

Non-stationarity of isomorphism between AF algebras defined by stationary Bratteli diagrams

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Abstract. We first study situations where the stable AF algebras defined by two square primitive non-singular incidence matrices with non-negative integer matrix elements are isomorphic, even though no powers of the associated automorphisms of the corresponding dimension groups are isomorphic. More generally we consider necessary and sufficient conditions for two such matrices to determine isomorphic dimension groups. We give several examples.

This paper was motivated by attempts in [BJO98] to classify certain AF algebras defined by constant incidence matrices. The specific incidence matrices considered in [BJO98] are of the form (18) below, and we shall see there that the first problem referred to in the abstract is most interesting for those matrices. The second problem referred to in the abstract is significant not only for AF algebras but also for, e.g.,

- classification of substitution minimal systems up to strong orbit equivalence [GPS95, For97, DHS].
- homeomorphism classification of domains of certain inverse limit hyperbolic systems [BD95, SV98]. The latter paper, which was written independently of this paper, and which was pointed out to us by the referee, makes contributions in the same direction as we do here. Our C^* -equivalence of matrices corresponds to weak equivalence of (the transposed) matrices in that paper. Theorems 2.3 (which is [BD95, Corollary 3.5]) and 2.4 in [SV98] correspond more or less to our Theorem 10. Their Theorem 3.2 is similar to our Theorem 6. While the latter part of

their paper is focused on a class of incidence matrices arising from periodic kneading sequences, our focus here and in [BJO98] is on matrices of the form (18) below, which arises in the representation theory of Cuntz algebras.

- cohomology of subshifts of finite type [BH96, Po89].

In a forthcoming paper we will show that the isomorphism problem for stationary AF algebras is decidable. It is already known that shift equivalence is decidable in this setting [KR79, KR88]. This is interesting in view of the fact that isomorphism between two AF algebras is known not to be decidable in general, i.e. there is no recursive algorithm to decide if two given effective presentations of Bratteli diagrams yield equivalent diagrams in the general (non-stationary) case, see [MP98].

First we will survey some terminology and basic facts in the fields of operator algebras and symbolic dynamics. Recall from [Bra72] that a C^* -algebra $\mathfrak{A}$ is called AF (approximately finite dimensional) if it is the closure of the union of an increasing sequence $\mathfrak{A}_n$ of finite-dimensional subalgebras. It is known from [Bra72, Theorem 2.7] that two AF algebras $\mathfrak{A} = \overline{\bigcup_n \mathfrak{A}_n}$, $\mathfrak{B} = \overline{\bigcup_n \mathfrak{B}_n}$ are isomorphic if and only if there are increasing sequences k_i, l_i of natural numbers and injections $\alpha_i : \mathfrak{A}_{k_i} \rightarrow \mathfrak{B}_{l_i}$, $\beta_i : \mathfrak{B}_{l_i} \rightarrow \mathfrak{A}_{k_{i+1}}$ such that the following diagram commutes:

$$\begin{array}{ccc}
 \mathfrak{A}_{k_1} & \xrightarrow{\alpha_1} & \mathfrak{B}_{l_1} \\
 \downarrow & \searrow \beta_1 & \downarrow \\
 \mathfrak{A}_{k_2} & \xrightarrow{\alpha_2} & \mathfrak{B}_{l_2} \\
 \downarrow & \searrow \beta_2 & \downarrow \\
 \mathfrak{A}_{k_3} & \xrightarrow{\alpha_3} & \mathfrak{B}_{l_3}
 \end{array} \tag{1}$$

(The ‘if’ part is trivial.) This is easily translated into the fact that there exists a complete isomorphism invariant for AF algebras $\mathfrak{A}$, namely the dimension group, [Ell76]. In the case that $\mathfrak{A}$ has a unit this is the triple $(K_0(\mathfrak{A}), K_0(\mathfrak{A})_+, [\mathbf{1}])$, where $K_0(\mathfrak{A})$ is an abelian group, $K_0(\mathfrak{A})_+$ are the positive elements in $K_0(\mathfrak{A})$ relative to an order making $K_0(\mathfrak{A})$ into a Riesz ordered group without perforation, and $[\mathbf{1}]$ is the class of the identity in $K_0(\mathfrak{A})$ (if $\mathfrak{A}$ is non-unital, replace $[\mathbf{1}]$ by the hereditary subset $\{[p] \mid p \text{ projection in } \mathfrak{A}\}$ of $K_0(\mathfrak{A})_+$). It is also customary to apply the term *dimension group* for just the couple $(K_0(\mathfrak{A}), K_0(\mathfrak{A})_+)$, and this will be done in the sequel. (In dynamical systems theory this term is used slightly differently; see the comments prior to Lemma 1, below.) See [Eff81], as well as [LM95, Wal92, Tor91, BMT87], for details on this and the following. Let us now specialize to the case when $\mathfrak{A}$ is given by a constant $N \times N$ incidence matrix J (with non-negative integer entries) which is primitive, i.e. J^n has only positive entries for some $n \in \mathbb{N}$, [LM95, Theorem 4.5.8]. Then $\mathfrak{A}$ is simple with a unique trace state τ . In the case

when $K_0(\mathfrak{A}) \cong \mathbb{Z}^N$, or, equivalently, when J is unimodular, this class of AF algebras (or, rather, dimension groups) has been characterized intrinsically in [Han81, Theorems 3.3 and 4.1]. We do not assume unimodularity in the sequel.

In general, when J is an $N \times N$ matrix with non-negative entries, the dimension group is the inductive limit

$$\mathbb{Z}^N \xrightarrow{J} \mathbb{Z}^N \xrightarrow{J} \mathbb{Z}^N \xrightarrow{J} \dots \quad (2)$$

with order generated by the order defined by

$$(m_1, \dots, m_n) \geq 0 \iff m_i \geq 0 \text{ on } \mathbb{Z}^N. \quad (3)$$

This group can be computed explicitly as a subgroup of $\mathbb{Q}^N$ as follows when $\det(J) \neq 0$ (as it will be in our examples). Put

$$G_m = J^{-m}(\mathbb{Z}^N), \quad m = 0, 1, \dots, \quad (4)$$

and equip G_m with the order

$$G_m^+ = J^{-m}((\mathbb{Z}^N)^+). \quad (5)$$

Then

$$G_0 \subset G_1 \subset G_2 \subset \dots \quad (6)$$

and

$$K_0(\mathfrak{A}_L) = \bigcup_m G_m \quad (7)$$

is a subgroup of $\mathbb{Q}^N$ (containing $\mathbb{Z}^N$), with order defined by

$$g \geq 0 \text{ if } g \geq 0 \text{ in some } G_m. \quad (8)$$

The action of the trace state τ on $K_0(\mathfrak{A})$ may be computed as follows. If λ is the Frobenius eigenvalue of J , and $\alpha = (\alpha_1, \dots, \alpha_N)$ is a corresponding eigenvector in the sense

$$\alpha J = \lambda \alpha \quad (9)$$

(i.e. $J^t \alpha^t = \lambda \alpha^t$, see [Eff81, pp. 33–37]), then if α is suitably normalized (by multiplying with a positive factor), the trace applied to an element g at the m th stage of

$$\begin{array}{ccccccc} & & & & g & & \\ & & & & \cap & & \\ \mathbb{Z}_1^N & \longrightarrow & \mathbb{Z}_2^N & \longrightarrow & \mathbb{Z}_3^N & \longrightarrow & \dots \longrightarrow \mathbb{Z}_m^N \longrightarrow \dots \end{array} \quad (10)$$

is

$$\tau(g) = \lambda^{-m+1} \langle \alpha | g \rangle, \quad (11)$$

where $\langle \cdot | \cdot \rangle$ denotes the usual inner product in $\mathbb{R}^N$, i.e. $\langle \alpha | g \rangle = \sum_{i=1}^N \alpha_i g_i$. Taking α as the Frobenius eigenvector makes the ansatz well defined: if $g \in G_m \subset G_{m+1}$, then

$$\lambda^{-m+1} \langle \alpha | g \rangle = \lambda^{-(m+1)+1} \langle \alpha | Jg \rangle. \quad (12)$$

Thus τ is an additive character on $K_0(\mathfrak{A})$, and up to normalization the unique positive such. If $\mathfrak{A}$ is unital we may normalize α by requiring $\langle \alpha | [\mathbf{1}]_0 \rangle = 1$, and it can then be shown that the range of the trace on projections is $\tau(K_0(\mathfrak{A})) \cap [0, 1]$.

When $K_0(\mathfrak{A})$ is given concretely in $\mathbb{Q}^N$ as above, the trace can be computed as

$$\tau(g) = \langle \alpha | g \rangle, \quad (13)$$

where $g \in m$ th term $\mathbb{Z}^N$ is identified with its image $J^{-m+1}g$ in $\mathbb{Q}^N$, and the positive cone in $K_0(\mathfrak{A}) \subset \mathbb{Q}^N$ identifies with those g such that $\tau(g) > 0$, or $g = 0$.

If one forgets about $[\mathbf{1}]$, or the hereditary subset of $K_0(\mathfrak{A})_+$, then $(K_0(\mathfrak{A}), K_0(\mathfrak{A})_+)$ is a complete invariant for stable isomorphism, i.e. isomorphism of $\mathfrak{A} \otimes \mathcal{K}(L^2(\mathbb{Z}))$. In the rest of this paper we will only consider stable isomorphism, and not study the position of $[\mathbf{1}]$ inside $(K_0(\mathfrak{A}), K_0(\mathfrak{A})_+)$. Let us just mention that in the applications in [BJO98], the element $[\mathbf{1}]$ is represented by $(1, 0, 0, \dots, 0)^T$ in the concrete representation (7), and to take care of $[\mathbf{1}]$ one has to assume that the matrices A_i, B_i in (15) below preserves the class of $[\mathbf{1}]$. In Theorems 6 and 7 below this amounts to the added condition that

$$A(1)J^{n_0}(1, 0, 0, \dots, 0)^T = K^{m_0}(1, 0, 0, \dots, 0)^T$$

for some non-negative integers n_0, m_0 . For more details on isomorphisms as opposed to stable isomorphisms in this setting, see [BJO98].

If $\mathfrak{A}, \mathfrak{B}$ are AF algebras defined by constant (necessarily square) incidence matrices J, K , it follows from [Bra72, Theorem 2.7] that $\mathfrak{A}$ and $\mathfrak{B}$ are stably isomorphic if and only if there exist natural numbers $n_1, n_2, n_3, \dots, m_1, m_2, m_3, \dots$, and matrices $A_1, A_2, \dots, B_1, B_2, \dots$ with non-negative integer matrix elements such that the following diagram commutes:

$$\begin{array}{ccc}
 \bullet & & \bullet \\
 \downarrow J^{n_1} & \searrow A_1 & \downarrow K^{m_1} \\
 \bullet & & \bullet \\
 \downarrow J^{n_2} & \searrow A_2 & \downarrow K^{m_2} \\
 \bullet & & \bullet \\
 \downarrow J^{n_3} & \searrow A_3 & \downarrow K^{m_3} \\
 \bullet & & \bullet \\
 \vdots & \searrow A_4 & \vdots \\
 & & \vdots
 \end{array}
 \quad (14)$$

This means that

$$\begin{aligned} J^{n_k} &= B_k A_k, \\ K^{m_k} &= A_{k+1} B_k \end{aligned} \tag{15}$$

for $k = 1, 2, \dots$. The first aim of this paper is to show that the sequences A, B, n, m cannot, in general, be taken to be constant when they exist.

In the covariant version of this isomorphism problem, it is known from a theorem of Krieger that the sequences can be taken to be constant. Let $G(J)$ be the dimension group associated to J , and $(\sigma_J)_*$ the shift automorphism of $G(J)$ determined by J [Eff81, pp. 36–37]. Now let $\mathfrak{A}$ be the stable AF algebra associated to $G(J)$, and σ_J an automorphism of $\mathfrak{A}$ such that the corresponding automorphism of $G(J)$ is $(\sigma_J)_*$. Then Krieger's theorem [Kri80] says that $(G(J), (\sigma_J)_*)$ is isomorphic to $(G(K), (\sigma_K)_*)$ if and only if there is a $k \in \mathbb{N}$ and non-negative rectangular matrices A, B such that

$$\begin{aligned} AJ &= KA \\ BK &= JB \\ BA &= J^k \\ AB &= K^k. \end{aligned} \tag{16}$$

If also $N > 1$, it was proved recently in [BrKi98] that this is also equivalent to outer conjugacy of σ_J and σ_K . (This was proved in [EvKi97] for the case when $(K_0(\mathfrak{A}), K_0(\mathfrak{A})_+)$ has no infinitesimal elements, i.e. this ordered group is totally ordered.)

So, in dynamical system language, the problem is: given primitive square matrices J, K over the non-negative integers such that there exist sequences $m(i), n(i) \in \mathbb{N}$ and matrices $A(i), B(i)$ over the non-negative integers for $i \in \mathbb{N}$ with

$$\begin{aligned} J^{n(i)} &= B(i)A(i), \\ K^{m(i)} &= A(i+1)B(i), \end{aligned} \tag{17}$$

are some positive powers of J, K elementary shift equivalent?

We will show in Propositions 2 and 5 that the answer to this question is no in general. The difference between these two propositions is that the matrices in Proposition 5 have the special form (18) below. Note that the pair of matrices in Proposition 5 are not unimodular, and

$$G_m \subsetneq G_{m+1},$$

where G_m is defined by (4). Moreover, the two matrices have the form (18) which was the one required for the [BJO98] analysis. In general, we have $(G_{m+1} : G_m) = |\det J|$, and for the J, K pair in Proposition 5, G_{m+1}/G_m is $\mathbb{Z}/32\mathbb{Z}$ for J , and $\mathbb{Z}/16\mathbb{Z}$ for K .

A few words about terminology. In the theory of symbolic dynamics the term ‘dimension group’ is used slightly differently from the usage in C^* -theory introduced above, namely for the abelian group $G(J)$ without order structure, and it is used even in the wider context of non-positive matrices, see [LM95, Definition 7.5.1, BMT87]. The shift automorphism $(\sigma_J)_*$ is called the *dimension group automorphism* in this context. If J is non-negative, the positive part $G(J)_+$ is called the *dimension semigroup*, and the triple

$(G(J), G(J)_+, (\sigma_J)_*)$ is called the *dimension triple*. In the rest of this paragraph, let the term ‘matrix’ mean ‘matrix over the non-negative integers’. Then two square matrices J, K are *elementary shift equivalent* if there exist matrices A, B such that $J = BA$ and $K = AB$. We say that J, K are *shift equivalent* of lag k if (16) holds for some matrices A, B , and they are *shift equivalent* if they are shift equivalent of some lag k in $\mathbb{N}$. Thus shift equivalence of lag 1 is the same as elementary shift equivalence [LM95, Proposition 7.3.2]. The matrices J, K are *strongly shift equivalent* if they can be connected by a finite chain of elementary shift equivalent matrices. (Note that elementary shift equivalence is not an equivalence relation; it is not transitive.) Strong shift equivalence trivially implies shift equivalence [LM95, Theorem 7.3.3], but the converse is the long standing Williams conjecture, and it is not true [KR92], even when J and K are irreducible [KiRo98]. Since the stable AF algebras defined by J, K are isomorphic if and only if (17) holds for some sequences A, B, n, m , the equivalence defined by (17), or (14)–(15), could be termed *C*-equivalence*.

To summarize, in this paper we consider the following notions of equivalence of two non-negative matrices J, K , where each notion is strictly stronger than the next one. The notions 2, 3, 4 and 5 are equivalence relations, and 2 is the equivalence relation generated by 1.

1. Elementary shift equivalence = shift equivalence of lag 1.
2. Strong shift equivalence.
3. Shift equivalence = isomorphism of dimension triples (by Krieger’s theorem).
4. Shift equivalence of some positive powers J^n and K^m = elementary shift equivalence of some powers.
5. C*-equivalence = isomorphism of (ordered) dimension groups = isomorphism of associated stable AF algebras.

(It is well known that 2, 3 and 5 are equivalence relations, and for 4 the argument is as follows. If J, K and K, L are related as in 4, then there exist positive powers m, n, k, j such that each of the pairs J^n, K^m and K^k, L^j are elementary shift equivalent. But then each of the pairs J^{nk}, K^{mk} and K^{mk}, L^{mj} are elementary shift equivalent, and thus shift equivalent. Since shift equivalence is transitive, it follows that J^{nk}, L^{mj} are shift equivalent.)

Recall that elementary shift equivalence of non-singular square matrices implies conjugacy in the usual matrix sense over $\mathbb{Q}$, even in the absence of positivity. This is even true for shift equivalence, by an argument in the next paragraph. Note also that if J, K are primitive matrices, then shift equivalence in the sense of (16) is equivalent to the corresponding relation where A, B are only assumed to be matrices over $\mathbb{Z}$ (not assuming non-negativity), see [PW77, KR79, Theorem 2.1, LM95, Theorem 7.3.6].

The problem addressed in this paper arose in [BJO98] for incidence matrices J and K of the form

$$J = \begin{pmatrix} m_1 & 1 & 0 & \cdots & 0 & 0 \\ m_2 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ m_{N-1} & 0 & 0 & \cdots & 0 & 1 \\ m_N & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \quad (18)$$

where m_i are non-negative integers, $m_N \neq 0$ and the greatest common divisor of the set of k such that $m_k \neq 0$ is 1. The last conditions ensure that these matrices are non-singular and primitive, and whenever we refer to a ‘matrix of type (18)’ we assume that these additional conditions are satisfied. At the outset, it was not even clear if there were different such matrices with isomorphic C^* -algebras. The paper [BJO98] does, however, contain examples of pairs J, K of distinct matrices of type (18) which are C^* -equivalent. It is ironic that while the present paper started in a quest for pairs of the type (18) defining stably isomorphic AF algebras, i.e. pairs of matrices satisfying condition (17), but not with any constant sequences A, B, n, m , it is not so easy to find an example of a pair of distinct such matrices satisfying the condition with constant A, B, n, m . We give such an example between (32) and (33), and another example which is close in that J^{12} is conjugate to K^{12} in (31). Note in this connection that two non-negative square matrices J, K of the form (18) which are shift equivalent are identical by the following reasoning. They are non-singular and primitive, and since they are C^* -equivalent they have the same size $N \times N$, where N is the rank of the associated dimension group. The two last relations in (16) then imply that A and B are non-singular $N \times N$ matrices. Hence any of the first two relations in (16) implies that J and K are conjugate over $\mathbb{Q}$, and thus they have the same characteristic polynomial. But as explained in the beginning of Example 9, below, the characteristic polynomial uniquely determines these matrices, and hence $J = K$. Thus for matrices J, K of type (18), shift equivalence is the same as equality, C^* -equivalence is *not* the same as equality (by Proposition 5 and Example 9), while the situation for the intermediate equivalence 4 is also that shift equivalence of some powers do not imply equality; see the penultimate example in Example 9, or [BJO98, §7]. (The matrices (31) are C^* -equivalent with conjugate twelfth powers, but apparently do not satisfy 4.)

LEMMA 1. *Suppose that two given non-negative square $d \times d$ matrices J, K are primitive, non-singular and equal at their largest eigenvalue, that is, they have the same Perron eigenvalue and row and column eigenvectors. Then there exists a positive integer c such that $J^{cn} K^{-n}$ is non-negative for all $n > 0$.*

Proof. The parts at the largest eigenvalue will multiply, be positive, and will swamp all the others, since they grow at an exponential rate corresponding to this eigenvalue. More precisely, we can conjugate and then write $J = J_1 + J_2$, $K = K_1 + K_2$ corresponding to the eigenspace for the maximal eigenvalue λ_1 , and the eigenspaces for all other eigenvalues. Let the maximum of the absolute values of those eigenvalues be λ_2 and the maximum absolute value for an eigenvalue of their inverses be, say, λ_3 . Then

$$J^{ck} K^{-k} = J_1^{ck} K_1^{-k} + J_2^{ck} K_2^{-k}. \quad (19)$$

After we conjugate back, the entries in the first summand contribute entries proportional to the fixed row and column eigenvectors, which are at least $C_1 \lambda_1^{ck-k}$. The entries in the second part are at most $C_2 \lambda_2^{ck} \lambda_3^k$. Choose c large enough so that $\lambda_1^{c-1} > \lambda_2^c \lambda_3$ and the inequality will eventually hold. $\square$

Let us now consider the following two matrices:

$$J = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix} = 1 + P \quad (20)$$

$$K = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} = QJ, \quad (21)$$

where P and Q denote the matrices of the cyclic permutations which send 1, 2, 3, 4, 5 respectively to 2, 3, 4, 5, 1 and 2, 3, 5, 1, 4.

PROPOSITION 2. *The matrices J, K are C^* -equivalent, but no two powers are conjugate over the rational numbers.*

Proof. The C^* -equivalence is the existence of an infinite sequence $A(i), B(i)$ of non-negative matrices, and suitable powers, such that we have $J^{n(i)} = B(i)A(i)$, $K^{m(i)} = A(i+1)B(i)$. Let $A(1) = I$. We can solve recursively

$$\begin{aligned} B(1) &= J^{n(1)} \\ A(2) &= K^{m(2)} J^{-n(1)} \\ B(2) &= J^{n(2)+n(1)} K^{-m(2)} \\ A(3) &= K^{m(3)+m(2)} J^{-n(2)-n(1)} \end{aligned} \quad (22)$$

and so on. It follows that if for any c there exists d such that $J^d K^{-c}, K^d J^{-c}$ are non-negative integer matrices, the results hold. Non-negativity follows from Lemma 1: the matrices are primitive, their Perron eigenvalue is 2 and their left and right Perron eigenvectors are $(1, 1, 1, 1, 1)$ and $(1, 1, 1, 1, 1)^T$, respectively. To verify integrality, we compute the determinants as both 2; then the row spaces of J^n, K^n each lie within the space of vectors w whose product with the column vector $(1, 1, \dots, 1)^T$ is a positive integer multiple of 2^n . But since 2^n is their determinant, this is their exact row spaces, and the same holds for column spaces. It follows that each matrix is the product of the other matrix and a unimodular matrix, since the rows of the powers of each lie in the row spaces of the powers of the other.

To show powers of J, K can never be conjugate over the rational numbers (which the last equation implies) we compute that their characteristic polynomials are

$$\det(t\mathbf{1} - J) = (t - 2)(t^4 - 3t^3 + 4t^2 - 2t + 1), \quad \det(t\mathbf{1} - K) = (t - 2)(t^4 + t^3 + 1)$$

and the degree 4 factors are irreducible in $\mathbb{Q}[t]$. We restrict to the eigenspaces associated with the degree 4 factors. The discriminants of the degree 4 factors and of their algebraic

number fields are 229, 125. These are relatively prime, and the root field of J is cyclotomic (5th roots of unity from its circulant form), its only non-trivial proper subfield is quadratic and can be determined also to have discriminant a multiple of 5. Therefore the only intersection of the fields is the rational numbers [W63]. Suppose we have powers which are conjugate over the rational numbers, so have the same eigenvalues. These powers of eigenvalues all are in the intersection field, the rational numbers, and their product is 1 by the determinant. So powers of the eigenvalues are rational units, ± 1 and all eigenvalues of both matrices (other than the Perron eigenvalues) must be roots of unity. But this is false. $\square$

We will next show that even within the class of algebras with incidence matrices of the form (18) there are non-stationary isomorphisms. To this end we need a more general version of Lemma 1.

LEMMA 3. *Suppose the matrices J, K are primitive, non-singular and of equal size. Let V_1, V_2 be the sums of all column eigenspaces for eigenvalues other than the PF eigenvalue of J, K , respectively, and let w_1, w_2 be positive eigenvectors corresponding to the PF eigenvalues of J, K , respectively. Suppose there is a non-negative integer matrix $A(1)$ such that:*

- (1) $A(1)V_1 = V_2$;
- (2) $A(1)w_1, A(1)^{-1}w_2$ project non-trivially to positive multiples of w_2, w_1 in the direct sum of eigenspaces.

Then there exists a positive integer c such that $J^{cn}A(1)^{-1}K^{-n}, K^{cn}A(1)J^{-n}$ are non-negative for all sufficiently large n .

Proof. The parts at the largest eigenvalue will multiply, be positive, and will swamp all the others, since they grow at an exponential rate corresponding to this eigenvalue. More precisely, we can conjugate and then write $J = J_1 + J_2, K = K_1 + K_2$ corresponding to the eigenspace for the maximal eigenvalue λ_1 , and the eigenspaces for all other eigenvalues. Let the maximum of the absolute values of those eigenvalues be λ_2 and the maximum absolute value for an eigenvalue of their inverses be, say, λ_3 . Then by (1),

$$K^{ck}A(1)J^{-k} = K_1^{ck}A(1)J_1^{-k} + K_2^{ck}A(1)J_1^{-k} + K_2^{ck}A(1)J_2^{-k}. \quad (23)$$

After we conjugate back, the entries in the first summand contribute entries proportional to the fixed row and column eigenvectors, which are at least $C_1\lambda_1^{ck-k}$. The entries in the second and third parts are at most $C_2\lambda_2^{ck}\lambda_3^k$. Choose c large enough so that $\lambda_1^{c-1} > \lambda_2^c\lambda_3$ and the inequality will eventually hold. The same holds true for the $A(1)^{-1}$ term. $\square$

LEMMA 4. *If for all primes p dividing $\det(J)\det(K)$ the non-singular matrices J, K are nilpotent modulo p and the conditions of Lemma 3 are true for some $A(1)$ whose determinant involves only these primes, then the equations (15) hold for $A(1)$ and appropriate sequences $n(k), m(k)$. (This includes the case when there are no such p .)*

Proof. We write out the equations for a general $A(1)$ which is assumed to be a non-negative unit:

$$\begin{aligned} B(1) &= J^{n(1)} A(1)^{-1} \\ A(2) &= K^{m(1)} A(1) J^{-n(1)} \\ B(2) &= J^{n(2)+n(1)} A(1)^{-1} K^{-m(1)} \\ A(2) &= K^{m(2)+m(1)} A(1) J^{-n(2)-n(1)}. \end{aligned} \quad (24)$$

The nilpotence modulo p guarantees that some powers of J, K are divisible by p , hence any sufficiently large power is divisible by the determinant of a given power of the other matrix, so that if the powers increase sufficiently rapidly and $A(1)$ has determinant dividing some power of the determinants of J, K , these matrices exist over the integers, and positivity follows from Lemma 3 (except for the first equation, which follows by a similar argument or can be checked step by step). $\square$

PROPOSITION 5. *The matrices below are C^* -equivalent, but no powers are shift equivalent over the rationals:*

$$\begin{pmatrix} 4 & 1 \\ 32 & 0 \end{pmatrix}, \quad \begin{pmatrix} 6 & 1 \\ 16 & 0 \end{pmatrix}. \quad (25)$$

Proof. We check that the condition in Lemma 4 holds with $A(1) = I$ and that the matrices have eigenvalues respectively $8, -4$; $8, -2$. At the negative eigenvalues both column eigenvectors are $(1, -8)^T$ so the identity maps one to the other. One can alternatively check by the recipe (4)–(13) that the two matrices define the same dimension group (see below). But the values of the two pairs of eigenvalues prevents any power of one matrix being conjugate to a power of the other matrix over the rationals. $\square$

Note that in both the examples $J = \begin{pmatrix} 4 & 1 \\ 32 & 0 \end{pmatrix}$ and $K = \begin{pmatrix} 6 & 1 \\ 16 & 0 \end{pmatrix}$ above, the concrete realization of the dimension group G as a subgroup of $\mathbb{Q}^2$ is $\mathbb{Z}[1/2]^2$, and $(x, y) \in \mathbb{Z}[1/2]^2$ is positive and non-zero iff $8x + y > 0$.

This is proved as follows. One shows that

$$J = \begin{pmatrix} 4 & 1 \\ 32 & 0 \end{pmatrix} \quad (26)$$

has eigenvalues $8, -4$ with left eigenvectors $(8, 1), (4, -1)$ and right eigenvectors $\begin{pmatrix} 1 \\ 4 \end{pmatrix}, \begin{pmatrix} -1 \\ 8 \end{pmatrix}$. Hence

$$\begin{aligned} J^n &= \begin{pmatrix} \frac{2}{3}8^n + \frac{1}{3}(-4)^n & \frac{1}{12}8^n - \frac{1}{12}(-4)^n \\ \frac{8}{3}8^n - \frac{8}{3}(-4)^n & \frac{1}{3}8^n + \frac{2}{3}(-4)^n \end{pmatrix} \\ &= \frac{4^{n-1}}{3} \begin{pmatrix} 4(2 \cdot 2^n + (-1)^n) & 2^n - (-1)^n \\ 32(2^n - (-1)^n) & 4(2^n + 2(-1)^n) \end{pmatrix} \end{aligned} \quad (27)$$

for all $n \in \mathbb{Z}$. But as $2 \equiv -1 \pmod{3}$ and hence $2^n \equiv (-1)^n \pmod{3}$ for $n \in \mathbb{Z}$ one computes that the matrix elements for J^n are contained in $\mathbb{Z}[1/2]$ for both negative and positive n , and it follows that the dimension group $G(J) = \bigcup_{n=0}^{\infty} J^{-n}\mathbb{Z}^2$ is contained

in $\mathbb{Z}[1/2]^2$. But since J^n is 4^{n-1} times a matrix with integer coefficients by the formula above, it follows that $4^{1-n}(\mathbb{Z} \times \mathbb{Z}) \subseteq G(J)$ for $n = 1, 2, \dots$, and hence $\mathbb{Z}[1/2]^2 \subseteq G(J)$. Thus $G(J) = \mathbb{Z}[1/2]^2$. If $(x, y) \in G(J)$ it follows from the form of the left eigenvector that $(x, y) > 0$ if and only if $8x + y > 0$.

Correspondingly,

$$K = \begin{pmatrix} 6 & 1 \\ 16 & 0 \end{pmatrix} \quad (28)$$

has eigenvalues 8, -2 with left eigenvectors $(8, 1)$ and $(2, -1)$ and right eigenvectors $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 8 \end{pmatrix}$. Hence

$$K^n = \frac{2^{n-1}}{5} \begin{pmatrix} 2(4 \cdot 4^n + (-1)^n) & 4^n - (-1)^n \\ 16(4^n - (-1)^n) & 2(4^n + 4(-1)^n) \end{pmatrix} \quad (29)$$

for all $n \in \mathbb{Z}$, and using $4 \equiv -1 \pmod{5}$ and hence $4^n \equiv (-1)^n \pmod{5}$ for $n \in \mathbb{N}$, one proceeds as in the previous case.

The criteria in Theorems 6 and 7, below, reduce the question of isomorphism of this kind of AF algebra to standard questions in matrix theory somewhat like those in [KR79], that is, existence of a non-negative integer matrix $A(1)$ which maps certain computable linear spaces associated with J over extension fields isomorphically to corresponding linear spaces associated with K . In fairly simple cases computations should be practical, and we will show in a forthcoming paper that, as in the case of shift equivalence [KR79, KR88], an algorithm exists which will always decide isomorphism of the algebras. The hypotheses that J and K are non-singular in these propositions could be removed with some reformulation and a longer proof.

THEOREM 6. *Let J, K be primitive, non-singular, non-negative square matrices. Let V_1, V_2 be the sums of all column eigenspaces for eigenvalues other than the PF eigenvalue of J, K respectively, and let w_1, w_2 be positive eigenvectors corresponding to the PF eigenvalues of J, K , respectively. In order for the stationary C^* -algebras defined by J, K to be isomorphic, it is necessary that there be a non-negative integer matrix $A(1)$ such that:*

- (1) $A(1)(V_1) = V_2$;
- (2) $A(1)w_1, A(1)w_2$ project non-trivially to positive multiples of w_2, w_1 by the projection from the sum of eigenspaces to the eigenspace for the maximal eigenvector for K, J , respectively;
- (3) for the row (left) PF eigenvector v_1 of J , $v_1 A(1)^{-1}$ is non-negative.

These conditions are necessary for the non-negativity of $A(i)$, $i > 1$, $B(i)$, $i > 0$, and are sufficient if we allow replacement of $A(1)$ by some $A(1)J^{s_0}$.

Proof. If (1) does not hold true, then $A(1)$ will map vectors from V_1 into the maximal column eigenspace of K non-trivially. At the largest eigenvalue where this occurs, these terms will become dominant in $K^r A(1)J^{-s}$ and give the asymptotic value of the entire matrix. This will make the limit of $K^r A(1)J^{-s}$ as $r, s \rightarrow \infty$ in any way, a limit of matrices whose column vectors come from the non-maximal eigenspace of K . But this

is impossible, since other non-maximal eigenspaces of a positive matrix contain no non-negative vectors (if they did, multiplication by powers of the matrix would increase them at a rate which is asymptotically the maximal eigenvalue, which means that they would have components in the maximal eigenspace).

This proves (1), and given (1) we have

$$K^r A(1) J^{-s} = K_1^r A(1) J_1^{-s} + K_2^r A(1) J_1^{-s} + K_2^r A(1) J_2^{-s}.$$

If $A(1)w_1$ projects to a negative multiple of w_2 , then those terms will be dominant and make the entirety negative. If it projects to a zero multiple, then $A(1)w_1 \subset V_2$ which makes equality in (2) impossible. This proves (2). Sufficiency of (1), (2) are proved in Lemma 3.

For the condition (3), it will suffice that $J^{-s} A(1)^{-1}$ is eventually positive, looking at the dominant maximal eigenspaces. Conversely, if the vector is negative then the dominant part is negative, which is impossible. Suppose it is non-negative but not positive. Then the indicated replacement continues to allow solution of the other equations with altered exponents, but it maps the non-negative vector to a positive one. $\square$

Define p -adic limits of the powers of integer matrices A whose determinants divide p as follows: modulo each power p^m , there is a unique power $A^{e(m)}$ which is idempotent, since $\{A^n \mid n \in \mathbb{Z}^+\}$ is a cyclic finite semigroup. These idempotent powers agree to reductions modulo the lower of the two powers of p , by uniqueness. Therefore $A^{e(m)}$ have a p -adic limit. Modulo each power of p , any sufficiently large powers of the original matrix have the same row spaces as the idempotents, since each is a power of A times the other.

The result in the next theorem also holds if $\mathbb{Z}$ is replaced by the ring of algebraic integers of any algebraic number field, and p by any prime ideal of that ring. The same proof goes through. We recall here and in Proposition 10 that every algebraic number field contains the unique subring of algebraic integers, all elements in the field which satisfy monic polynomial equations. Primes refer, in general, to prime ideals in this subring (which give rise to valuations on the field). Given an element in the field, we can factor the ideals generated by its numerator and denominator into prime ideals uniquely, subtract, and hence up to units write it as a product of positive and negative powers of primes. These prime ideals will not, in general, be principal and hence arise from single elements, but a finite index subgroup of their multiplicative group are principal ideals; the finite quotient group is called the (ideal) class group of the field.

THEOREM 7. *In order for a pair of non-singular non-negative square matrices J, K to be C^* -equivalent, it is necessary that there exists a non-negative integer matrix $A(1)$ sending the p -adic row space of J_1 to the p -adic row space of K_1 isomorphically, where J_1, K_1 are the p -adic limits of powers of J, K respectively. This condition, taken over all primes dividing the determinants of J, K , is necessary and sufficient that we can (possibly altering exponents and taking a replacement of the matrix $A(1)$) make all the matrices $A(i), B(i)$ in (17) and (24) have integral entries.*

Proof. If $A(1)$ gives an isomorphism then for arbitrary large powers of J, K , the matrices $K^c A(1) J^{-d} = U, J^e A(1)^{-1} K^{-f} = V$ are integral. Modulo any fixed power of p , we can arrange by increasing these powers and altering U, V to other integral matrices that

the powers are in each case those giving rise to the idempotent limits K_1, J_1 . Therefore modulo each power of p ,

$$K_1 A(1) = U J_1, \quad J_1 = V K A(1).$$

Hence these matrices $K_1 A(1), J_1$ have equal row spaces.

Conversely, suppose that this condition holds. Then we can find p -adic U, V satisfying the last equations. Hence they satisfy them modulo each power of p . Consider a term like $K^r A(1) J^{-s}$ and the problem of making it integral at the prime p for sufficiently large r . In order for it to be integral, it suffices that it be so modulo the power of p dividing the determinant of J^s . Modulo this power of p , increase r until we may replace K^r by K_1 and use

$$K^r A(1) = U J_1 = U_1 J^s$$

and

$$K^r A(1) J^{-s} \det(J)^s \equiv U_1 \det(J)^s \pmod{\det(J)^s}.$$

This guarantees the left-hand side is divisible by $\det(J)^{-s}$, so that fractions in

$$K^r A(1) J^{-s}$$

have no denominators p . Taking all these primes means we have no denominators at all.

A special case is the first equation

$$B(1) = J^{n(1)} A(1)^{-1}.$$

For this we allow a replacement similar to the above of $A(1)$ by $K^r A(1)$ which will not affect solvability of the other equations. $\square$

PROPOSITION 8. *Consider the matrix*

$$K = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \quad (30)$$

Let $A(1) = K^{20}(K - 1)$, $J = K A(1)$ which is non-negative. Then the matrices J, K give isomorphic AF-algebras and are unimodular with irreducible characteristic polynomial, but no powers of them are shift equivalent.

Proof. These matrices come from [KRW, Example 4.1] and [KR92, main theorem]. It can be checked that they are unimodular and are units in the field generated by a root of the characteristic polynomial of K . By diagonalizing the field, it follows that multiplication by $A(1)$ sends V_1, w_1 to V_2, w_2 in the notation of Lemma 3, so that the isomorphism conditions are satisfied. The eigenvalues of J, K can be identified with the matrices themselves and their conjugates, under the map sending K to its maximal eigenvalue. If powers of J, K were shift equivalent, then the maximal eigenvalues would correspond to maximal eigenvalues up to powers. Hence we would have some equation $K^r = (K^{21}(K - 1))^s$. But $K, K - 1$ are independent in the group of units of this degree 4 field, so there can be no such equation. $\square$

Example 9. Since the characteristic polynomial of a matrix of the form (18) has the form

$$t^N - m_1 t^{N-1} - m_2 t^{N-2} - \dots - m_{N-1} t - m_N$$

it follows that two such matrices are conjugate (over $\mathbb{C}$) if and only if they are equal. Proposition 5 shows that two such matrices may satisfy the isomorphism condition (14)–(15) without being equal. Let us mention an example of two matrices satisfying (14)–(15) such that their twelfth powers are conjugate over $\mathbb{C}$, but which are not shift equivalent:

$$J = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (31)$$

Their joint Perron–Frobenius eigenvalue is the real root of $t^3 - t - 1$ and since they are unimodular, the dimension groups are isomorphic by [BJO98, Corollary 6.2]. One computes that their spectra are non-degenerate, and their twelfth powers have the eigenvalue 1 with multiplicity 2 in addition to the twelfth powers of the roots of the equation above. Thus these powers are conjugate over $\mathbb{C}$. But their twelfth powers also seem to be not shift equivalent. The steps involved were to make the two matrices block triangular using the two eigenvalues 1 and their eigenvectors over the integers. Then any conjugacy of the matrices over the integers must also be triangular in this form and for the 3×3 block we must have a unit of the field generated by a root of $z^3 - 29z^2 - 6z - 1$, and the units are generated by our matrix, say U_1 , and $(7/55)U_1^2 - (10/11)U_1 - (57/55)$. We have an equation for the three non-zero blocks of the block triangular matrix giving a shift equivalence, and modulo 7 one block in effect cancels. For the other we calculate up to scalar multiples 64 units reduced modulo 7 and check for each that the equation is impossible. We expect that what happens for twelfth powers should also be true for all higher powers by a result in [KR79].

Let us consider a more clearcut example:

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (32)$$

The characteristic polynomials are

$$\det(t\mathbf{1} - J) = (t^3 - t - 1)(t^3 + 1), \quad \det(t\mathbf{1} - K) = (t^3 - t - 1)(t^3 + t + 1)$$

and hence the Perron–Frobenius eigenvalue is the real root of $t^3 - t - 1$ in both cases, and the associated dimension groups are isomorphic by [BJO98, Corollary 6.2]. All six roots of each of the two polynomials are distinct. But we note that the three additional roots of the first polynomial all have modulus one, and since all the roots of the latter polynomial

has moduli different from one, J^n is not conjugate to K^m over $\mathbb{C}$ for any positive powers n, m .

We now give an example of two matrices J, K of the form (18) which are distinct, but have shift equivalent second powers, so in particular the equivalence relation 4 is satisfied. To begin with let J, K be the 6×6 matrices

$$J = \begin{pmatrix} 6 & 1 & 0 & 0 & 0 & 0 \\ 16 & 0 & 1 & 0 & 0 & 0 \\ 197 & 0 & 0 & 1 & 0 & 0 \\ 90 & 0 & 0 & 0 & 1 & 0 \\ 2200 & 0 & 0 & 0 & 0 & 1 \\ 12000 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 8 & 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 \\ 97 & 0 & 0 & 1 & 0 & 0 \\ 370 & 0 & 0 & 0 & 1 & 0 \\ 3400 & 0 & 0 & 0 & 0 & 1 \\ 12000 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

respectively. Their characteristic polynomials are

$$\det(t\mathbf{1} - J) = (t - 10)(t + 3)(t^2 - 4t + 16)(t^2 + 5t + 25)$$

$$\det(t\mathbf{1} - K) = (t - 10)(t + 3)(t^2 + 4t + 16)(t^2 - 5t + 25)$$

respectively. The squares J^2, K^2 of these matrices have the same characteristic polynomial:

$$(t - 100)(t - 9)(t^2 + 16t + 256)(t^2 + 25t + 625)$$

which has six distinct roots. Thus there are non-trivial intertwiners between each of J, K and each of the matrices

$$(100), \quad (9), \quad \begin{pmatrix} -8 & 14 \\ -14 & -8 \end{pmatrix}, \quad \begin{pmatrix} -12 & 67 \\ -7 & -13 \end{pmatrix},$$

which are all irreducible over $\mathbb{Q}$. At the outset, these intertwiners may be taken to have matrix elements in a finite field extension of order 1 or 2 over $\mathbb{Q}$, but then by linearity they may be taken to have matrix elements in $\mathbb{Q}$. Thus J^2 and K^2 are similar over $\mathbb{Q}$, and hence they are elementary shift equivalent over $\mathbb{Q}$. It follows by an argument using some rational shift equivalences with denominator d , that if we replace J, K by

$$J = \begin{pmatrix} 6d & 1 & 0 & 0 & 0 & 0 \\ 16d^2 & 0 & 1 & 0 & 0 & 0 \\ 197d^3 & 0 & 0 & 1 & 0 & 0 \\ 90d^4 & 0 & 0 & 0 & 1 & 0 \\ 2200d^5 & 0 & 0 & 0 & 0 & 1 \\ 12000d^6 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 8d & 1 & 0 & 0 & 0 & 0 \\ 2d^2 & 0 & 1 & 0 & 0 & 0 \\ 97d^3 & 0 & 0 & 1 & 0 & 0 \\ 370d^4 & 0 & 0 & 0 & 1 & 0 \\ 3400d^5 & 0 & 0 & 0 & 0 & 1 \\ 12000d^6 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

for an appropriate d in $\mathbb{N}$ we obtain shift equivalence of the squares J^2, K^2 over the integers. (The details of the argument are written out in, for example, [BJO98, §7].) But then J^2, K^2 are shift equivalent over the non-negative integers by [KR79, Theorem 2.1] and [LM95, Theorem 7.3.6]. (As explained after (18), J, K themselves cannot be shift equivalent.)

Let us end with an example which is maximally clearcut, and clears the way for Theorem 10:

$$J = [6], \quad K = [12]. \tag{33}$$

We leave it to the reader to decide that J, K are C^* -equivalent, i.e. (15) holds, but no powers are shift equivalent or even conjugate.

We next give a necessary condition for C^* -equivalence. Recall that if λ is an algebraic number, then $\mathbb{Q}[\lambda]$ is a field. ($\mathbb{Q}[\lambda]$ is by definition a ring, all polynomials in λ over $\mathbb{Q}$, and multiplicative inverses of non-zero elements exist because of the Euclidean algorithm in $\mathbb{Q}[t]$. Let $f(t)$ be the prime polynomial corresponding to λ (so that $\mathbb{Q}[\lambda] = \mathbb{Q}[t]/f[t]$) and let $g(t)$ be any polynomial non-zero modulo f . Then there exist polynomials $r, s \in \mathbb{Q}[t]$ such that $rg + sf = 1$, hence modulo f , r is an inverse of g .) The term ‘prime’ in this field still means a prime in the associated subring of algebraic integers.

THEOREM 10. *Suppose that the two non-negative square $d \times d$ matrices J, K are non-singular and primitive. Let π_1, π_2 be the projections from all row vectors to the maximal eigenspaces of J, K . Then a necessary condition for isomorphism of the AF algebras determined by J, K is that:*

- (1) *[BD95, Corollary 3.5] the fields $\mathbb{Q}[\lambda_1], \mathbb{Q}[\lambda_2]$ generated by the maximal eigenvalues λ_1, λ_2 of J, K are the same and λ_1, λ_2 are products of the same primes over this field;*
- (2) *the dimension group quotients denoted $\pi_i(G) = G^{\text{Per}}$ in [BMT87] (quotients by non-maximal eigenspaces) are isomorphic as modules over $\mathbb{Z}[1/\lambda_i^n]$ for sufficiently large n .*

This is a sufficient condition if the characteristic polynomials are irreducible, otherwise Theorem 7 gives some additional necessary conditions.

Proof. First we argue for the necessity. Let the maximal and non-maximal eigenspaces of J, K be $\langle w_1 \rangle, V_1, \langle w_2 \rangle, V_2$ as in Lemma 3. The matrix $A(1)$ must map V_1 to V_2 non-trivially and send w_1 to w_2 non-trivially by Theorem 6. Since $A(1)$ is rational, it commutes with Galois actions among the different conjugates of the maximal eigenvalue. This already implies that the two fields must be the same (they have the same set of non-trivial Galois actions under some finite Galois extension containing both). It follows essentially by Theorem 7 that the prime factors of the two maximal eigenvalues must be the same over algebraic number fields. The intersection of all conjugates of V_1 goes to the corresponding intersection for V_2 by this isomorphism consistent with Galois action, so that the quotient of the rational dimension group by non-conjugate eigenvalues is mapped from the one to the other. The mapping $A(1)$ taken over the field $\mathbb{Q}[\lambda]$ gives an isomorphism between the maximal eigenspaces. This will be multiplication by some element $x \in \mathbb{Q}[\lambda]$. Let G_1, G_2 be $\pi_1(\mathbb{Z}^d), \pi_2(\mathbb{Z}^d)$. Multiplication by x will take G_1 into G_2 , by the effect of $A(1)$, and its inverse will do the reverse, up to multiplication by powers of the primes in λ_i which represent multiplication by J, K . Therefore

$$xG_1 \subset G_2, \quad \lambda_1^{e_0} x^{-1} G_2 \subset G_1 \quad \text{for some positive } e_0. \quad (\text{DG})$$

Recall that the dimension groups of J, K can be viewed as the direct limit of $\mathbb{Z}^d$ sent to itself by J, K ; if we embed them in the maximum eigenspace this direct limit is equivalent to making λ_1 or λ_2 , or all the primes in them invertible so that (DG) implies isomorphism. This implies that the dimension groups of the matrices J, K are isomorphic as modules over $\mathbb{Z}[1/\lambda_i^n]$ for sufficiently large n . (This condition does not depend on i .) This proves necessity.

Now assume the characteristic polynomials of J , K are irreducible and (1) and (2) hold, hence (DG). Choose such an isomorphism as in (DG), and adjust its sign so that on the maximal eigenvector it is positive. Expanding out the coefficients gives a map $A(1)$ over the integers which preserves all the conjugates of the rational eigenspace. The effect of $A(1)$ on all vectors over $\mathbb{Q}$ is isomorphically mirrored in its effect on vectors over $\mathbb{Q}[\lambda]$ on the maximal eigenspace there. This fact together with Lemma 3 ensures positivity of the other $A(i)$, $B(i)$. To get positivity of $A(1)$ we replace it by some product $K^{r_1} A(1) J^{r_2}$ and note that its effect on the maximum row eigenvector is positive and this dominates the product asymptotically. The assumption on the primes and (DG), implies that for all n there exists an m such that $K^m A(1) J^{-n}$, $K^m A(1)^{-1} J^n$ involves only non-negative powers of all the primes and is divisible by any given power of each λ_i . This implies that all the $A(i)$, $B(i)$ exist over the integers. $\square$

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REFERENCES

- [BD95] M. Barge and B. Diamond. Homeomorphisms of inverse limit spaces of one-dimensional maps. *Fundamenta Math.* **146** (1995), 171–187.
- [BH96] M. Boyle and D. Handelman. Orbit equivalence, flow equivalence, and ordered cohomology. *Israel J. Math.* **95** (1996), 169–210.
- [BMT87] M. Boyle, B. Marcus and P. Trow. Resolving maps and the dimension group for shifts of finite type. *Memoirs Amer. Math. Soc.* **377** (1987).
- [Bra72] O. Bratteli. Inductive limits of finite dimensional C^* -algebras. *Trans. Amer. Math. Soc.* **171** (1972), 195–234.
- [BJO98] O. Bratteli, P. E. T. Jørgensen and V. Ostrovsky. Representation theory and numerical AF-invariants: the representations and centralizers of certain states on $\mathcal{O}_d$. *Preprint*, 1999.
- [BrKi98] O. Bratteli and A. Kishimoto. Trace scaling automorphisms of certain stable AF-algebras, II. *Quarterly J. Math.* **51** (2000), 131–154.
- [DHS] F. Durand, B. Host and C. Skau. Substitution dynamical systems, Bratteli diagrams and dimension groups. *Ergod. Th. & Dynam. Sys.* **19** (1999), 953–993.
- [Eff81] E. G. Effros. *Dimensions and C^* -algebras* (CBMS Regional Conference Series in Mathematics, 46). Conference Board of the Mathematical Sciences, Washington, 1981.
- [Ell76] G. A. Elliott. On the classification of inductive limits of sequences of semi-simple finite dimensional algebras. *J. Algebra* **38** (1976), 29–44.
- [EvKi97] D. E. Evans and A. Kishimoto. Trace scaling automorphisms of certain stable AF algebras. *Hokkaido Math. J.* **26** (1997), 211–224.
- [For97] A. Forrest. K-groups associated with substitution minimal systems. *Israel J. Math.* **98** (1997), 101–139.
- [Fro08] G. Frobenius. Über Matrizen aus positiven Elementen, Sitzungsber. Preuß. Akad. Wiss. Physikal.-Math. Kl. (1908), 471–476.
- [Fro09] G. Frobenius. Über Matrizen aus positiven Elementen, Sitzungsber. Preuß. Akad. Wiss. Physikal.-Math. Kl. (1909), 514–518.
- [Gan59] F. R. Gantmacher. *Applications of the Theory of Matrices*. Interscience Publishers, New York, 1959.

- [GPS95] T. Giordano, I. Putnam and C. Skau. Topological orbit equivalence and C^* -crossed products. *J. Reine Angew. Math.* **469** (1995), 51–111.
- [Han81] D. Handelman. Positive matrices and dimension groups affiliated to C^* -algebras and topological Markov chains. *J. Operator Theory* **6** (1981), 55–74.
- [KR79] K. H. Kim and F. W. Roush. Some results on decidability of shift equivalence. *J. Combinatorics, Information & System Sci.* **4** (1979), 123–146.
- [KR88] K. H. Kim and F. W. Roush. Decidability of shift equivalence. *Dynamical Systems (Lecture Notes in Mathematics, 1342)*. Ed. J. C. Alexander. Springer, 1988, pp. 374–424.
- [KR92] K. H. Kim and F. W. Roush. Williams’s conjecture is false for reducible subshifts. *J. Amer. Math. Soc.* **5** (1992), 213–215.
- [KiRo98] K. H. Kim and F. W. Roush. The Williams conjecture is false for irreducible subshifts. *Ann. Math.* **149** (1999), 545–558.
- [KRW] K. H. Kim, F. W. Roush and J. Wagoner. Automorphisms of the dimension group and gyration numbers. *J. Amer. Math. Soc.* **5** (1992), 191–212.
- [Kri80] W. Krieger. On dimension functions and topological Markov chains. *Invent. Math.* **56** (1980), 239–250.
- [LM95] D. Lind and B. Marcus. *Symbolic Dynamics and Coding*. Cambridge University Press, 1995.
- [MP98] D. Mundici and G. Panti. The equivalence problem for Bratteli diagrams. *Preprint*, 1998.
- [Per07] O. Perron. Zur Theorie der Matrices. *Math. Ann.* **64** (1907), 248–263.
- [PW77] W. Parry and R. F. Williams. Block coding and a zeta function for finite Markov chains. *Proc. London Math. Soc.* **35** (1977), 483–495.
- [Po89] Y. T. Poon. A K -theoretic invariant for dynamical systems. *Trans. Amer. Math. Soc.* **311** (1989), 515–533.
- [SV98] R. Swanson and H. Volkmer. Invariants of weak equivalence in primitive matrices. *Ergod. Th. & Dynam. Sys.* **20** (2000), 611–626.
- [Tor91] A. Török. AF-algebras with unique trace. *Acta Sci. Math. (Szeged)* **55** (1991), 129–139.
- [W63] E. Weiss. *Algebraic Number Theory*. McGraw-Hill, New York, 1963.
- [Wal92] D. Walters (Ed.). *Symbolic Dynamics and its Applications (Contemporary Mathematics, 135)*. American Mathematical Society, 1992.