completion* of X we mean a complete metric space $\hat{X}$ together with an isometric (norm preserving) embedding of X onto a dense subspace of $\hat{X}$.
(a) Show that the following construction gives a completion on X. Let S be the set of Cauchy sequences in X. Define an equivalence relation $\sim$ in S by setting $(x_n) \sim (y_n)$ if $\lim_{n\to\infty} d(x_n, y_n) = 0$. Let $\hat{X} = S/\sim$, let $[x_n]$ denote the equivalence class of (x_n) and define a metric $\hat{d}$ on $\hat{X}$ by $\hat{d}([\hat{x}_n], [\hat{y}_n]) = \lim_{n\to\infty} d(x_n, y_n)$. Embed X in $\hat{X}$ by $x \to [x, x, \ldots]$.
(b) Show that a completion of X is unique up to isometry.

8.5 Let $A_1 \supseteq A_2 \supseteq \ldots \supseteq A_n \supseteq \ldots$ be a decreasing sequence of nonempty, closed sets in a complete metric space such that diam $A_n \to 0$. Show that $\bigcap_{n=1}^{\infty} A_n$ consists of one point.

8.6 Consider the differential equation problem $y' = y$, $y(0) = y_0$. Find successive approximations y_n to a solution by starting with the constant y_0. Find the solution by going to the limit.

8.7 Let X be a compact space and $C(X)$ the space of all continuous, real functions on X with the supremum norm. A set $A \subseteq C(X)$ is called a *subalgebra* of $C(X)$ if A is a linear subspace and f, $g \in A$ imply $fg \in A$. It contains the constant functions if $1 \in A$. We say that A *separates points* in X if for all x,$y \in X$ where $x \neq y$, there is an $f \in A$ such that $f(x) \neq f(y)$. Assume this is satisfied. Let $\overline{A} = \text{cl } A$. Prove the following:
(a) There is a sequence of polynomials $p_n(x)$ with real coefficients such that $p_n(x) \to \sqrt{x}$ uniformly for $0 \leq x \leq 1$.
(b) If $f \in A$, then $|f| \in \overline{A}$.
(c) If f,$g \in A$, then $\max(f, g)$, $\min(f, g) \in \overline{A}$.
(d) If x,$y \in X$, $x \neq y$, and α, $\beta \in \mathbb{R}$, then there is an $f \in \overline{A}$ such that $f(x) = \alpha$ and $f(y) = \beta$.
(e) For all $f \in C(X)$, $x \in X$ and $\epsilon > 0$ there is a $g \in \overline{A}$ such that $g(x) = f(x)$

and $g \leq f + \epsilon$.
(f) $\overline{A} = C(X)$ (Stone-Weierstrass).

8.8 Show that every continuous, real function on a compact set in $\mathbb{R}^n$ is a uniform limit for a sequence of polynomials (Weierstrass).

8.9 Consider $C = C(X, Y)$ where X is an arbitrary space and Y a metric space.
(a) Show that the sets $U(f; K, \epsilon) = \{g \in C : |f(x) - g(x)| < \epsilon \text{ for } x \in K\}$ where $K \subseteq X$ is compact and $\epsilon > 0$, constitute neighborhood bases for a topolgoy τ on C, *the topology for uniform convergence on compacts*.
(b) Show that τ is identical to the compact-open topology on C.

8.10 Let A be a dense subspace of a metric space X and $f : A \longrightarrow Y$ a uniformly continuous map to a complete metric space Y. Show that there exists a unique continuous map $\overline{f} : X \longrightarrow Y$ which coincides with f on A. Show that $\overline{f}$ is uniformly continuous (extension by continuity).

8.11 Let $C^1[a, b]$ denote the vector space of real functions with continuous derivative on $[a, b]$.
(a) Show that $C^1[a, b]$ may be given a norm defined by

$$||f|| = \max(|f|_{sup}, |f'|_{sup})$$

(b) Show that $C^1[a, b]$ is complete with respect to the norm $||\ ||$.

8.12 Show that the set of accumulation points for a point sequence (x_n) in a topological space is $\bigcap_n F_n$ where $F_n = \text{cl}\ \{x_k : k \geq n\}$. Use this to show that the set of accumulation points is closed, and nonempty if the space is compact.

8.13 Show that in a compact space a point sequence with precisely one accumulation point is convergent to that point.

8.14 Let $K : [a, b] \times [a, b] \longrightarrow \mathbb{R}$ and $f : [a, b] \longrightarrow \mathbb{R}$ be continuous functions. Prove that the integral equation

$$\phi(x) = f(x) + \int_a^b K(x, y)\phi(y)dy$$

has a unique solution $\phi : [a, b] \longrightarrow \mathbb{R}$ if $|K|_{\text{sup}} < \frac{1}{b-a}$.

8.15 Show that every locally compact space is a Baire space.

8.16 Show that every open subspace of a Baire space is a Baire space.

8.17
(a) Show that every normed space V can be embedded as a dense subspace of a Banach space $\hat{V}$ by a linear isometry. In what sense is the completion $\hat{V}$ unique?
(b) Show that every inner product space may similarly be embedded in a Hilbert space.

8.18 In the following we use the topology for uniform convergence.
(a) Show that the space $C[0,1]$ is separable.
(b) Investigate whether the space $BC(\mathbb{R})$ is separable.

8.19 Show that in the space $C[0,1]$ with the topology for uniform convergence, functions which at least are differentiable at one point constitute a meager set. Hint: Let A_n denote the set of all $f \in C[0,1]$ with the following property; there is an $x \in [0,1]$ such that for all $x' \in [0,1]$ where $|x'-x| \leq 1/n$, we have $|f(x')-f(x)| \leq n|x'-x|$. Then A_n is closed, without interior points, and $\bigcup_n A_n$ contains all functions in $C[0,1]$ which at least are differentiable at one point. (S. Banach 1931).

8.20 Let V and W be normed vector spaces and assume W is complete. Show that $L(V,W)$ with the operator norm is complete.

Index

accumulation point, 68
adjunction space, 44
antipodal points, 44

Baire space, 73
Baire's theorem, 73
Banach space, 68
basis for the topology, 4
basis of neighborhoods, 15
Bolzano-Weierstrass' theorem, 69
boundary, 12
bounded set, 7

C[a,b], 8
Cauchy sequence, 65
Cauchy's convergence principle, 65
cell complex, 44
chart, 28
closed
    equivalence relation, 42
    map, 25
    set, 1
closure, 12
closure point, 13
coarser, 9
collapse of subspace, 45
compact space, 48
compatible with an equivalence relation, 41
complete metric space, 66
components of a space, 59
connected, 57
continuous map, 19
contraction, 70
contraction principle, 71
convergence, 13
convex, 58
countable neighborhood basis, 15

De Morgan's laws, 11
dense, 13
descending countable basis of neighborhoods, 15
diagonal, 32
diameter, 7
disconnected, 57
discrete
    metric, 6
    topology, 1
divergence of sequence, 13

embedding, 25
equivalent norms, 6
Euclidean
    neighborhood, 3
    norm, 3
exterior, 12
extremal value theorem, 47

finer, 9
fixed point, 70
form, 23

general linear group GL(n), 24
generated topology, 4

Hausdorff space, 13
Heine-Borel's theorem, 47
Hilbert space, 67
Hilbert-Schmidt norm, 7
homeomorphic spaces, 25

homeomorphism, 25
homeomorphism type, 28
homogenous coordinates, 43
homogenous polynomial, 23

identification, 41
induced map, 41
interior, 12
interior point, 12

Klein's bottle, 43

limit of a sequence, 13
Lipschitz-requirement, 72
local homeomorphism, 25
local parametrizations, 28
locally
    compact, 52
    connected, 61
    Euclidean, 28
    path connected, 61

M(m,n), 7
M(n), 24
manifold, 28
meager set, 73
metric, 6
metric space, 6
metrizable, 15
monomial, 23

neighborhood, 12
norm, 3
normed vector space, 5

open
    equivalence relation, 42
    map, 25
    set, 1
operator norm, 8
orthogonal
    group, 24
    matrix, 24

partially continuous, 34
path, 61
path component, 61
path connected, 61
period, 39
periodic function, 39
product space, 31
product topology, 31
projective space, 43

quadratic form, 23
quotient space, 39
quotient topology, 40

rational function, 23
residual set, 73
restriction, 22

saturated, 40
special
    linear group SL(n), 24
    orthogonal group SO(n), 25
splitting, 57
standard topology, 27
stereographic projection, 27
subspace, 9
sum of topological spaces, 44
supremum
    neighorhood, 5
    norm, 5

topological space, 1
topology, 1
    for pointwise convergence, 9
    for uniform convergence, 8
totally
    bounded metric space, 69
totally disconnected, 60
trivial topology, 1