

Rohlin flows on the Cuntz algebra $\mathcal{O}_\infty$

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Abstract

It is shown that certain quasi-free flows on the Cuntz algebra $\mathcal{O}_\infty$ have the Rohlin property and therefore are cocycle-conjugate with each other. This, in particular, shows that any unital separable nuclear purely infinite simple C^* -algebra has a Rohlin flow.

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1. Introduction

A flow α on a unital C^* -algebra A is said to have the Rohlin property, or to be a Rohlin flow, if for any $p \in \mathbf{R}$ there is a central sequence (u_n) in $\mathcal{U}(A)$, the unitary group of A , such that $\max_{|t| \leq 1} \|\alpha_t(u_n) - e^{ipt} u_n\| \rightarrow 0$ as $n \rightarrow \infty$. A major consequence of this property can be paraphrased as *any α -cocycle is almost a coboundary*. This consequence, combined with enough information on $\mathcal{U}(A)$, could lead to a classification theory of Rohlin flows up to cocycle conjugacy. This is a goal we have in mind (see [14,17–20]).

Since the Rohlin property is rather stringent, it is not easy to present a Rohlin flow in general. Nevertheless, one of us (A.K.) managed to construct Rohlin flows on the Cuntz algebra $\mathcal{O}_n$ with n finite. Moreover, one can identify the quasi-free flows which have the Rohlin property. In this paper we show that certain quasi-free flows on $\mathcal{O}_\infty$ have the Rohlin property. Hence it follows

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that any unital C^* -algebra A with $A \cong A \otimes \mathcal{O}_\infty$ has Rohlin flows. The class of such A includes all unital separable nuclear purely infinite simple C^* -algebras, by a result of Kirchberg.

We have left undecided whether or not certain quasi-free flows on $\mathcal{O}_\infty$ have the Rohlin property. But, as in [17], we show that all the Rohlin flows on $\mathcal{O}_\infty$ are cocycle conjugate with each other in the class of quasi-free flows. This is true for a wider class of flows. Noting that there is a certain maximal abelian C^* -subalgebra $\mathcal{C}_\infty$ of $\mathcal{O}_\infty$ whose elements are fixed by the quasi-free flows, we show that Rohlin flows are cocycle-conjugate in the class of flows which are $C^{1+\varepsilon}$ on $\mathcal{C}_\infty$ (see below for details and note that our definition of quasi-free flows is restrictive).

We now describe the contents of the paper more precisely.

For each $n = 2, 3, \dots$ the Cuntz algebra $\mathcal{O}_n$ is generated by n isometries $s_1, s_2, \dots, s_n$ such that $\sum_{k=1}^n s_k s_k^* = 1$. For $n = \infty$ the Cuntz algebra $\mathcal{O}_\infty$ is generated by a sequence $(s_1, s_2, \dots)$ of isometries such that $\sum_{k=1}^n s_k s_k^* \leq 1$ for all n . It is shown in [6] that $\mathcal{O}_n$ with $n = 2, 3, \dots$ or $n = \infty$ is a simple purely infinite nuclear C^* -algebra.

For a finite (respectively infinite) sequence $(p_1, p_2, \dots, p_n)$ in $\mathbf{R}$ we define a flow α , called a quasi-free flow, on $\mathcal{O}_n$ (respectively $\mathcal{O}_\infty$) by

$$\alpha_t(s_k) = e^{ip_k t} s_k.$$

(In the case $n = \infty$, a more general flow can be induced by a unitary flow U on the closed linear subspace $\mathcal{H}$ spanned by $s_1, s_2, \dots$, where the inner product $\langle \cdot, \cdot \rangle$ is given by $y^* x = \langle x, y \rangle 1$, $x, y \in \mathcal{H}$, if the generator of U is not diagonal. But we will exclude them from the quasi-free flows in this paper, except in Corollary 5.12.) It is shown in [11,21] that if $p_1, p_2, \dots$ generate $\mathbf{R}$ as a closed subsemigroup, then the crossed product $\mathcal{O}_n \rtimes_\alpha \mathbf{R}$ is simple and purely infinite (whether n is finite or infinite). It is also known [17,20] that if n is finite, the flow α has the Rohlin property if and only if $\mathcal{O}_\infty \rtimes_\alpha \mathbf{R}$ is simple and purely infinite. Alternatively for $n = \infty$, it is known [11,14] that if α has the Rohlin property then $\mathcal{O}_\infty \rtimes_\alpha \mathbf{R}$ is simple and purely infinite. In this paper we shall give a partial converse to this fact.

Theorem 1.1. *Let (p_k) in $\mathbf{R}$ be an infinite sequence such that $p_1, p_2, \dots, p_n$ generate $\mathbf{R}$ as a closed subsemigroup for some n . Then the quasi-free flow α on $\mathcal{O}_\infty$ defined by $\alpha_t(s_k) = e^{ip_k t} s_k$ has the Rohlin property.*

We shall prove that each α_t is α -invariantly approximately inner, i.e., for each $t \in \mathbf{R}$ there is a sequence (u_n) in $\mathcal{U}(\mathcal{O}_\infty)$ such that $\alpha_t(x) = \lim \text{Ad } u_n(x)$, $x \in \mathcal{O}_\infty$, and $\max_{s \in [0,1]} \|\alpha_s(u_n) - u_n\| \rightarrow 0$. Then we can derive the above theorem, by [18,19], from the fact that $\mathcal{O}_\infty \rtimes_\alpha \mathbf{R}$ is simple and purely infinite.

Let $\mathcal{E}_n$ be the C^* -subalgebra of $\mathcal{O}_\infty = C^*(s_1, s_2, \dots)$ generated by $s_1, s_2, \dots, s_n$. Then $\mathcal{E}_n$ is left invariant under α and the union $\bigcup_n \mathcal{E}_n$ is dense in $\mathcal{O}_\infty$. Hence, to prove the assertion in the previous paragraph, it suffices to show that $\alpha|_{\mathcal{E}_n}$ is α -invariantly approximately inner for all large n . Let us state formally:

Proposition 1.2. *Let $s_1, s_2, \dots, s_n$ be isometries such that*

$$\sum_{k=1}^n s_k s_k^* \leq 1$$

and let $\mathcal{E}_n$ be the C^* -algebra generated by these $s_1, \dots, s_n$. Further let $(p_1, p_2, \dots, p_n)$ be a finite sequence in $\mathbf{R}$ such that $p_1, \dots, p_n$ generate $\mathbf{R}$ as a closed subsemigroup and define a quasi-free flow α on $\mathcal{E}_n$ by $\alpha_t(s_k) = e^{ip_k t} s_k$. Then each α_t is α -invariantly approximately inner.

To prove this we use the following facts. Let $\mathcal{J}_n$ be the ideal of $\mathcal{E}_n$ generated by $e_n^0 = 1 - \sum_{k=1}^n s_k s_k^*$. Then $\mathcal{J}_n$ is isomorphic to the C^* -algebra $\mathcal{K}$ of compact operators (on a separable infinite-dimensional Hilbert space) and is left invariant under α . The quotient $\mathcal{E}_n/\mathcal{J}_n$ is isomorphic to $\mathcal{O}_n$ by mapping $s_k + \mathcal{J}_n$ into s_k (the latter s_k satisfy the equality $\sum_{k=1}^n s_k s_k^* = 1$ and generate $\mathcal{O}_n$). By the assumption on (p_k) the induced flow $\dot{\alpha}$ on $\mathcal{O}_n$ has the Rohlin property [20], from which follows that each $\dot{\alpha}_t$ is α -invariantly approximately inner. We will translate this property to α_t on $\mathcal{E}_n$ by using the fact that $\mathcal{J}_n \cong \mathcal{K}$. See Section 3 for details.

Before embarking on the proof of the above proposition, we have to prove that if α is a Rohlin flow on $\mathcal{O}_n$, then each α_t is not only α -invariantly approximately inner but also α -invariantly asymptotically inner, i.e., there is a continuous map $u: [0, \infty) \rightarrow \mathcal{U}(\mathcal{O}_n)$ such that $\alpha_t(x) = \lim_{s \rightarrow \infty} \text{Ad } u(s)(x)$ for $x \in \mathcal{O}_n$ and $\max_{t_1 \in [0, 1]} \|\alpha_{t_1}(u(s)) - u(s)\| \rightarrow 0$. This will be proved for a wider class of C^* -algebras (see 2.2 for details). (As a matter of fact we do not know of a single example of α without the above property of α -invariant asymptotical innerness if it has covariant irreducible representations. We expect that this property holds fairly generally whether α has the Rohlin property or not.)

As a corollary to the above theorem we deduce that any purely infinite simple separable nuclear C^* -algebra has a Rohlin flow. The reason is that a C^* -algebra A of this type satisfies $A \cong A \otimes \mathcal{O}_\infty$ by a result of Kirchberg (see [9]). Hence a flow α on $\mathcal{O}_\infty$ induces a flow on A via $\text{id} \otimes \alpha$ on $A \otimes \mathcal{O}_\infty$ and the latter flow has the Rohlin property whenever α has the property.

Let $\mathcal{C}_\infty$ denote the C^* -subalgebra of $\mathcal{O}_\infty$ generated by $s_{i_1} s_{i_2} \cdots s_{i_k} s_{i_k}^* \cdots s_{i_1}^*$ with all finite sequences $(i_1, i_2, \dots, i_k)$ in $\mathbf{N}$. Then $\mathcal{C}_\infty$ is a weakly regular maximal abelian C^* -subalgebra of $\mathcal{O}_\infty$ (weakly regular in the sense that $\{u \in \mathcal{PI}(\mathcal{O}_\infty): uu^*, u^*u \in \mathcal{C}_\infty, u\mathcal{C}_\infty u^* = \mathcal{C}_\infty uu^*\}$ generates $\mathcal{O}_\infty$, where $\mathcal{PI}(\mathcal{O}_\infty)$ is the set of partial isometries of $\mathcal{O}_\infty$). Moreover, there is a projection of norm one from $\mathcal{O}_\infty$ onto $\mathcal{C}_\infty$ and there is a character of $\mathcal{C}_\infty$ which extends uniquely to a state of $\mathcal{O}_\infty$. (When a weakly regular masa (maximal abelian self-adjoint algebra) satisfies these two additional conditions, we will say that it is a *weak Cartan* masa.) Note that if α is a quasi-free flow (in our sense) then α_t is the identity on $\mathcal{C}_\infty$. In other words, if δ_α denotes the generator of α , then $D(\delta_\alpha) \supset \mathcal{C}_\infty$ and $\delta_\alpha|_{\mathcal{C}_\infty} = 0$. We consider the following condition for a flow γ on $\mathcal{O}_\infty$: $D(\delta_\gamma) \supset \mathcal{C}_\infty$ and $\sup_{x \in \mathcal{C}_\infty, \|x\| \leq 1} \|(\gamma_t - \text{id})\delta_\gamma(x)\|$ converges to zero as $t \rightarrow 0$. Equivalently we say that γ is $C^{1+\varepsilon}$ on $\mathcal{C}_\infty$. This is obviously satisfied if γ is C^2 on $\mathcal{C}_\infty$ or $D(\delta_\gamma^2) \supset \mathcal{C}_\infty$ (because then $\delta_\alpha^2|_{\mathcal{C}_\infty}$ is bounded).

We also show:

Corollary 1.3. *Any two Rohlin flows on $\mathcal{O}_\infty$ which are $C^{1+\varepsilon}$ on $\mathcal{C}_\infty$ are cocycle conjugate.*

The proof consists of two parts. In the first part we show that if the flow γ is $C^{1+\varepsilon}$ on $\mathcal{C}_\infty$ then $\delta_\gamma|_{\mathcal{C}_\infty}$ is inner, i.e., there is an $h = h^* \in \mathcal{O}_\infty$ such that $\delta_\gamma(x) = \text{ad } ih(x)$, $x \in \mathcal{C}_\infty$ (see Corollary 5.6). Thus we can assume, by inner perturbation, that $\delta_\gamma|_{\mathcal{C}_\infty} = 0$. In the second part we show that any two Rohlin flows are cocycle-conjugate to each other if they fix each element of $\mathcal{C}_\infty$ (see Corollary 5.11).

2. Rohlin property

In this section we consider the class of purely infinite simple nuclear separable C^* -algebra satisfying the universal coefficient theorem, which is classified by Kirchberg and Phillips [9,10] in terms of K -theory.

Let A be a unital C^* -algebra of the above class. Let $\ell^\infty(A)$ be the C^* -algebra of bounded sequences in A and for each free ultrafilter ω on $\mathbf{N}$ let $c_\omega(A)$ be the ideal of $\ell^\infty(A)$ consisting of $x = (x_n)$ with $\lim_\omega \|x_n\| = 0$. If α is a flow on A , i.e., a strongly continuous one-parameter automorphism group of A , we can define an action of $\mathbf{R}$ on $\ell^\infty(A)$ by $t \mapsto (\alpha_t(x_n))$ for $x = (x_n)$. Let $\ell_\alpha^\infty(A)$ be the strong continuity subspace of this action, i.e., the maximal C^* -subalgebra of $\ell^\infty(A)$ on which the action is continuous. We also denote this flow by α . Set

$$A^\omega = \ell^\infty(A)/c_\omega(A), \quad A_\alpha^\omega = \ell_\alpha^\infty(A)/c_\omega(A).$$

Embed A into $\ell_\alpha^\infty(A)$ by constant sequences. Since $A \cap c_\omega(A) = \{0\}$, we regard A as a C^* -subalgebra of $A_\alpha^\omega \subset A^\omega$.

We recall the following result.

Theorem 2.1. [18,19] *Let A be a unital separable nuclear purely infinite simple C^* -algebra satisfying the universal coefficient theorem and let α be a flow on A . Then the following conditions are equivalent.*

1. *The flow α has the Rohlin property.*
2. *The fixed point algebra $(A' \cap A_\alpha^\omega)^\alpha$ is purely infinite and simple, $K_0((A' \cap A_\alpha^\omega)^\alpha) \cong K_0(A' \cap A^\omega)$ induced by the embedding, and $\text{Spec}(\alpha|_{A' \cap A_\alpha^\omega}) = \mathbf{R}$.*
3. *The crossed product $A \rtimes_\alpha \mathbf{R}$ is purely infinite and simple and the dual action $\hat{\alpha}$ has the Rohlin property.*
4. *The crossed product $A \rtimes_\alpha \mathbf{R}$ is purely infinite and simple and α_{t_0} is α -invariantly approximately inner for every $t_0 \in \mathbf{R}$.*

If the above conditions are satisfied, it also follows that $K_1((A' \cap A_\alpha^\omega)^\alpha) \cong K_1(A' \cap A^\omega)$, and this isomorphism is induced by the embedding.

Recall that α_{t_0} , for a fixed t_0 , is α -invariantly approximately inner if there is a sequence (u_n) in $\mathcal{U}(A)$ such that $\alpha_{t_0} = \lim \text{Ad } u_n$ and $\max_{t \in [0,1]} \|\alpha_t(u_n) - u_n\| \rightarrow 0$. We strengthen this condition as follows.

Lemma 2.2. *Let α be a Rohlin flow on a unital C^* -algebra A of the above class (or in particular $\mathcal{O}_n$). Then each α_{t_0} is α -invariantly asymptotically inner, i.e., there is a continuous map $u : [0, \infty) \rightarrow \mathcal{U}(A)$ such that $\alpha_{t_0} = \lim_{s \rightarrow \infty} \text{Ad } u(s)$ and*

$$\lim_{s \rightarrow \infty} \max_{t \in [0,1]} \|\alpha_t(u(s)) - u(s)\| = 0.$$

Proof. Since $KK(\alpha_{t_0}) = KK(\text{id})$, α_{t_0} is asymptotically inner [25], i.e., there is a continuous map $v : [0, \infty) \rightarrow \mathcal{U}(A)$ such that

$$\alpha_{t_0} = \lim_{s \rightarrow \infty} \text{Ad } v(s).$$

Let $w(s, t) = v(s)\alpha_t(v(s)^*)$, $s \in [0, \infty)$, $t \in \mathbf{R}$. Then for each $s \in [0, \infty)$, the map $t \mapsto w(s, t)$ is an α -cocycle, i.e., $t \mapsto w(s, t)$ is a continuous function into $\mathcal{U}(A)$ such that $w(s, t_1 + t_2) = w(s, t_1)\alpha_{t_1}(w(s, t_2))$, $t_1, t_2 \in \mathbf{R}$. Since for each $x \in A$,

$$\| [w(s, t), x] \| \leq \| \text{Ad } v(s)(\alpha_{-t_0-t}(x)) - \alpha_{-t}(x) \| + \| \text{Ad } v(s)(\alpha_{-t_0}(x)) - x \|,$$

we find, for any (large) positive T and for any $x \in A$, that

$$\sup_{0 \leq t \leq T} \| [w(s, t), x] \| \rightarrow 0$$

as $s \rightarrow \infty$.

More specifically let $\mathcal{F}$ be a finite subset of A and $\varepsilon > 0$. Then there exists an $a > 0$ such that if $s \geq a$, then $\| \text{Ad } v(s)(\alpha_{-t_0-t}(x)) - \alpha_{-t}(x) \| < \varepsilon/22$ for $x \in \mathcal{F}$ and $t \in [0, T]$, which entails that $\| [w(s, t), x] \| < \varepsilon/11$ for $x \in \mathcal{F}$ and $t \in [0, T]$.

Furthermore, for any bounded interval I of $[a, \infty)$, there is a continuous map $z: I \times [0, T] \rightarrow \mathcal{U}(A)$ such that

$$\begin{aligned} z(s, 0) &= 1, & z(s, T) &= w(s, T), \\ \| z(s, t_1) - z(s, t_2) \| &\leq (16\pi/3 + \varepsilon)|t_1 - t_2|/T, \\ \| [z(s, t), x] \| &< 10\varepsilon/11, \quad x \in \mathcal{F}, \end{aligned}$$

for $s \in I$ and $t, t_1, t_2 \in [0, T]$. (Here we used the estimate

$$\| [z(s, t), x] \| < 9 \max_{0 \leq t_1 \leq T} \| [w(s, t_1), x] \| + \varepsilon'$$

of [24] or 2.7 of [18] which is valid for a particular construction of $z(s, t)$ and for any $\varepsilon' > 0$.) Using this z we define a continuous map $U: I \rightarrow \mathcal{U}(A)$ such that

$$\begin{aligned} \| w(s, t) - U(s)\alpha_t(U(s)^*) \| &\leq 6\pi|t|/T + \varepsilon, \\ \| [U(s), x] \| &\leq \varepsilon, \quad x \in \mathcal{F}, \end{aligned}$$

where we have assumed that $16\pi/3 + \varepsilon < 6\pi$.

We recall from [14] how $U_T(s) = U(s)$ is defined. First we define a unitary $\tilde{U}_T$ in $C(\mathbf{R}/\mathbf{Z}) \otimes A$ by

$$\tilde{U}_T(t) = w(s, Tt)\alpha_{T(t-1)}(z_T(s, Tt)^*),$$

where $\mathbf{R}/\mathbf{Z}$ is identified with $[0, 1]/\{0, 1\}$ and $z_T(s, t) = z(s, t)$ where z is given above. Secondly we embed $C(\mathbf{R}/\mathbf{Z}) \otimes A$ into A *approximately* by using the Rohlin property. (If τ is the flow on $C(\mathbf{R}/\mathbf{Z})$ induced by translations on $\mathbf{R}/\mathbf{Z}$, then $\| 1 \otimes w(s, t) - \tilde{U}(\tau_{t/T} \otimes \alpha_t)(\tilde{U}^*) \| \leq 6\pi|t|/T$. We find an approximate homomorphism ϕ of $C(\mathbf{R}/\mathbf{Z}) \otimes A$ into A such that $\phi \circ (\tau_{t/T} \otimes \alpha_t) \approx \alpha_t \phi$ and $\phi(1 \otimes x) \approx x$, $x \in A$.) Since z_T is defined in terms of $w(s, t)$, $t \in [0, T]$, and other elements which almost commute with them we may assume that $S \in [0, T] \rightarrow z_S$ is continuous. Hence we may assume that $S \in [0, T] \rightarrow \tilde{U}_S \in \mathcal{U}(C(\mathbf{R}/\mathbf{Z}) \otimes A)$ is continuous. Note also that the commutators of $\tilde{U}_S$ and $\tilde{U}_T$ with a general element satisfy the same uniform estimate. Since $\tilde{U}_0 = 1$, we

may thus assume that there is a continuous path, U_t , $t \in [0, 1]$, in the space of continuous maps of I into $\mathcal{U}(A)$ such that $U_0(s) = 1$, $U_1(s) = U(s)$, and $\|[U_t(s), x]\| < \varepsilon$, $x \in \mathcal{F}$. Finally define $v_1(s) = U(s)^*v(s)$ for $s \in I$. Then v_1 satisfies

$$\begin{aligned} \|\operatorname{Ad} v_1(s)(x) - \alpha_{t_0}(x)\| &\leq 2\varepsilon, \quad x \in \alpha_{-t_0}(\mathcal{F}), \\ \max_{0 \leq t \leq 1} \|\alpha_t(v_1(s)) - v_1(s)\| &\leq 6\pi/T + \varepsilon. \end{aligned}$$

Thus we have established the following. Choose $v: [0, \infty) \rightarrow \mathcal{U}(A)$ such that $\alpha_{t_0} = \lim_{s \rightarrow \infty} \operatorname{Ad} v(s)$. Then for any finite subset $\mathcal{F}$ of A and $\varepsilon > 0$ there exists an $a \in [0, \infty)$ such that for any compact interval I of $[a, \infty)$ we can find a continuous $v_I: I \times [0, 1] \rightarrow \mathcal{U}(A)$ such that

$$\begin{aligned} \|\operatorname{Ad} v_I(s, t)(x) - \alpha_{t_0}(x)\| &< \varepsilon, \quad x \in \mathcal{F}, \quad (s, t) \in I \times [0, 1], \\ v_I(s, 0) &= v(s), \quad s \in I, \end{aligned}$$

and

$$\max_{0 \leq t \leq 1} \|\alpha_t(v_I(s, 1)) - v_I(s, 1)\| < \varepsilon, \quad s \in I.$$

Let $(\mathcal{F}_n)$ be an increasing sequence of finite subsets of A such that $\bigcup_n \mathcal{F}_n$ is dense in A and (ε_k) a decreasing sequence in $(0, \infty)$ such that $\lim_k \varepsilon_k = 0$. We choose an increasing sequence (a_k) in $(0, \infty)$ such that if I is a compact interval of $[a_k, \infty)$ then there is a continuous $v: I \times [0, 1] \rightarrow \mathcal{U}(A)$ such that the above conditions are satisfied for $\mathcal{F} = \mathcal{F}_k$ and $\varepsilon = \varepsilon_k$.

Let $a_0 = 0$ and $I_k = [a_k, a_{k+1}]$ for $k = 0, 1, 2, \dots$. For each $k = 1, 2, \dots$ we choose $v_k: I_k \times [0, 1] \rightarrow \mathcal{U}(A)$ for $\mathcal{F}_k$ and ε_k as above and define $v_0: I_0 \times [0, 1] \rightarrow \mathcal{U}(A)$ by $v_0(s, t) = v(s)$, $s \in I_0$. If $v_{k-1}(a_k, 1) = v_k(a_k, 1)$ for $k = 1, 2, \dots$ the proof would be completed by defining a continuous function $v: [0, \infty) \rightarrow \mathcal{U}(A)$ such that $v|_{I_k} = v_k(\cdot, 1)$. But note that $v_k(a_k, 1)v_{k-1}(a_k, 1)^*$ is connected to 1 by a continuous path $w_k(s) = v_k(a_k, s)v_{k-1}(a_k, s)^*$, $s \in [0, 1]$, such that $w_k(0) = 1$, $w_k(1) = v_k(a_k, 1)v_{k-1}(a_k, 1)^*$ and

$$\|[w_k(s), x]\| < 2\varepsilon_{k-1}, \quad x \in \alpha_{t_0}(\mathcal{F}_{k-1}),$$

for $s \in [0, 1]$. In addition to this property, by modifying the path $w_k(s)$, $s \in [0, 1]$, we may ensure that $\max_{0 \leq t \leq 1} \|\alpha_t(w_k(s)) - w_k(s)\|$ is small. Then the path $s \mapsto w_k(s)v_{k-1}(a_k, 1)$ connects $v_{k-1}(a_k, 1)$ with $v_k(a_k, 1)$ and has the desired property with respect to α . Thus it suffices to prove the following lemma assuming that $\alpha_{t_0}(\mathcal{F}_k)$ is sufficiently rapidly increasing and ε_k is sufficiently rapidly decreasing with k . $\square$

Lemma 2.3. *For any finite subset $\mathcal{F}$ of A and $\varepsilon > 0$ there exist a finite subset $\mathcal{G}$ of A and $\delta > 0$ satisfying the following condition. If a continuous $v: [0, 1] \rightarrow \mathcal{U}(A)$ satisfies that*

$$\begin{aligned} v(0) &= 1, \\ \|[v(s), x]\| &< \delta, \quad s \in [0, 1], \quad x \in \mathcal{G}, \\ \|\alpha_t(v(1)) - v(1)\| &< \delta, \quad t \in [0, 1], \end{aligned}$$

then there exists a continuous $u : [0, 1] \rightarrow \mathcal{U}(A)$ such that

$$\begin{aligned} u(0) &= 1, & u(1) &= v(1), \\ \|[u(s), x]\| &< \varepsilon, & s \in [0, 1], x \in \mathcal{F}, \\ \|\alpha_t(u(s)) - u(s)\| &< \varepsilon, & t \in [0, 1], s \in [0, 1]. \end{aligned}$$

Proof. Suppose that v satisfies $v(0) = 1$ and

$$\begin{aligned} \|[v(s), \alpha_{-t}(x)]\| &< \delta, & x \in \mathcal{G}, t \in [0, 1], \\ \max_{0 \leq t \leq 1} \|\alpha_t(v(1)) - v(1)\| &< \delta. \end{aligned}$$

We define $w(s, t) = v(s)\alpha_t(v(s)^*)$. Then $t \mapsto w(s, t)$ is an α -cocycle for each $s \in [0, 1]$ and satisfies $w(0, t) = 1$ and

$$\begin{aligned} \max_{0 \leq t \leq 1} \|[w(s, t), x]\| &< 2\delta, & x \in \mathcal{G}, \\ \max_{0 \leq t \leq 1} \|w(1, t) - 1\| &< \delta. \end{aligned}$$

From the latter condition there are $b, h \in A_{sa}$ such that $b \approx 0$, $h \approx 0$, and $w(1, t) = e^{ib} z_t^{(h)} \alpha_t(e^{-ib})$, where $z^{(h)}$ is a differentiable α -cocycle such that $dz_t^{(h)}/dt|_{t=0} = ih$ (see [15]). Connecting the α -cocycle $t \mapsto w(1, t)$ with the trivial α -cocycle 1 by the path of α -cocycles $s \mapsto (t \mapsto e^{isb} z_t^{(sh)} \alpha_t(e^{-isb}))$ and modifying it around $0 \in \mathbf{T} = \mathbf{R}/\mathbf{Z}$ we obtain an α -cocycle W in $C(\mathbf{T}) \otimes A$ with respect to the flow $\text{id} \otimes \alpha$ such that $W(0, t) = 1$ and $W(s, t) \approx w(s, t)$. Hence the proof is completed by the following lemma. Because we can then find a unitary $Z \in C(\mathbf{T}) \otimes A$ with appropriate commutativity such that $Z(0) = 1$ and $W(\cdot, t) \approx Z\alpha_t(Z)^*$ and replace v by the path $s \mapsto Z(s)^*v(s)$ which is almost α -invariant and moves from $v(0) = 1$ to $v(1)$. $\square$

Lemma 2.4. For any finite subset $\mathcal{F}$ of A and $\varepsilon > 0$ there exists a finite subset $\mathcal{G}$ of A and $\delta > 0$ satisfying the following condition. Define a flow $\bar{\alpha} = \text{id} \otimes \alpha$ on $C(\mathbf{T}) \otimes A$ and let $t \mapsto W_t$ be an $\bar{\alpha}$ -cocycle such that $W_t(0) = 1$ at $0 \in \mathbf{T} = [0, 1]/\{0, 1\}$ and

$$\max_{0 \leq t \leq 1} \|[W_t, 1 \otimes x]\| < \delta, \quad x \in \mathcal{G}.$$

Then there exists a unitary Z in $C(\mathbf{T}) \otimes A$ such that $Z(0) = 1$ and

$$\begin{aligned} \|[Z, 1 \otimes x]\| &< \varepsilon, & x \in \mathcal{F}, \\ \max_{0 \leq t \leq 1} \|W_t - Z\bar{\alpha}_t(Z^*)\| &< \varepsilon. \end{aligned}$$

Proof. We just sketch the proof (see [14] or the first part of the proof of Lemma 2.2 for details).

To ensure the last condition we choose $T \in \mathbf{N}$ such that $T^{-1} < \varepsilon/6\pi$. Then we impose the condition that $\max_{0 \leq t \leq 1} \|[W_t, 1 \otimes x]\| < \delta/T$ for $x \in \bigcup_{-T \leq t \leq 0} \alpha_t(\mathcal{F})$. Note that the union can be replaced by a finite subset because it is compact. Since $\max_{0 \leq t \leq T} \|[W_t, 1 \otimes x]\| < \delta$ we can find a continuous path U_t , $t \in [0, T]$, in $\mathcal{U}(C(\mathbf{T}) \otimes A)$ such that $U_0 = 1$, $U_T = W_T$, $U_t(0) = 1$,

$\|U_{t_1} - U_{t_2}\| \leq 6\pi|t_1 - t_2|/T$, and $\|[U_t, x]\| < 9\delta$. By using W and U and the Rohlin property for α we then define a unitary $Z \in C(\mathbf{T}) \otimes A$ such that $Z(0) = 1$, $\max_{0 \leq t \leq 1} \|W_t - Z\alpha_t(Z^*)\| < \varepsilon$, and $\|[Z, 1 \otimes x]\| < 10\delta$, $x \in \mathcal{F}$. $\square$

This completes the proof of Lemma 2.2 and the strengthened statement of condition 4 in Theorem 2.1.

We now give several technical results which will be used in the next section. We assume that α is a Rohlin flow on A as before.

Lemma 2.5. *For any finite subset $\mathcal{F}$ of A and $\varepsilon > 0$ there exists a finite subset $\mathcal{G}$ of A and $\delta > 0$ satisfying the following condition. If $u(s)$, $s \in [0, 1]$, is a continuous path in $\mathcal{U}(A)$ such that*

$$\begin{aligned} \|[u(s), x]\| &< \delta, \quad x \in \mathcal{G}, \\ \max_{0 \leq t \leq 1} \|\alpha_t(u(s)) - u(s)\| &< \delta, \quad s \in [0, 1], \end{aligned}$$

then there exists a rectifiable path $v(s)$, $s \in [0, 1]$, such that $v(0) = u(0)$, $v(1) = u(1)$,

$$\begin{aligned} \|[v(s), x]\| &< \varepsilon, \quad x \in \mathcal{F}, \\ \max_{0 \leq t \leq 1} \|\alpha_t(v(s)) - v(s)\| &< \varepsilon, \quad s \in [0, 1], \end{aligned}$$

and the length of the path v is less than $17\pi/3$. If $\mathcal{F} = \emptyset$ then $\mathcal{G} = \emptyset$ is possible.

Proof. This is shown in [24], without the conditions with respect to α .

To define v we use certain elements of A which almost commute with $u(s)$, $s \in [0, 1]$. They form a compact subset of $\mathcal{O}_\infty$ which is then embedded centrally in A in [24], using a result due to Kirchberg and Phillips. In the present case, to meet the condition of almost α -invariance, those elements embedded in A should be almost invariant under α . To establish this we use the fact that $(A' \cap A_\alpha^\omega)^\alpha$ is purely infinite and simple [18].

Explicitly we assume that the above elements of $\mathcal{O}_\infty = C^*(s_1, s_2, \dots)$, before the embedding into A , is in the linear subspace spanned by a finite number of monomials in $s_1, \dots, s_k$ and their adjoints for some k . We find a finite sequence $(T_1, \dots, T_k)$ of isometries in $(A' \cap A_\alpha^\omega)^\alpha$ such that $\sum_{i=1}^k T_i T_i^* \leq 1$. Each T_i is represented by a central sequence $(t_i(m))$ of isometries in A such that $\sum_{i=1}^k t_i(m) t_i(m)^* \leq 1$ and $\max_{t \in [0, 1]} \|\alpha_t(t_i(m)) - t_i(m)\| \rightarrow 0$ as $m \rightarrow \infty$. We then express those elements in terms of $t_1(m), \dots, t_k(m)$ in place of $s_1, \dots, s_k$ respectively for a sufficiently large m . Thus we establish the required condition involving α . $\square$

We denote by δ_α the generator of α . It is a closed derivation from a dense $*$ -subalgebra $D(\delta_\alpha)$ into A . (See [2,4,27] for the theory of generators and derivations.)

Lemma 2.6. *Let $u(s)$, $s \in [0, \infty)$, be a continuous path in $\mathcal{U}(A)$ such that*

$$\max_{0 \leq t \leq 1} \|\alpha_t(u(s)) - u(s)\|$$

converges to zero as $s \rightarrow \infty$. It follows that there is a continuous path $v(s)$, $s \in [0, \infty)$, of unitaries such that $v(s) \in D(\delta_\alpha)$ and $\delta_\alpha(v(s))$ and $u(s) - v(s)$ converge to zero as $s \rightarrow \infty$.

Proof. Let f be a non-negative C^∞ -function on $\mathbf{R}$ of compact support such that the integral is 1. We set

$$z(s) = b(s) \int f(b(s)t) \alpha_t(u(s)) dt,$$

where $b: [0, \infty) \rightarrow (0, \infty)$ is a continuous decreasing function such that $\lim_s b(s) = 0$, $\|z(s) - u(s)\| < 1$, and $\|z(s) - u(s)\| \rightarrow 0$. Then it follows that $z(s) \in D(\delta_\alpha)$ and

$$\|\delta_\alpha(z(s))\| \leq b(s) \int |f'(t)| dt,$$

which converges to zero as $s \rightarrow \infty$. We set $v(s) = z(s)|z(s)|^{-1}$, which satisfies the required conditions. $\square$

Lemma 2.7. For a finite subset $\mathcal{F}$ of A and $\varepsilon > 0$ there exists a finite subset $\mathcal{G}$ of A and $\delta > 0$ satisfying the following condition. Let u be a unitary in $C[0, 1] \otimes A$ such that $u(0) = 1$, $u(t) \in D(\delta_\alpha)$, $\|\delta_\alpha(u(t))\| < \delta$, and $\|[u(t), x]\| < \delta$, $x \in \mathcal{G}$. Then there exist $h_i \in D(\delta_\alpha) \cap A_{sa}$ for $i = 1, 2, \dots, 10$ such that

$$u(1) = e^{ih_1} e^{ih_2} \dots e^{ih_{10}},$$

$$\|h_i\| < \pi, \quad \|\delta_\alpha(h_i)\| < \varepsilon, \quad \|[h_i, x]\| < \varepsilon, \quad x \in \mathcal{F}.$$

If $\mathcal{F} = \emptyset$ then $\mathcal{G} = \emptyset$ is possible.

Proof. We may assume, by Lemma 2.5, that the length of the path u is smaller than $17\pi/3 < 18$. Then we choose $0 < s_1 < s_2 < \dots < s_9 < 1$ such that $\|u(s_i) - u(s_{i-1})\| < 9/5 < 2$ for $i = 1, 2, \dots, 10$, where $s_0 = 0$ and $s_{10} = 1$. Note that

$$u(1) = u(s_0)^* u(s_1) \cdot u(s_1)^* u(s_2) \dots u(s_9)^* u(s_{10}).$$

Since $\|u(s_{i-1})^* u(s_i) - 1\| < 9/5$ the spectrum of $u(s_{i-1})^* u(s_i)$ is contained in $S = \{e^{i\theta}: |\theta| < \theta_0\}$, where $\theta_0 = \pi - 2\cos^{-1}(9/10) < \pi$. Let Arg denote the function $e^{i\theta} \mapsto \theta$ from S onto the interval $(-\theta_0, \theta_0)$ and set $h_i = \text{Arg}(u(s_{i-1})^* u(s_i))$. Then it follows that $\|h_i\| < \pi$ and $u(1) = e^{ih_1} e^{ih_2} \dots e^{ih_{10}}$. We shall show that these h_i satisfy the other conditions for a sufficiently small $\delta > 0$.

In general if v is a unitary with $\text{Spec}(v) \subset S$ then $h = \text{Arg}(v)$ can be obtained as

$$h = \frac{1}{2\pi i} \oint_C (\log z)(z - v)^{-1} dz,$$

where $\log z$ is the logarithmic function on $\mathbf{C} \setminus (-\infty, 0]$ with values in $\{z: |\Im z| < \pi\}$ and C is a simple rectifiable path surrounding S in the domain of $\log$. We fix C and let r be the distance between C and S . Since

$$\delta_\alpha(h) = \frac{1}{2\pi i} \oint_C \log z (z - v)^{-1} \delta_\alpha(v) (z - v)^{-1} dz,$$

we have the estimate

$$\|\delta_\alpha(h)\| \leq (2\pi)^{-1} M|C|r^{-2} \|\delta_\alpha(v)\|,$$

where M is the maximum of $|\log z|$, $z \in C$, and $|C|$ is the length of C . Similarly we have the estimate $\|[h, x]\| \leq (2\pi)^{-1} M|C|r^{-2} \|[v, x]\|$ for any $x \in A$. (See [4,27] for details.) This establishes the desired conclusion. $\square$

3. Proof of Proposition 1.2

We recall that $\mathcal{E}_n = C^*(s_1, \dots, s_n)$, where $s_1, \dots, s_n$ are isometries such that $e_n^0 = 1 - \sum_{k=1}^n s_k s_k^*$ is a non-zero projection, and that $\mathcal{J}_n$ is the ideal of $\mathcal{E}_n$ generated by e_n^0 . Let $\mathcal{S} = \{1, 2, \dots, n\}^*$ denote the set of all finite sequences including an empty sequence, denoted by $\emptyset$. For $I = (i_1, i_2, \dots, i_m) \in \mathcal{S}$ with $m = |I|$, we set $s_I = s_{i_1} s_{i_2} \cdots s_{i_m}$, where $|I|$ is the length of I ; if $|I| = 0$ or $I = \emptyset$, then $s_I = 1$. It then follows that $\{s_I e_n^0 s_J^* : I, J \in \mathcal{S}\}$ forms a family of matrix units and spans $\mathcal{J}_n$. Thus, in particular, $\mathcal{J}_n$ is isomorphic to the C^* -algebra $\mathcal{K}$ of compact operators on an infinite-dimensional separable Hilbert space. Hence there is a unique, up to unitary equivalence, irreducible representation π_0 of $\mathcal{E}_n$ such that $\pi_0|_{\mathcal{J}_n}$ is non-zero or $\pi_0(e_n^0)$ is a one-dimensional projection. We call this representation the Fock representation and denote by $\mathcal{H}_0$ the representation Hilbert space of π_0 .

We recall that the flow α is defined as $\alpha_t(s_k) = e^{ip_k t} s_k$ for $k = 1, 2, \dots, n$. For $I = (i_1, \dots, i_m) \in \mathcal{S}$ let $p(I) = \sum_{k=1}^m p_{i_k} \in \mathbf{R}$. We set $H_0 = \sum_{I \in \mathcal{S}} p(I) \pi_0(s_I e_n^0 s_I^*)$, which is a well-defined self-adjoint operator on $\mathcal{H}_0$. Then it follows that $\text{Ad } e^{itH_0} \pi_0(x) = \pi_0 \alpha_t(x)$, $x \in \mathcal{E}_n$, and, by the assumption on $p_1, \dots, p_n$, that the spectrum of H_0 is the whole real line $\mathbf{R}$. Note that $\mathcal{E}_n/\mathcal{J}_n \cong \mathcal{O}_n = C^*(\dot{s}_1, \dots, \dot{s}_n)$, where $\dot{s}_k = s_k + \mathcal{J}_n$. (Subsequently we denote $\dot{s}_k$ by s_k .) Note that α induces a flow on $\mathcal{O}_n$, also denoted by α . We denote by Q the quotient map of $\mathcal{E}_n$ onto $\mathcal{O}_n$.

Lemma 3.1. *For any $h \in (\mathcal{O}_n)_{sa} \cap D(\delta_\alpha)$ and $\varepsilon > 0$ there exists a $b \in (\mathcal{E}_n)_{sa} \cap D(\delta_\alpha)$ such that $Q(b) = h$, $\|b\| < \|h\| + \varepsilon$, and $\|\delta_\alpha(b)\| < \|\delta_\alpha(h)\| + \varepsilon$.*

Proof. Since $Q\alpha_t = \alpha_t Q$ on $\mathcal{E}_n$, it follows that $(1 + \delta_\alpha)^{-1} Q = Q(1 + \delta_\alpha)^{-1}$, which implies that $Q(D(\delta_\alpha)) = D(\delta_\alpha)$. (We will use the same symbol δ_α for the generator of $\alpha|_{\mathcal{E}_n}$ and of $\alpha|_{\mathcal{O}_n}$.)

Thus, for any h as above, there is a $b \in (\mathcal{E}_n)_{sa} \cap D(\delta_\alpha)$ such that $Q(b) = h$. By C^∞ -functional calculus we may suppose that $\|b\| < \|h\| + \varepsilon$.

Since H_0 is diagonal there exists an approximate identity (p_k) for $\mathcal{J}_n$ consisting of projections in $\mathcal{J}_n \cap D(\delta_\alpha)$ such that $\delta_\alpha(p_k) = 0$. Since $Q(p_k) = 0$, we may replace b by $(1 - p_k)b(1 - p_k)$. We choose a p_k such that $\|(1 - p_k)\delta_\alpha(b)(1 - p_k)\| < \|\delta_\alpha(h)\| + \varepsilon$ (since $\|Q(\delta_\alpha(b))\| = \|\delta_\alpha(h)\| = \lim_k \|(1 - p_k)\delta_\alpha(b)(1 - p_k)\|$). Then it follows that $b_1 = (1 - p_k)b(1 - p_k)$ belongs to $(\mathcal{E}_n)_{sa} \cap D(\delta_\alpha)$ and satisfies that $\|b_1\| \leq \|b\| \leq \|h\| + \varepsilon$ and

$$\|\delta_\alpha(b_1)\| = \|(1 - p_k)\delta_\alpha(b)(1 - p_k)\| < \|\delta_\alpha(h)\| + \varepsilon.$$

Thus b_1 satisfies the required conditions. $\square$

Lemma 3.2. *For any finite subset $\mathcal{F}$ of $\mathcal{E}_n$ and $\varepsilon > 0$ there exists a finite subset $\mathcal{G}$ of $\mathcal{O}_n$ and $\delta > 0$ satisfying the following condition. If $h \in (\mathcal{O}_n)_{sa} \cap D(\delta_\alpha)$ such that $\|h\| < \pi$, $\|\delta_\alpha(h)\| < \delta$, and $\|[h, x]\| < \delta$, $x \in \mathcal{G}$, then there is a $b \in (\mathcal{E}_n)_{sa} \cap D(\delta_\alpha)$ such that*

$$\begin{aligned}
 Q(b) &= h, \\
 \|b\| &< \pi + \varepsilon, \quad \|\delta_\alpha(b)\| < \varepsilon, \quad \|[b, x]\| < \varepsilon, \quad x \in \mathcal{F}, \\
 be_n^0 &= 0.
 \end{aligned}$$

Proof. Note that in the above statement we may allow $\mathcal{F}$ and $\mathcal{G}$ to be compact subsets instead of finite subsets.

Note that $\mathcal{E}_n$ is nuclear as well as $\mathcal{O}_n$ and $\mathcal{J}_n$. Let $T > 0$ be so large that $\pi/T < \varepsilon/2$. Since $\mathcal{F}_1 = \bigcup_{-T \leq t \leq T} \alpha_t(\mathcal{F})$ is compact, there is a $w = (w_1, \dots, w_K) \in M_{1K}(\mathcal{E}_n)$ for some $K \in \mathbb{N}$ such that $ww^* = 1$ and

$$\|[wxw^*, a]\| \leq (\varepsilon/\pi)\|x\|, \quad x \in \mathcal{E}_n,$$

for any $a \in \mathcal{F}_1$, where $wxw^* = \sum_{i=1}^K w_i x w_i^*$. Note also that $x \mapsto wxw^*$ is a kind of unital averaging map of A into A (this is due to Haagerup [8]; see also [22]). We set $\mathcal{G} = \{Q(\alpha_t(w_i)): i = 1, 2, \dots, K, |t| \leq T\}$, which is a compact subset of $\mathcal{O}_n$. Let $\delta \in (0, \varepsilon/2)$ and let $h \in (\mathcal{O}_n)_{sa} \cap D(\delta_\alpha)$ be such that $\|h\| < \pi$, $\|\delta_\alpha(h)\| < \delta$, and $\|[h, x]\| < \delta$, $x \in \mathcal{G}$. Assume that $\delta > 0$ is so small that

$$\|Q(\alpha_t(w))hQ(\alpha_t(w)) - h\| < \varepsilon, \quad t \in [-T, T].$$

Let $b \in (\mathcal{E}_n)_{sa} \cap D(\delta_\alpha)$ be such that $Q(b) = h$, $\|b\| < \pi$, and $\|\delta_\alpha(b)\| < \delta$.

Define

$$b_1 = \frac{1}{2T} \int_{-T}^T \alpha_t(w) b \alpha_t(w)^* dt.$$

Then it follows that $\|b_1\| < \pi$ and $\|Q(b_1) - h\| < \varepsilon$. It also follows that $b_1 \in (\mathcal{E}_n)_{sa} \cap D(\delta_\alpha)$ and

$$\delta_\alpha(b_1) = (2T)^{-1} (\alpha_T(w) b \alpha_T(w)^* - \alpha_{-T}(w) b \alpha_{-T}(w)^*) + \frac{1}{2T} \int_{-T}^T \alpha_t(w) \delta_\alpha(b) \alpha_t(w)^* dt,$$

which implies that $\|\delta_\alpha(b_1)\| < \pi/T + \delta < \varepsilon$.

Let $a \in \mathcal{F}$. Since $\|[\alpha_t(w) b \alpha_t(w)^*, a]\| = \|[w \alpha_{-t}(b) w^*, \alpha_{-t}(a)]\| \leq (\varepsilon/\pi)\|b\| < \varepsilon$ for $t \in [-T, T]$ we deduce that $\|[b_1, a]\| < \varepsilon$, $a \in \mathcal{F}$.

To ensure the condition $b_1 e_n^0 = 0$, let (p_k) be an approximate identity for $\mathcal{J}_n$ in $\mathcal{J}_n \cap D(\delta_\alpha)$ such that $p_k \geq e_n^0$, $\delta_\alpha(p_k) = 0$ and $\|[p_k, x]\| \rightarrow 0$ for all $x \in \mathcal{E}_n$. We replace b_1 by $(1 - p_k)b_1(1 - p_k)$ for a sufficiently large k .

In this way we construct a $b \in (\mathcal{E}_n)_{sa} \cap D(\delta_\alpha)$ which satisfies all the required conditions except for $Q(b) = h$. Instead $\|Q(b) - h\| < \varepsilon$. Since, however, $\|\delta_\alpha(Q(b) - h)\| < \varepsilon + \delta$, by the previous lemma, there is a $c \in (\mathcal{E}_n)_{sa} \cap D(\delta_\alpha)$ such that $Q(c) = h - Q(b)$, $\|c\| < \varepsilon$, and $\|\delta_\alpha(c)\| < \varepsilon + \delta$. We may also suppose that $ce_n^0 = 0$. Thus we can take $b + c$ for b and this satisfies the required conditions if we start with a smaller ε . $\square$

Fix $t_0 \in \mathbf{R}$. We choose, by Lemmas 2.2 and 2.6, a continuous $u : [0, \infty) \rightarrow \mathcal{U}(\mathcal{O}_n) \cap D(\delta_\alpha)$ such that $\alpha_{t_0} = \lim_{s \rightarrow \infty} \text{Ad } u(s)$ and $\lim_{s \rightarrow \infty} \delta_\alpha(u(s)) = 0$. Since the unitary group of $\mathcal{O}_n$ is connected, we may suppose that $u(0) = 1$.

Let $(\mathcal{F}_k)$ be an increasing sequence of finite subsets of $\mathcal{E}_n$ such that the union $\bigcup_k \mathcal{F}_k$ is dense in $\mathcal{E}_n$ and (ε_k) a decreasing sequence of positive numbers such that $\sum_k \varepsilon_k \equiv \varepsilon$ is very small. We choose, by Lemma 3.2, $\mathcal{G}_k = \mathcal{G}$ and $\delta_k = \delta$ for $\mathcal{F} = \mathcal{F}_k$ and $\varepsilon = \varepsilon_k$. We may also suppose that $\mathcal{G}_k$ is increasing and δ_k is decreasing to zero.

For the above continuous map $u : [0, \infty) \rightarrow \mathcal{U}(\mathcal{O}_\infty) \cap D(\delta_\alpha)$ we choose an increasing sequence (s_k) in $[0, \infty)$ with $s_0 = 0$ such that $\|\delta_\alpha(u(s_k)^* u(s_{k+1}))\|$ is sufficiently small for $k \geq 0$ and $u(s_k)^* u(s_{k+1})$ is sufficiently central for $k \geq 1$. Specifically, by Lemma 2.7, we may assume that $u(s_k)^* u(s_{k+1})$ has the following factorization:

$$u(s_k)^* u(s_{k+1}) = e^{ih_{k,1}} e^{ih_{k,2}} \dots e^{ih_{k,10}},$$

$$\|h_{ki}\| < \pi, \quad \|\delta_\alpha(h_{ki})\| < \delta_k, \quad \|[h_{ki}, x]\| < \delta_k, \quad x \in \mathcal{G}_k,$$

where $\mathcal{G}_0 = \emptyset$. Then by Lemma 3.2 we may choose $b_{ki} \in (\mathcal{E}_n)_{sa} \cap D(\delta_\alpha)$ such that

$$Q(b_{ki}) = h_{ki},$$

$$\|b_{ki}\| < \pi + \varepsilon_k, \quad \|\delta_\alpha(b_{ki})\| < \varepsilon_k, \quad \|[b_{ki}, x]\| < \varepsilon_k, \quad x \in \mathcal{F}_k,$$

$$\pi_0(b_{ki})\Omega_0 = 0,$$

where $\mathcal{F}_0 = \emptyset$ and Ω_0 is a unit vector in $\mathcal{H}_0$ such that $\pi_0(e_n^0)\Omega_0 = \Omega_0$.

Set $w_0 = 1$. Then one can lift $u(s_k)$ ($= u(s_{k-1}) \cdot u(s_{k-1})^* u(s_k)$) for each $k \geq 1$ to a unitary $w_k \in \mathcal{E}_n \cap D(\delta_\alpha)$ by

$$w_k = w_{k-1} e^{ib_{k,1}} e^{ib_{k,2}} \dots e^{ib_{k,10}}.$$

It follows that $\text{Ad } w_k$ converges on $\mathcal{E}_n$ as $k \rightarrow \infty$ and $Q \circ (\lim \text{Ad } w_k) = \alpha_{t_0} \circ Q$. If the $\mathcal{G}_k$ and δ_k are chosen for $\mathcal{F} = \mathcal{F}_k \cup \text{Ad } w_{k-1}^*(\mathcal{F}_k)$ and $\varepsilon = \varepsilon_k$ then this ensures that $\text{Ad } w_k^*$ also converges. In this way we have $\beta = \lim \text{Ad } w_k$, as an automorphism of $\mathcal{E}_n$, which satisfies that

$$\beta \circ Q = Q \circ \alpha_{t_0}.$$

Since $\|\alpha_t(w_k) - w_k\|$ is dominated by

$$\|\alpha_t(w_{k-1}) - w_{k-1}\| + \sum_{j=1}^{10} \|\alpha_t(e^{ib_{kj}}) - e^{ib_{kj}}\|$$

and since $\|\alpha_t(e^{ib_{kj}}) - e^{ib_{kj}}\| \leq \|\delta_\alpha(b_{kj})\| |t| \leq \varepsilon_k |t|$ we deduce that

$$\|\alpha_t(w_k) - w_k\| \leq \|\alpha_t(w_{k-1}) - w_{k-1}\| + 10\varepsilon_k |t|.$$

Thus it follows that if $|t| \leq 1$ then

$$\|\alpha_t(w_k) - w_k\| \leq 10 \sum_{i=1}^k \varepsilon_i < 10\varepsilon.$$

In the Fock representation π_0 , since $\pi_0(w_k)\Omega_0 = \Omega_0$, it follows that

$$\pi_0(w_k)\pi_0(x)\Omega_0 = \pi_0(\text{Ad } w_k(x))\Omega_0$$

converges strongly to $\pi_0(\beta(x))\Omega_0$ for any $x \in \mathcal{E}_n$. Hence $\pi_0(w_k)$ converges strongly to a unitary, which we denote by W . Note that $\text{Ad } W\pi_0(x) = \pi_0\beta(x)$, $x \in \mathcal{E}_n$, and $W\Omega_0 = \Omega_0$.

As we remarked before, the unitary flow $U_t \equiv e^{itH_0}$ implements α in π_0 , where $H_0 = \sum_{I \in \mathcal{S}} p(I)\pi_0(s_I e_n^0 s_I^*)$. Since $\|U_t\pi_0(w_k)U_t^* - \pi_0(w_k)\| < 10\varepsilon$ for $|t| \leq 1$ we obtain

$$\|U_t W U_t^* - W\| \leq 10\varepsilon, \quad t \in [-1, 1].$$

We also have

$$\pi_0\beta^{-1}\alpha_t\beta = \text{Ad}(W^*U_t W U_t^*)\pi_0\alpha_t.$$

On the other hand, $\beta^{-1}\alpha_t\beta$ is obtained as the limit of $\text{Ad}(w_k^*\alpha_t(w_k))\alpha_t$, which implies that

$$\|\alpha_t - \beta^{-1}\alpha_t\beta\| \leq 20\varepsilon, \quad t \in [-1, 1].$$

Hence there exists an α -cocycle u in $\mathcal{E}_n$ such that $\text{Ad } u_t\alpha_t = \beta^{-1}\alpha_t\beta$ and $\max_{t \in [-1, 1]} \|u_t - 1\|$ is at most of order of 400ε (see [4, p. 296]). Combining the observation in the previous paragraph and noting that π_0 is irreducible, this implies

$$\pi_0(u_t) = c(t)W^*U_t W U_t^*$$

for some constant $c(t) \in \mathbf{T}$. Then it follows by simple computation that $c(t) = e^{ipt}$ for some $p \in \mathbf{R}$. Thus we know that $t \mapsto U_t W U_t^*$ is continuous in norm and that $W^*U_t W U_t^* \in \pi_0(\mathcal{E}_n)$. Since $Q \text{Ad } u_t\alpha_t = \text{Ad } Q(u_t)\alpha_t Q = \alpha_t Q$ on $\mathcal{E}_n$, we also have that $Q(u_t) \in \mathbf{C}1 \subset \mathcal{O}_n$, which implies that

$$W^*U_t W U_t^* \in \pi_0(\mathcal{J}_n + \mathbf{C}1).$$

Since any automorphism of $\mathcal{E}_n$ is weakly inner in π_0 there is a unitary V on $\mathcal{H}_0$ such that $\text{Ad } V\pi_0 = \pi_0\beta^{-1}\alpha_{t_0}$. Then as the vector state of $\mathcal{E}_n$ defined through Ω_0 is left invariant under by $\beta^{-1}\alpha_{t_0}$ we may define V by $V\pi_0(x)\Omega_0 = \pi_0(\beta^{-1}\alpha_{t_0}(x))\Omega_0$, $x \in \mathcal{E}_n$. Since $Q\beta^{-1}\alpha_{t_0} = Q$ we have $[V, \pi_0(x)] \in \mathcal{K}(\mathcal{H}_0)$ for $x \in \mathcal{E}_n$. But the multiplier algebra $M(\mathcal{J}_n)$ of $\mathcal{J}_n$ identifies with $B(\mathcal{H}_0)$ through π_0 so we may regard V as an element of $M(\mathcal{J}_n)$. Similarly $\mathcal{E}_n \subset M(\mathcal{J}_n)$. Then $\beta^{-1}\alpha_t\beta = \text{Ad}(V U_t V^* U_t^*)\alpha_t$. But $\beta^{-1}\alpha_t\beta = \text{Ad}(W^*U_t W U_t^*)\alpha_t$, from the above, and $V U_t V^* U_t^* \Omega_0 = \Omega_0 = W^*U_t W U_t^* \Omega_0$. Therefore one has

$$V U_t V^* U_t^* = W^*U_t W U_t^*, \quad t \in \mathbf{R}.$$

This implies that $t \mapsto U_t V U_t^*$ is norm-continuous and

$$\|U_t V U_t^* - V\| = \|U_t W U_t^* - W\|.$$

In particular

$$\max_{t \in [0,1]} \|U_t V U_t^* - V\| \leq 10\varepsilon.$$

Let us denote by $M(\mathcal{J}_n)_\alpha$ the C^* -subalgebra consisting of $x \in M(\mathcal{J}_n)$ such that $t \mapsto \alpha_t(x) = U_t x U_t^*$ is norm-continuous. Summing up the above we have established the following.

Lemma 3.3. *Let $(p_1, \dots, p_n)$ be a finite sequence in $\mathbf{R}$ and define a quasi-free flow α on $\mathcal{E}_n = C^*(s_1, \dots, s_n)$ by*

$$\alpha_t(s_j) = e^{ip_j t} s_j.$$

Suppose that $p_1, \dots, p_n$ generate $\mathbf{R}$ as a closed subsemigroup. (Hence the flow $\dot{\alpha}$ on the quotient $\mathcal{O}_n$ induced by α has the Rohlin property and each $\dot{\alpha}_t$ is α -invariantly asymptotically inner.)

Fix $t_0 \in \mathbf{R}$. For any $\varepsilon > 0$ there exists an automorphism β of $\mathcal{E}_n$, a sequence (w_k) in $\mathcal{U}(\mathcal{E}_n)$, and a unitary $V \in M(\mathcal{J}_n)_\alpha$ such that $t \mapsto V^* \alpha_t(V) \in \mathcal{J}_n + \mathbf{C}1$ is an α -cocycle,

$$\begin{aligned} Q(V) &\in (\mathcal{O}_n)', & Q\beta &= Q\alpha_{t_0}, \\ \beta^{-1}\alpha_{t_0} &= \text{Ad } V, & \beta &= \lim_k \text{Ad } w_k, \\ \max_{|t| \leq 1} \|\alpha_t(w_k) - w_k\| &< \varepsilon, & \max_{|t| \leq 1} \|\alpha_t(V) - V\| &< \varepsilon, \end{aligned}$$

where Q denotes the quotient map of $M(\mathcal{J}_n)$ onto $M(\mathcal{J}_n)/\mathcal{J}_n$, which maps $\mathcal{E}_n$ onto $\mathcal{O}_n$.

To show that α_{t_0} is α -invariantly approximately inner we have to approximate $\beta^{-1}\alpha_{t_0}$ by $\text{Ad } v$ where v is a unitary in $\mathcal{J}_n + 1$ which is almost α -invariant. We will use the following result whose proof we postpone to the next section.

Lemma 3.4. *For any $\varepsilon > 0$ there exists a $\delta > 0$ satisfying the following condition. Let $V \in M(\mathcal{J}_n)_\alpha$ be a unitary such that $Q(V) \in (\mathcal{O}_n)'$ and*

$$\max_{|t| \leq 1} \|\alpha_t(V) - V\| < \delta.$$

Then there exists a rectifiable path V_s , $s \in [0, 1]$, in $\mathcal{U}(M(\mathcal{J}_n)_\alpha)$ such that $V_0 = 1$, $V_1 = V$,

$$\|\lambda(Q(V_s)) - Q(V_s)\| < \varepsilon, \quad \sup_{s \in [0,1]} \max_{|t| \leq 1} \|\alpha_t(V_s) - V_s\| < \varepsilon,$$

where λ is the unital endomorphism of $M(\mathcal{J}_n)/\mathcal{J}_n$ defined by

$$\lambda(x) = \sum_{i=1}^n Q(s_i) x Q(s_i)^*.$$

The following is a key lemma for the proof of Proposition 1.2.

Lemma 3.5. *For any $\varepsilon > 0$ there exists a $\delta > 0$ satisfying the following condition. If V is a unitary in $M(\mathcal{J}_n)_\alpha$ such that $\beta = \text{Ad } V$ is an automorphism of $\mathcal{E}_n$, $Q(V) \in (\mathcal{O}_n)'$, and $\max_{|t| \leq 1} \|\alpha_t(V) - V\| < \delta$, then there is a unitary v in $\mathcal{J}_n + 1$ such that*

$$\|\beta(s_i) - vs_i v^*\| < \varepsilon, \quad i = 1, 2, \dots, n, \quad \max_{|t| \leq 1} \|\alpha_t(v) - v\| < \varepsilon.$$

Proof. We define a non-unital endomorphism λ of $M(\mathcal{J}_n)_\alpha$ by $\lambda(x) = \sum_{i=1}^n s_i x s_i^*$. Note that $Q\lambda = \lambda Q$, where the latter λ is the unital endomorphism defined in Lemma 3.4.

Let $\varepsilon > 0$. We choose $\delta > 0$ small enough that there exists a continuous path V_s , $s \in [0, 1]$, in $M(\mathcal{J}_n)_\alpha$ such that $V_0 = V$, $V_1 = 1$, $\|\lambda(V_s) - V_s + \mathcal{J}_n\| < \varepsilon$ and $\max_{|t| \leq 1} \|\alpha_t(V_s) - V_s\| < \varepsilon$.

Let $\varepsilon' > 0$. We choose an increasing sequence (μ_k) in $[0, 1]$ such that $\mu_0 = 0$, $\lim_k \mu_k = 1$, and $\|V_{\mu_k} - V_{\mu_{k+1}}\| < \varepsilon'$. We then set $V_k = V_{\mu_k}$. Note that $p_m = \sum_{|I| \leq m} s_I e_n^0 s_I^*$ is an α -invariant projection in $\mathcal{J}_n$ for each $m \in \mathbb{N}$ and that (p_m) forms an approximate identity for $\mathcal{J}_n$.

Since $V_s s_k V_s^* = V_s \lambda(V_s^*) s_k$ and $\|1 - V_s \lambda(V_s^*) + \mathcal{J}_n\| < \varepsilon$ it follows that $\|V_s s_k V_s^* - s_k + \mathcal{J}_n\| < \varepsilon$. Since $\{V_s s_k V_s^* - s_k : s \in [0, 1]\}$ is compact there is a projection $p \in \mathcal{J}_n$ such that $\|(V_s s_k V_s^* - s_k)(1 - p)\| < \varepsilon$ and $\|(1 - p)(V_s s_k V_s^* - s_k)\| < \varepsilon$ for $s \in [0, 1]$ and $k = 1, \dots, n$. We may suppose that $p = p_m$ for some m . Then we can construct an approximate unit (e_k) for $\mathcal{J}_n$ from convex combinations of $\{p_k : k \geq m\}$ such that $e_0 = 0 \leq p \leq e_1$, $\alpha_t(e_k) = e_k$, $e_{k+1} e_k = e_k$ and

$$\|[(e_{k+1} - e_k)^{1/2}, s_i]\| < \varepsilon' 2^{-k-1}, \quad i = 1, 2, \dots, n,$$

$$\|[(e_{k+1} - e_k)^{1/2}, V_j]\| < \varepsilon', \quad j \leq k + 1,$$

$$\|[(e_k - e_k^2)^{1/2}, V_j]\| < \varepsilon', \quad j \leq k + 1.$$

Since $-1 \leq \sum_{k=0}^K (e_{k+1} - e_k)^{1/2} x_k (e_{k+1} - e_k)^{1/2} \leq 1$ for any K and any $x_k = x_k^*$ with $\|x_k\| \leq 1$ we can define

$$z = \sum_{k=0}^{\infty} (e_{k+1} - e_k)^{1/2} V_k (e_{k+1} - e_k)^{1/2}.$$

The sum converges in the strict topology in $M(\mathcal{J}_n)$ and $\|z\| \leq 2$. Since $\|V_k - 1\| \rightarrow 0$ it follows that $z - 1 \in \mathcal{J}_n$. We claim that z is close to a unitary by writing

$$\begin{aligned} z z^* &= \sum_k (e_{k+1} - e_k)^{1/2} V_k (e_{k+1} - e_k) V_k^* (e_{k+1} - e_k)^{1/2} \\ &\quad + \sum_k (e_{k+1} - e_k)^{1/2} V_k (e_{k+1} - e_{k+1}^2)^{1/2} V_{k+1}^* (e_{k+2} - e_{k+1})^{1/2} \\ &\quad + \sum_k (e_{k+2} - e_{k+1})^{1/2} V_{k+1} (e_{k+1} - e_{k+1}^2)^{1/2} V_k^* (e_{k+1} - e_k)^{1/2}, \end{aligned}$$

where we used $(e_{k+1} - e_k)^{1/2}(e_{j+1} - e_j)^{1/2} = 0$ if $|k - j| > 1$ and $(e_{k+1} - e_k)^{1/2}(e_{k+2} - e_{k+1})^{1/2} = (e_{k+1} - e_{k+1}^2)^{1/2}$. By splitting each summation into a sum over the even integers and a sum over the odd integers and noting that

$$\|V_k(e_{k+1} - e_k)V_k^* - (e_{k+1} - e_k)\| < 2\varepsilon'$$

and

$$\|V_k(e_{k+1} - e_{k+1}^2)^{1/2}V_{k+1}^* - (e_{k+1} - e_{k+1}^2)^{1/2}\| < 2\varepsilon',$$

we estimate

$$\|zz^* - 1\| < 2(2\varepsilon' + 2\varepsilon' + 2\varepsilon') = 12\varepsilon'.$$

Similarly we find that $\|zz^* - 1\| < 12\varepsilon'$. Thus if ε' is sufficiently small, then $v = z|z^*z|^{-1/2}$ is a unitary in $\mathcal{J}_n + 1$ satisfying

$$\|v - z\| \leq \|v\| \|1 - |z^*z|^{1/2}\| < \varepsilon'',$$

with $\varepsilon'' \approx 6\varepsilon'$.

Since

$$\left\| s_i z^* - \sum_{k=0}^{\infty} (e_{k+1} - e_k)^{1/2} s_i V_k^* (e_{k+1} - e_k)^{1/2} \right\| < \varepsilon',$$

one has

$$\begin{aligned} z s_i z^* &\approx \sum_k (e_{k+1} - e_k)^{1/2} (e_{k+1} - e_k) V_k s_i V_k^* (e_{k+1} - e_k)^{1/2} \\ &+ \sum_k (e_{k+1} - e_k)^{1/2} (e_{k+1} - e_{k+1}^2)^{1/2} V_{k+1} s_i V_{k+1}^* (e_{k+2} - e_{k+1})^{1/2} \\ &+ \sum_k (e_{k+2} - e_{k+1})^{1/2} (e_{k+1} - e_{k+1}^2)^{1/2} V_k s_i V_k^* (e_{k+1} - e_k)^{1/2}, \end{aligned}$$

with the norm difference less than $\|z\|\varepsilon' + 12\varepsilon' \leq 14\varepsilon'$. Then using $\|(V_k s_i V_k^* - s_i)(1 - p)\| < \varepsilon$ etc., for all terms except for the $k = 0$ term of the first summation one finds that

$$\begin{aligned} z s_i z^* &\approx e_1^{3/2} V s_i V^* e_1^{1/2} + \sum_{k \geq 1} (e_{k+1} - e_k)^{3/2} s_i (e_{k+1} - e_k)^{1/2} \\ &+ \sum_{k \geq 0} (e_{k+1} - e_k)^{1/2} (e_{k+1} - e_{k+1}^2)^{1/2} s_i (e_{k+2} - e_{k+1})^{1/2} \\ &+ \sum_{k \geq 0} (e_{k+2} - e_{k+1})^{1/2} (e_{k+1} - e_{k+1}^2)^{1/2} s_i (e_{k+1} - e_k)^{1/2}, \end{aligned}$$

with the norm difference less than $14\varepsilon' + 6\varepsilon$. Since $\|[Vs_i V^*, e_1^{1/2}]\| < 3\varepsilon'$ and $\|(e_{k+1} - e_k)^{1/2}, s_i]\| < \varepsilon' 2^{-k-1}$ it follows that

$$zs_i z^* \approx e_1^2 V s_i V^* + (1 - e_1^2) s_i,$$

with the norm difference less than $14\varepsilon' + 6\varepsilon + 3\varepsilon' + 3\varepsilon' = 6\varepsilon + 20\varepsilon'$. Since $e_1^2 V s_i V^* + (1 - e_1^2) s_i - \beta(s_i) = (1 - e_1^2)(s_i - \beta(s_i))$, it follows that $\|zs_i z^* - \beta(s_i)\| < 7\varepsilon + 20\varepsilon'$, which implies that $\|vs_i v^* - \beta(s_i)\| < \varepsilon''(\|z\| + 1) + 7\varepsilon + 20\varepsilon' \approx 7\varepsilon + 32\varepsilon'$.

On the other hand, we have that $\|\alpha_t(z) - z\| \leq 2 \sup_k \|\alpha_t(V_k) - V_k\|$, which implies that $\max_{|t| \leq 1} \|\alpha_t(v) - v\| < 2\varepsilon + 2\varepsilon''$. This concludes the proof. $\square$

Proof of Proposition 1.2. This follows by combining Lemmas 3.3–3.5.

Let $t_0 \in \mathbf{R}$ and $\varepsilon > 0$. Then choose $\delta > 0$ corresponding to ε as in Lemma 3.5. By Lemma 3.3 one has an automorphism β of $\mathcal{E}_n$ and a unitary $V \in M(\mathcal{J}_n)_\alpha$ such that $Q\beta = \alpha_{t_0} Q$ and $\beta^{-1}\alpha_{t_0} = \text{Ad } V$, $\max_{|t| \leq 1} \|\alpha_t(V) - V\| < \delta$ etc. Then by Lemma 3.5 we obtain a unitary $v \in \mathcal{J}_n + 1$ such that $\|\beta^{-1}\alpha_{t_0}(s_i) - vs_i v^*\| < \varepsilon$ and $\max_{|t| \leq 1} \|\alpha_t(v) - v\| < \varepsilon$. Since $\beta = \lim_k \text{Ad } w_k$ we deduce that $\|\alpha_{t_0}(s_i) - w_k vs_i v^* w_k\| < \varepsilon$ for a large k and $\max_{|t| \leq 1} \|\alpha_t(w_k v) - w_k v\| < \varepsilon + \delta$ from the properties imposed on w_k . Thus we can conclude that α_{t_0} is α -invariantly approximately inner. $\square$

4. Proof of Lemma 3.4

Recall that H_0 is a self-adjoint operator on $\mathcal{H}_0$ with $\text{Spec}(H_0) = \mathbf{R}$. Moreover, $\alpha_t = \text{Ad } e^{itH_0}$ on $M(\mathcal{J}_n)_\alpha$, where $M(\mathcal{J}_n)$ is identified with $B(\mathcal{H}_0)$ and $M(\mathcal{J}_n)_\alpha$ is the largest C^* -subalgebra of $M(\mathcal{J}_n)$ on which $t \mapsto \text{Ad } e^{itH_0}$ is strongly continuous. The following can be proved by adopting the arguments in [5,16].

Lemma 4.1. *For any $\varepsilon > 0$ there exists a $\delta > 0$ satisfying the following condition. If $Z \in \mathcal{U}(M(\mathcal{J}_n)_\alpha)$ satisfies $\max_{|t| \leq 1} \|\alpha_t(Z) - Z\| < \delta$ then there exists a rectifiable path Z_s , $s \in [0, 1]$, in $\mathcal{U}(M(\mathcal{J}_n)_\alpha)$ such that $Z_0 = 1$, $Z_1 = Z$,*

$$\sup_{s \in [0, 1]} \max_{|t| \leq 1} \|\alpha_t(Z_s) - Z_s\| < \varepsilon,$$

and the length of Z_s , $s \in [0, 1]$, is less than $2\pi + \varepsilon$.

Proof. Choose a non-negative C^∞ -function on $\mathbf{R}$ such that $\hat{f}(0) = 1$ and $\text{supp } \hat{f} \subset (-\delta_0, \delta_0)$ for a small $\delta_0 > 0$ and modify Z by setting

$$X = \int f(t) \alpha_t(Z) dt \in M(\mathcal{J}_n)_\alpha.$$

Then X is still close to Z , i.e., $\|X - Z\| < \mu$ where μ , which depends on δ , can be arbitrarily close to zero. Let E be the spectral measure of H_0 and define, for any $k \in \mathbf{Z}$, the infinite rank projection

$$E_k = E[2k\delta_0, 2(k+1)\delta_0) \in M(\mathcal{J}_n)_\alpha.$$

Noting that the α -spectrum of X is contained in $(-\delta_0, \delta_0)$, we have, as in the proof of [16, 4.1], that

$$\begin{aligned} X E_k X^* &\leq E[(2k-1)\delta_0, (2k+3)\delta_0), \\ X E_k X^* - (X E_k X^*)^2 &\leq 2\mu \cdot X E_k X^* \leq 2\mu 1, \\ \text{Spec}(X E_k X^*) &\subset \{0\} \cup [1-2\mu, 1], \\ Y_k^+ - (Y_k^+)^2 &\leq 6\mu 1, \\ \text{Spec}(Y_k^+) &\subset [0, 6\mu'] \cup [1-6\mu', 1], \\ Y_k^- - (Y_k^-)^2 &\leq 6\mu 1, \\ \text{Spec}(Y_k^-) &\subset [0, 6\mu'] \cup [1-6\mu', 1], \end{aligned}$$

where $\mu \approx \mu'$ and

$$\begin{aligned} Y_k^+ &= E[(2k+1)\delta_0, (2k+3)\delta_0) X E_k X^* E[(2k+1)\delta_0, (2k+3)\delta_0), \\ Y_k^- &= E[(2k-1)\delta_0, (2k+1)\delta_0) X E_k X^* E[(2k-1)\delta_0, (2k+1)\delta_0). \end{aligned}$$

Define $F_k^\pm$ to be the spectral projection of $Y_k^\pm$ corresponding to $[1-6\mu', 1]$ and let F_k denote the projection $F_k^+ + F_k^-$. We note that these projections are of infinite rank, because $E(I)$ is a projection of infinite rank for any non-empty open subset I of $\mathbf{R}$. Further

$$\begin{aligned} F_k^+ &\leq E[(2k+1)\delta_0, (2k+3)\delta_0), \\ F_k^- &\leq E[(2k-1)\delta_0, (2k+1)\delta_0), \\ \|F_k - X E_k X^*\| &< 14\mu. \end{aligned}$$

Define

$$\begin{aligned} G_k^+ &= E[(2k+1)\delta_0, (2k+3)\delta_0) - F_{k+1}^-, \\ G_k^- &= E[(2k-1)\delta_0, (2k+1)\delta_0) - F_{k-1}^+, \\ G_k &= G_k^- + G_k^+ \end{aligned}$$

and note that $G_k^\pm$ and G_k are projections of infinite rank satisfying

$$\|X E_k X^* - G_k\| < 34\mu.$$

Using $\sum_k E_k = 1$ and

$$\sum_k F_{2k} + \sum_k G_{2k-1} = 1,$$

one checks that

$$W = \sum_k F_{2k} X E_{2k} + \sum_k G_{2k-1} X E_{2k-1}$$

is close to a unitary. Denote by V the unitary part of the polar decomposition of W . Since

$$\begin{aligned} \|\alpha_t(F_{2k} X E_{2k}) - F_{2k} X E_{2k}\| &\leq \|\alpha_t(X) - X\| + 4\delta_0|t|, \\ \|\alpha_t(G_{2k-1} X E_{2k-1}) - G_{2k-1} X E_{2k-1}\| &\leq \|\alpha_t(X) - X\| + 4\delta_0|t|, \quad \text{and} \\ \|\alpha_t(W) - W\| &\leq \sup_k \|\alpha_t(F_{2k} X E_{2k}) - F_{2k} X E_{2k}\| + \sup_k \|\alpha_t(G_{2k-1} X E_{2k-1}) - G_{2k-1} X E_{2k-1}\|, \end{aligned}$$

it follows that $W, V \in M(\mathcal{J}_n)_\alpha$. Moreover, one estimates that $\|Z - V\| < 10\mu''$ where $\mu'' \approx \mu^{1/2}$. Note also that

$$\begin{aligned} V E_{2k} V^* &= F_{2k} \leq E[(4k-1)\delta_0, (4k+3)\delta_0], \\ V E_{2k-1} V^* &= G_{2k-1} \leq E[(4k-3)\delta_0, (4k+1)\delta_0]. \end{aligned}$$

It then follows that the α -spectrum of V is contained in $[-3\delta_0, 3\delta_0]$.

Since $\|V Z^* - 1\| < 10\mu''$ there is a self-adjoint $B \in M(\mathcal{J}_n)_\alpha$ such that $V Z^* = e^{iB}$ and $\|B\|$ is of the order $10\mu''$. Then the path $\gamma: [0, 1] \ni s \mapsto e^{isB} Z$ running from Z to V is of length $\|B\|$. Since $\|\alpha_t(e^{isB} Z) - e^{isB} Z\| \leq s\|\alpha_t(B) - B\| + \|\alpha_t(Z) - Z\|$ one has $\max_s \|\alpha_t(\gamma_s) - \gamma_s\| \rightarrow 0$ as $t \rightarrow 0$ and

$$\sup_s \max_{|t| \leq 1} \|\alpha_t(\gamma_s) - \gamma_s\| \leq 2\|B\| + \max_{|t| \leq 1} \|\alpha_t(Z) - Z\|,$$

which is arbitrarily small.

Then one can find a rectifiable path ζ_t , $t \in [0, 1]$, in $\mathcal{U}(M(\mathcal{J}_n))$ such that ζ_t commutes with $E[(2k-1)\delta_0, (2k+1)\delta_0]$ for all $k \in \mathbf{Z}$,

$$\begin{aligned} \text{Ad } \zeta_1(F_{2k}^+) &= E[(4k+1)\delta_0, (4k+2)\delta_0], \\ \text{Ad } \zeta_1(F_{2k}^-) &= E[4k\delta_0, (4k+1)\delta_0] \end{aligned}$$

and the length of ζ is at most π . Then we deduce that

$$\text{Ad}(\zeta_1 V) E_{2k} = E_{2k}.$$

Since $\text{Ad } \zeta_1(G_{2k+1}^-) = E[(4k+2)\delta_0, (4k+3)\delta_0]$ etc., we also deduce that

$$\text{Ad}(\zeta_1 V) E_{2k+1} = E_{2k+1}.$$

Note that the α -spectrum of ζ_t is contained in $[-2\delta_0, 2\delta_0]$, which implies that $\zeta_t \in M(\mathcal{J}_n)_\alpha$ and $\max_{|t| \leq 1} \|\alpha_t(\zeta_s) - \zeta_s\| \leq 2\delta_0$.

There is a rectifiable path η_t , $t \in [0, 1]$, in $\mathcal{U}(M(\mathcal{J}_n))$ such that η_t commutes with $E[(2k-1)\delta_0, (2k+1)\delta_0]$ for all $k \in \mathbf{Z}$, $\eta_0 = 1$, $\eta_1 = (\zeta_1 V)^*$, and the length of η is at most π . Note

also that the α -spectrum of η_t is contained in $[-2\delta_0, 2\delta_0]$, which implies that $\eta_t \in M(\mathcal{J}_n)_\alpha$ and $\max_{|t| \leq 1} \|\alpha_t(\eta_s) - \eta_s\| \leq 2\delta_0$.

Combining these paths we obtain the desired conclusion. $\square$

Next we prove another version of the above lemma. We replace $M(\mathcal{J}_n)_\alpha$ by the C^* -tensor product $C(\mathbf{T}) \otimes M(\mathcal{J}_n)_\alpha$ with the flow $\text{id} \otimes \alpha$, which will sometimes be denoted by α .

As preparation we present the following two lemmas which are just concerned with $B(\mathcal{H})$, the bounded operators on an infinite-dimensional Hilbert space $\mathcal{H}$, without a flow. We note that the C^* -tensor product $C(\mathbf{T}) \otimes B(\mathcal{H})$ identifies with the norm-continuous functions from $\mathbf{T}$ into $B(\mathcal{H})$.

Lemma 4.2. *Let E, F, P be projections in $C(\mathbf{T}) \otimes B(\mathcal{H})$ such that $E(s), F(s), P(s)$ are of infinite rank, $P(s) = P(0)$ for $s \in \mathbf{T} = [0, 1]/\{0, 1\}$, and $EP = 0 = FP$. Then there is a rectifiable path U_t , $t \in [0, 1]$, in $\mathcal{U}(C(\mathbf{T}) \otimes B(\mathcal{H}))$ such that $U_0 = 1$, $\text{Ad } U_1(E) = F$, and the length of U is at most π . If furthermore $E(0) = F(0)$ at $0 \in \mathbf{T}$, then the condition $U_t(0) = 1$, $t \in [0, 1]$, can be imposed.*

Moreover, if E_k, F_k, P_k are for each $k \in \mathbf{N}$ such a triple of projections in $C(\mathbf{T}) \otimes B(\mathcal{H})$ and such that $\mathbf{T} \ni s \mapsto \bigoplus_k E_k(s)$ and $\mathbf{T} \ni s \mapsto \bigoplus_k F_k(s)$ are continuous in $\prod_{k=1}^\infty C(\mathbf{T}) \otimes B(\mathcal{H})$ then there are rectifiable paths U_t^k , $t \in [0, 1]$, in $\mathcal{U}(C(\mathbf{T}) \otimes B(\mathcal{H}))$ as above such that $\mathbf{T} \ni s \mapsto \bigoplus_k U_t^k(s)$ is continuous and the length of $[0, 1] \ni t \mapsto \bigoplus_k U_t^k$ is at most π .

Proof. We have expressed the base space $\mathbf{T}$ as $[0, 1]/\{0, 1\}$. Since $t \mapsto E(t)$ is norm-continuous there exists a norm-continuous $V : [0, 1] \mapsto \mathcal{U}(B(\mathcal{H}))$ such that $V(0) = 1$ and

$$\text{Ad } V(t)(E(0)) = E(t), \quad t \in [0, 1].$$

More specifically we choose a finite number of points $t_0 = 1 < t_1 < t_2 < \dots < t_{m-1} < t_m = 1$ such that $\|E(s) - E(t)\| < 1/2$ for $s, t \in [t_{i-1}, t_i]$, for $i = 1, 2, \dots, m$. Then we define

$$Z_t^i = E(t)E(t_{i-1}) + (1 - E(t))(1 - E(t_{i-1}))$$

for $t \in [t_{i-1}, t_i]$. Since $\|(Z_t^i)^* Z_t^i - 1\| < 1/2$ etc., the polar decomposition of Z_t^i gives a unitary $V_t^i = Z_t^i |Z_t^i|^{-1}$. Then we define

$$V_t = V_t^i V_{t_{i-1}}^{i-1} V_{t_{i-2}}^{i-2} \dots V_{t_1}^1$$

for $t \in [t_{i-1}, t_i]$. Since $\text{Ad } V_t^i(E(t_{i-1})) = E(t)$ we deduce that $\text{Ad } V(t)(E(0)) = E(t)$. It is obvious that $t \mapsto V_t$ is norm-continuous. But more is true.

Note that

$$\|V_t^i - V_s^i\| \leq 2^{1/2} \|E(t) - E(s)\| + \| |Z_t^i|^{-1} - |Z_s^i|^{-1} \| \quad \text{for } s, t \in [t_{i-1}, t_i].$$

Let $\mathcal{S}$ be the set of self-adjoint elements $h \in B(\mathcal{H})$ such that $\text{Spec}(h) \subset [1/2, 1]$. Then $|Z_t^i|^2 = (Z_t^i)^* Z_t^i \in \mathcal{S}$. The map $\mathcal{S} \ni h \mapsto h^{-1/2}$ is uniformly continuous in $h \in \mathcal{S}$. Since $\| |Z_t^i|^2 - |Z_s^i|^2 \| \leq \|E(t) - E(s)\|$ the continuity of $t \mapsto V_t$ only depends on the continuity of $t \mapsto E(t)$ on each $[t_{i-1}, t_i]$, i.e., there is a non-decreasing continuous function φ_1 on $[0, 1]$ such that $\varphi_1(0) = 0$ and $\|V_s - V_t\| \leq \varphi_1(\|E(s) - E(t)\|)$ for $s, t \in [t_{i-1}, t_i]$ and all i . Since $[0, 1] \ni t \mapsto E(t)$ is

uniformly continuous there is a non-decreasing continuous function φ_2 on $[0, 1]$ into $[0, 1]$ such that $\varphi_2(0) = 0$ and $\|E(s) - E(t)\| \leq \varphi_2(|s - t|)$. Let $\varphi = \varphi_1 \circ \varphi_2$. Combining these estimates, we have that $\|V_s - V_t\| \leq \varphi(|s - t|)$ for $s, t \in [t_{i-1}, t_i]$.

Let $\Delta = \min_{1 \leq i \leq m} \{t_i - t_{i-1}\}$. Further let $\psi(s) = \max_{0 \leq t \leq s} \{\varphi(t) + \varphi(s - t)\}$ for $s \in [0, 1]$. Then ψ is a non-decreasing continuous function with $\psi(0) = 0$.

If $s, t \in [0, 1]$ satisfy $0 < s - t < 2\Delta$ then either $t_{i-1} < t < t_i < s < t_{i+1}$ or $t_{i-1} \leq t < s \leq t_i$ for some i . In the former case $\|V_s - V_t\| \leq \|V_s - V_{t_i}\| + \|V_{t_i} - V_t\| \leq \varphi_1(\|E(s) - E(t_i)\|) + \varphi_1(\|E(t_i) - E(t)\|) \leq \varphi(s - t_i) + \varphi(t_i - t) \leq \psi(s - t)$ with a similar estimate for the latter case. Therefore $\|V_s - V_t\| \leq \psi(|s - t|)$ if $|s - t| < 2\Delta$. By redefining ψ on $[2\Delta, 1]$, and denoting the redefined function by φ , we may suppose that $\|V_s - V_t\| \leq \varphi(|s - t|)$ is valid for all $s, t \in [0, 1]$.

We define a function F on $\mathbf{T} = \{z \in \mathbb{C} : |z| = 1\}$ by $F(e^{it}) = t$ for $t \in (-\pi, \pi]$ and let $H = F(V(1))$. Since $[V(1), E(0)] = 0$ one has $[H, E(0)] = 0$. Replacing V by $V' : t \mapsto V_t e^{-itH}$ one has $V'(1) = 1 = V'(0)$, $\text{Ad } V'(t)(E(0)) = E(t)$, and the continuity of $t \mapsto V'_t$ depends only on the continuity of $t \mapsto E(t)$. If we replace φ by $s \mapsto \varphi(s) + \pi s$ then $\|V'(s) - V'(t)\| \leq \varphi(|s - t|)$.

In this way we construct $V, W \in \mathcal{U}(C(\mathbf{T}) \otimes B((1 - P(0))\mathcal{H}))$ which satisfy

$$V(1 \otimes E(0))V^* = E, \quad V(0) = 1, \quad W(1 \otimes F(0))W^* = F \quad \text{and} \quad W(0) = 1.$$

Moreover, we may assume that there is a non-decreasing continuous function φ on $[0, 1]$ such that $\varphi(0) = 0$ and

$$\|V(s) - V(t)\| \leq \varphi(|s - t|), \quad \|W(s) - W(t)\| \leq \varphi(|s - t|).$$

Note that φ depends only on the choice of t_i and the continuity of $s \mapsto E(s)$ and of $s \mapsto F(s)$.

If $E(0) = F(0)$ the unitary $Z \equiv WV^*$ in $C(\mathbf{T}) \otimes B((1 - P(0))\mathcal{H})$ satisfies $\text{Ad}(Z)(E) = F$ and $Z(0) = 1 - P(0)$. Let $Y \in B(\mathcal{H})$ be such that $Y^*Y = P(0)$ and $YY^* = 1 - P(0)$ and let

$$X_t = \cos(\pi t/2) + (Y - Y^*) \sin(\pi t/2), \quad t \in [0, 1].$$

Then X_t is a unitary on $\mathcal{H}$. The path $U : t \mapsto (Z + P)(1 \otimes X_t)(Z^* + P)(1 \otimes X_t^*)$ in $\mathcal{U}(C(\mathbf{T}) \otimes B(\mathcal{H}))$ satisfies $U_0 = 1$, $U_1 = Z + Y^*Z^*Y$, $U_t(0) = 1$, and

$$\|U_{t_1} - U_{t_2}\| \leq 2\|X_{t_1} - X_{t_2}\| \leq \pi|t_1 - t_2|.$$

Moreover, U_t satisfies

$$\|U_t(s_1) - U_t(s_2)\| \leq 2\varphi(|s_1 - s_2|).$$

Since $\text{Ad } U_1(E) = F$, this concludes the proof for the case $E(0) = F(0)$.

In the case $E(0) \neq F(0)$ we choose a unitary T on $(1 - P(0))\mathcal{H}$ such that $TE(0)T^* = F(0)$. Then we can proceed as above with $Z = W(1 \otimes T)V^*$.

Suppose that E_k, F_k, P_k are given as in the statement of the lemma. Let $E_\infty = \bigoplus_k E_k$ and $F_\infty = \bigoplus_k F_k$ in $\prod_{k=1}^\infty C(\mathbf{T}) \otimes B(\mathcal{H})$. Then choose a finite number of points $t_0 = 0 < t_1 < t_2 < \dots < t_m = 1$ such that $\|E_\infty(s) - E_\infty(t)\| < 1/2$ and $\|F_\infty(s) - F_\infty(t)\| < 1/2$ for $s, t \in [t_{i-1}, t_i]$ with all i . By using these points in $[0, 1]$ we construct a path $U^k : [0, 1] \rightarrow \mathcal{U}(C(\mathbf{T}) \otimes B(\mathcal{H}))$ as above. Then there is a non-decreasing continuous function φ on $[0, 1]$ such that $\varphi(0) = 0$ and $\|U_t^k(s_1) - U_t^k(s_2)\| \leq 2\varphi(|s_1 - s_2|)$. Note that we can define φ based on the functions $s \mapsto E_\infty(s)$ and $s \mapsto F_\infty(s)$, i.e., φ is independent of k . Hence it follows that $\mathbf{T} \ni s \mapsto U_t^\infty(s) \equiv \bigoplus_k U_t^k(s)$ is

continuous for each $t \in [0, 1]$. Since $\|U_{t_1}^k - U_{t_2}^k\| \leq \pi |t_1 - t_2|$ we also deduce that $\|U_{t_1}^\infty - U_{t_2}^\infty\| \leq \pi |t_1 - t_2|$. This concludes the proof. $\square$

Lemma 4.3. *Let $U \in \mathcal{U}(C(\mathbf{T}) \otimes B(\mathcal{H}))$. If there is a projection $E \in B(\mathcal{H})$ such that E and $1 - E$ are of infinite rank and $[U, 1 \otimes E] = 0$, then there is a rectifiable path V_t , $t \in [0, 1]$, in $\mathcal{U}(C(\mathbf{T}) \otimes B(\mathcal{H}))$ such that $V_0 = 1$, $V_1 = U$, $\|V_t(s_1) - V_t(s_2)\| \leq 6\|U(s_1) - U(s_2)\|$, and the length of V is at most 3π .*

Proof. Let $Y \in B(\mathcal{H})$ be a partial isometry such that $Y^*Y = E$ and $YY^* = 1 - E$ and let $X_t = \cos(\pi t/2) + (Y - Y^*)\sin(\pi t/2)$ as in the proof of the previous lemma. Then the map $[0, 1] \ni t \mapsto (U(1 \otimes E) + 1 \otimes (1 - E))(1 \otimes X_t)(U(1 \otimes (1 - E)) + 1 \otimes E)(1 \otimes X_t^*)$ moves from U into $U_1 \equiv U(1 \otimes E)(1 \otimes Y)U(1 \otimes Y^*) + 1 \otimes (1 - E)$. Note that the length of this path is at most π .

Now $W = U(1 \otimes E)(1 \otimes Y^*)U(1 \otimes Y)$ is a unitary in $C(\mathbf{T}) \otimes B(E\mathcal{H})$. Since $1 - E$ is of infinite rank, letting $\mathcal{H}_k = E\mathcal{H}$ for all $k \in \mathbf{Z}$, we identify $\mathcal{H}$ with $\bigoplus_{k \in \mathbf{Z}} \mathcal{H}_k$ and $E\mathcal{H}$ with $\mathcal{H}_0$. Thus we regard $U_1(s)$ as

$$\cdots \oplus 1 \oplus 1 \oplus W(s) \oplus 1 \oplus \cdots,$$

where $W(s)$ is on $\mathcal{H}_0$. Let $W_r = \bigoplus_{k \leq 0} 1 \oplus \bigoplus_{k \geq 1} W \in C(\mathbf{T}) \otimes B(\bigoplus_k \mathcal{H}_k)$. Let H be a self-adjoint operator on $\bigoplus_k \mathcal{H}_k$ such that $\text{Spec}(H) = [-\pi, \pi]$, and e^{itH} induces the shift to the right. Then the map $[0, 1] \ni t \mapsto U_1 W_r (1 \otimes e^{-itH}) W_r^* (1 \otimes e^{itH})$ moves from U_1 into 1 . The length of this path is at most 2π .

Combining these two paths we obtain the desired path V_t , $t \in [0, 1]$. Since $\|W(s_1) - W(s_2)\| \leq 2\|U(s_1) - U(s_2)\|$ one has $\|V_t(s_1) - V_t(s_2)\| \leq 6\|U(s_1) - U(s_2)\|$ for any $t \in [0, 1]$. This concludes the proof. $\square$

Lemma 4.4. *For any $\varepsilon > 0$ there exists a $\delta > 0$ satisfying the following condition. If $Z \in \mathcal{U}(C(\mathbf{T}) \otimes M(\mathcal{J}_n)_\alpha)$ satisfies $\max_{|t| \leq 1} \|(\text{id} \otimes \alpha_t)(Z) - Z\| < \delta$ then there exists a rectifiable path Z_s , $s \in [0, 1]$, in $\mathcal{U}(C(\mathbf{T}) \otimes M(\mathcal{J}_n)_\alpha)$ such that $Z_0 = 1$, $Z_1 = Z$,*

$$\max_{s \in [0, 1]} \max_{|t| \leq 1} \|(\text{id} \otimes \alpha_t)(Z_s) - Z_s\| < \varepsilon,$$

and the length of Z_s , $s \in [0, 1]$, is less than $4\pi + \varepsilon$. Furthermore if $Z(0) = 1$ at $0 \in \mathbf{T}$ then the path can be chosen to satisfy $Z_s(0) = 1$ for $s \in [0, 1]$.

Proof. We prove this result for $C(\mathbf{T}) \otimes M(\mathcal{J}_n)_\alpha$ adapting the proof of Lemma 4.1 for $M(\mathcal{J}_n)_\alpha$ by use of Lemmas 4.2 and 4.3. We indicate below how to use Lemmas 4.2 and 4.3 in the adaptation of the proof of Lemma 4.1.

The proof of Lemma 4.1 consists of constructing two paths ζ and η . Now we apply Lemma 4.2 to construct the analogue of ζ necessary in the proof of Lemma 4.4. Explicitly we define $F_k^\pm$ as in the proof of Lemma 4.1, which entails that $F_k^\pm \in C(\mathbf{T}) \otimes M(\mathcal{J}_n)_\alpha$ and

$$F_k^+ \leq 1 \otimes E[(2k+1)\delta_0, (2k+3)\delta_0), \quad F_k^- \leq 1 \otimes E[(2k-1)\delta_0, (2k+1)\delta_0).$$

With $\mathcal{H} = E[(2k-1)\delta_0, (2k+1)\delta_0)\mathcal{H}_0$ we then construct a unitary path ζ^k in each $C(\mathbf{T}) \otimes B(\mathcal{H})$, where $B(\mathcal{H})$ equals

$$E[(2k-1)\delta_0, (2k+1)\delta_0)M(\mathcal{J}_n)_\alpha E[(2k-1)\delta_0, (2k+1)\delta_0).$$

For $\mathcal{H} = E[(4k+1)\delta_0, (4k+3)\delta_0]\mathcal{H}_0$ we have to find a path ζ^{2k+1} in $\mathcal{U}(C(\mathbf{T}) \otimes B(\mathcal{H}))$ such that $\zeta_0^{2k+1} = 1$,

$$\text{Ad } \zeta_1^{2k+1}(F_{2k}^+) = 1 \otimes E[(4k+1)\delta_0, (4k+2)\delta_0],$$

and

$$\|\zeta_{t_1}^{2k+1} - \zeta_{t_2}^{2k+1}\| \leq \pi |t_1 - t_2|.$$

We can apply Lemma 4.2 because $1 \otimes E[(4k+3)\delta_0 - \delta', (4k+3)\delta_0]$ is of infinite rank and orthogonal to F_{2k}^+ and $1 \otimes E[(4k+1)\delta_0, (4k+2)\delta_0]$ for a small $\delta' > 0$.

For $\mathcal{H} = E[(4k-1)\delta_0, (4k+1)\delta_0]\mathcal{H}_0$, we find a path ζ^{2k} in $\mathcal{U}(C(\mathbf{T}) \otimes B(\mathcal{H}))$ such that $\zeta_0^{2k} = 1$, $\text{Ad } \zeta^{2k}(F_{2k}^-) = 1 \otimes E[4k\delta_0, (4k+1)\delta_0]$, and $\|\zeta_{t_1}^{2k} - \zeta_{t_2}^{2k}\| \leq \pi |t_1 - t_2|$.

Note that $\mathbf{T} \ni s \mapsto Y_k(s)$ is equi-continuous in $k \in \mathbf{Z}$ (see the definition of Y_k^+ in the proof of Lemma 4.1). Since the spectrum of F_k^+ is contained in $[0, 6\mu'] \cup [1 - 6\mu', 1]$ and F_k^+ is the spectral projection of Y_k^+ corresponding to $[1 - 6\mu', 1]$ it follows that $\mathbf{T} \ni s \mapsto F_k^+(s)$ is equi-continuous in k . Similarly $\mathbf{T} \ni s \mapsto F_k^-(s)$ is equi-continuous in k . By Lemma 4.2 we can choose the paths ζ^k such that $\mathbf{T} \ni s \mapsto \bigoplus \zeta_t^k(s)$ is continuous for each $t \in [0, 1]$, which implies that $\zeta_t \equiv \sum_k \zeta_t^k \in C(\mathbf{T}) \otimes M(\mathcal{J}_n)$. Since ζ_t commutes with $E[(2k-1)\delta_0, (2k+1)\delta_0]$ it follows that $\zeta_t \in C(\mathbf{T}) \otimes M(\mathcal{J}_n)_\alpha$ and the α -spectrum of ζ_t is contained in $[-2\delta_0, 2\delta_0]$. Note that ζ_t , $t \in [0, 1]$, has length of at most π .

When we construct the path η we use Lemma 4.3.

More explicitly $\zeta_1 V$ commutes with $1 \otimes E_k$. We apply Lemma 4.3 and find a path η^k in $\mathcal{U}(C(\mathbf{T}) \otimes B((E_{2k} + E_{2k+1})\mathcal{H}_0))$ which connects $\zeta_1 V(E_{2k} + E_{2k+1})$ with $E_{2k} + E_{2k+1}$ for each k . Since $\|\eta_t^k(s_1) - \eta_t^k(s_2)\| \leq 6\|\zeta_1(s_1)V(s_1) - \zeta_1(s_2)V(s_2)\|$ it follows that $\mathbf{T} \ni s \mapsto \eta_t(s) \equiv \sum_k \eta_t^k(s)$ is continuous, i.e., $\eta_t \in C(\mathbf{T}) \otimes M(\mathcal{J}_n)$. Note that the length of η_t , $t \in [0, 1]$, is at most 3π and that the α -spectrum of η_t is contained in $[-4\delta_0, 4\delta_0]$. In this way we obtain the desired conclusion. $\square$

Recall that the flow α on $\mathcal{E}_n \subset M(\mathcal{J}_n)_\alpha$ induces a flow on the quotient $\mathcal{O}_n \subset M(\mathcal{J}_n)_\alpha / \mathcal{J}_n$, which we also denote by α . By using $Q(s_i) \in \mathcal{O}_n$ we define a unital endomorphism λ of $\mathcal{O}_n$ by $\lambda(x) = \sum_{i=1}^n Q(s_i)xQ(s_i)^*$. We know that λ has the Rohlin property (as a version for a single endomorphism or automorphism); see [12,13,26]. But more is true.

Lemma 4.5. *For any $N \in \mathbf{N}$ and $\varepsilon > 0$, there are n^N projections e_i in $\mathcal{O}_n$ for $i = 0, 1, \dots, n^N - 1$ such that $e_i \in D(\delta_\alpha)$,*

$$\sum_{i=0}^{n^N-1} e_i = 1, \quad \|\delta_\alpha(e_i)\| < \varepsilon, \quad \max_i \|\lambda(e_i) - e_{i+1}\| < \varepsilon,$$

with $e_{n^N} = e_0$.

Proof. Define an action γ of $\mathbf{T}^n$ on $\mathcal{O}_n$ by $\gamma_z(s_k) = z_k s_k$ for $z = (z_1, z_2, \dots, z_n) \in \mathbf{T}^n$ and embed $\mathbf{T}$ into $\mathbf{T}^n$ by $z \mapsto (z, z, \dots, z)$. Then the fixed point algebra of $\mathcal{O}_n$ under $\gamma|_{\mathbf{T}}$ is the closed linear span of $Q(s_I s_J^*)$, with $|I| = |J|$ and $I, J \in \{1, 2, \dots, n\}^*$, and is isomorphic to the UHF algebra

of type n^∞ . We denote it by UHF_n . Then the restriction of λ onto UHF_n is the one-sided shift and is known to have the Rohlin property.

Note that $\mathcal{O}_n^\gamma \subset \text{UHF}_n$, $\mathcal{O}_n^\gamma \subset \mathcal{O}_n^\alpha$, and by [20] that $\lambda|_{\mathcal{O}_n^\gamma}$ has the so-called one-cocycle property. (Since $\mathcal{O}_n^\gamma$ has n distinct characters, $\lambda|_{\mathcal{O}_n^\gamma}$ cannot have the Rohlin property but has an approximate Rohlin property.) In particular, for any $k \in \mathbf{N}$ one has mutually orthogonal projections f_i^k , $i = 0, \dots, M-1$, in $\mathcal{O}_n^\gamma$, with $M = n^N$, such that

$$\|\lambda(f_i^k) - f_{i+1}^k\| < 1/k$$

with $f_M^k = f_0$ and $[f_0^k] = [1]$ in $K_0(\mathcal{O}_n) = \mathbf{Z}/(n-1)\mathbf{Z}$. Let $e^k = \sum_{i=0}^{M-1} f_i^k \in \mathcal{O}_n^\gamma$. The e^k is an α -invariant projection. Since $\|\lambda(e^k) - e^k\| < M/k$ the (e^k) form a central sequence in $A = \mathcal{O}_n$. Hence (e^k) defines a projection E in $(A' \cap A_\alpha^\omega)^\alpha$ where ω is an ultrafilter on $\mathbf{N}$ and $(A' \cap A_\alpha^\omega)^\alpha$ is the α -fixed point algebra of the central sequence algebra divided by the ideal vanishing at ω . Since $[1] = [e^k]$ in $K_0(A) = \mathbf{Z}/(n-1)\mathbf{Z}$, one has $[1] = [E]$ in $K_0(A' \cap A^\omega)$ [19]. (This is not entirely trivial; we use the fact that $K_1(A) = 0$ for $A = \mathcal{O}_n$.)

By [18,19] one has $K_0((A \cap A_\alpha^\omega)^\alpha) = K_0(A' \cap A^\omega)$. Since $1, E \in (A' \cap A_\alpha^\omega)^\alpha$ there is a $W \in (A' \cap A_\alpha^\omega)^\alpha$ such that $WW^* = E$ and $W^*W = 1$. Suppose that (w_k) represent W . Then it follows that $\lim_\omega \max_{|t| \leq 1} \|\alpha_t(w_k) - w_k\| = 0$, $\lim_\omega \|w_k w_k^* - e^k\| = 0$, and $\lim_\omega \|w_k^* w_k - 1\| = 0$. Thus we can construct an isometry w in $A = \mathcal{O}_n$ from w_k for some k with $\varepsilon/3 > 1/k$ such that $\|\lambda(w) - w\| < \varepsilon/3$, $w \in D(\delta_\alpha)$, $\|\delta_\alpha(w)\| < \varepsilon/2$, $ww^* = e^k$, and $w^*w = 1$. We can take the projections $e_i = w^* f_i^k w$, $i = 0, 1, \dots, M-1$, because then $\sum_i e_i = 1$, $\|\lambda(e_i) - e_{i+1}\| < \varepsilon$ and $\|\delta_\alpha(e_i)\| < \varepsilon$ since the f_i^k are fixed by α . $\square$

Lemma 4.6. *For any $\varepsilon > 0$ there exists a $\delta > 0$ satisfying the following condition. Let H_σ be self-adjoint operators on the Hilbert spaces $\mathcal{H}_\sigma$ with $\sigma = 1, 2$ and such that $\dim(\mathcal{H}_1) = \dim(\mathcal{H}_2) < \infty$. If there is a unitary operator W from $\mathcal{H}_1$ onto $\mathcal{H}_2$ such that*

$$\max_{|t| \leq 1} \|e^{itH_2} W e^{-itH_1} - W\| < \delta,$$

then it follows that $|\lambda_i^{(1)} - \lambda_i^{(2)}| < \varepsilon$ for all i , where $(\lambda_i^{(\sigma)})$ is the increasing sequence of eigenvalues of H_σ (each repeated according to multiplicity) for $\sigma = 1, 2$.

Proof. Let f be a non-negative integrable continuous function on $\mathbf{R}$ such that $\hat{f}(0) = 1$ and $\text{supp}(\hat{f}) \subset [-\varepsilon, \varepsilon]$, where $\hat{f}(p) = \int f(t) e^{-ipt} dt$. Choose $\delta > 0$ sufficiently small so that $W_f = \int f(t) e^{itH_2} W e^{-itH_1} dt$ is close to W and is invertible for W satisfying the condition in the lemma. Then it follows that for any finite subset $\mathcal{F}$ of the eigenvalues of H_1 the number of eigenvalues λ of H_2 with $|\lambda - \mu| < \varepsilon$ for some $\mu \in \mathcal{F}$ is greater than or equal to the number of elements of $\mathcal{F}$. (Because if L is the linear subspace spanned by eigenvectors for H_1 whose eigenvalues are in $\mathcal{F}$, then the linear subspace $W_f L$ is contained in the linear subspace L' spanned by eigenvectors for H_2 whose eigenvalues are in $\{\lambda: \exists \mu \in \mathcal{F}, |\lambda - \mu| < \varepsilon\}$. Since W_f is invertible, the dimension of L' must be no less than the dimension of L .) This implies, by the matching theorem, that there is a bijection ϕ from the eigenvalues of H_1 to those of H_2 such that $|\phi(\lambda) - \lambda| < \varepsilon$ for any eigenvalue λ of H_1 . This implies the above conclusion. $\square$

Lemma 4.7. *For any $\varepsilon > 0$ there exists a $\delta > 0$ satisfying the following condition. If $v \in C(\mathbf{T}) \otimes M(\mathcal{J}_n)_\alpha / \mathcal{J}_n$ is a unitary such that $v(0) = 1$ at $0 \in \mathbf{T} = [0, 1]/\{0, 1\}$ and $\max_{|t| \leq 1} \|(\text{id} \otimes$*

$\alpha_t(v) - v\| < \delta$ then there is a unitary $Z \in C(\mathbf{T}) \otimes M(\mathcal{J}_n)_\alpha$ such that $Q(Z) = v$, $Z(0) = 1$ and $\max_{|t| \leq 1} \|(\text{id} \otimes \alpha_t)(Z) - Z\| < \varepsilon$.

Proof. Regarding v as a continuous function from $[0, 1]$ into $\mathcal{U}(M(\mathcal{J}_n)_\alpha/\mathcal{J}_n)$ we can find a continuous function V from $[0, 1]$ into $\mathcal{U}(M(\mathcal{J}_n)_\alpha)$ such that $V(0) = 1$ and $Q(V(s)) = v(s)$ for $s \in [0, 1]$. Since $V(1) \in 1 + \mathcal{J}_n$ and $\mathcal{U}(1 + \mathcal{J}_n)$ is connected we may suppose that $V(1) = 1$, i.e., we have a unitary $V \in C(\mathbf{T}) \otimes M(\mathcal{J}_n)_\alpha$ such that $V(0) = 1$ and $Q(V) = v$. If $\max_{|t| \leq 1} \|(\text{id} \otimes \alpha_t)(v) - v\| < \delta$ then $\max_{|t| \leq 1} \|\alpha_t(V(s)) - V(s) + \mathcal{J}_n\| < \delta$ for all $s \in \mathbf{T}$. Thus we may start with such a unitary $V \in C(\mathbf{T}) \otimes M(\mathcal{J}_n)_\alpha$ instead of $v \in C(\mathbf{T}) \otimes M(\mathcal{J}_n)_\alpha/\mathcal{J}_n$.

Let (p_k) be an increasing sequence of projections in $\mathcal{J}_n$ such that $x p_k \rightarrow x$ for any $x \in \mathcal{J}_n$ and $\alpha_t(p_k) = p_k$ for all k . (Note that $\alpha_t = \text{Ad } e^{iH_0 t}$ on $M(\mathcal{J}_n)_\alpha$ and H_0 is diagonal.)

Since $\mathbf{T} \times [-1, 1] \ni (s, t) \mapsto \alpha_t(V(s)) - V(s) \in M(\mathcal{J}_n)_\alpha$ is norm-continuous the range $\{\alpha_t(V(s)) - V(s) : s \in \mathbf{T}, t \in [-1, 1]\}$ is compact. Suppose that $\|\alpha_t(V(s)) - V(s) + \mathcal{J}_n\| < \delta$ for all $(s, t) \in \mathbf{T} \times [-1, 1]$ for a sufficiently small $\delta > 0$. Then there is a $p = p_k$ such that

$$\|(\alpha_t(V(s)) - V(s))(1 - p)\| < \delta$$

for all $(s, t) \in \mathbf{T} \times [-1, 1]$.

Let $P(s) = V(s)pV(s)^*$ for $s \in \mathbf{T}$. Then $s \mapsto P(s)$ defines a projection P in $C(\mathbf{T}) \otimes \mathcal{J}_n$ such that

$$\max_{|t| \leq 1} \|(\text{id} \otimes \alpha_t)(P) - P\| < 2\delta.$$

Hence, for any $\varepsilon' > 0$ by choosing a sufficiently small $\delta > 0$, we may suppose that there is a unitary $U \in C(\mathbf{T}) \otimes (\mathcal{J}_n + \mathbf{C}1)$ and a self-adjoint $b \in C(\mathbf{T}) \otimes \mathcal{J}_n$ such that $\|U - 1\| < \varepsilon'$, $\|b\| < \varepsilon'$ and $(\text{id} \otimes \delta_\alpha + \text{ad } ib)(UPU^*) = 0$. We may also suppose that $U(0) = 1$ and $b(0) = 0$ since $P(0) = p$ is α -invariant. Let $P' = UPU^* = UV(1 \otimes p)V^*U^*$, which implies that $P'(0) = p$. Since $H_0 + b(s)$ commutes with $P'(s)$ we can compute $\text{Spec}((H_0 + b(s))P'(s))$. Let $E_0 = \min_{s \in \mathbf{T}} \min \text{Spec}((H_0 + b(s))P'(s))$ and let $E_1 = \max_{s \in \mathbf{T}} \max \text{Spec}((H_0 + b(s))P'(s))$.

One can find a projection $q \in \mathcal{J}_n$ such that $q \leq 1 - p$, $[H_0, q] = 0$, $\text{Spec}(H_0 q) \subset [E_0, E_1]$, $\text{rank}(q) \gg \text{rank}(p)$ and the eigenvalues of $H_0 q$ are almost equally distributed in $[E_0, E_1]$. In particular, by the last condition we have the following. If $(\lambda_k)_{k=1}^{\text{rank}(q)}$ is the increasing sequence of eigenvalues of $H_0 q$, each repeated according to multiplicity, then $|\lambda_k - \lambda_\ell| < \varepsilon$ for any k and ℓ with $|k - \ell| \leq \text{rank}(p)$.

Let $Q = V(1 \otimes q)V^*$. Then $Q(0) = q$ and

$$\max_{|t| \leq 1} \|(\text{id} \otimes \alpha_t)(Q) - Q\| < 2\delta.$$

One can choose the projection q so that $\|(1 \otimes q)P'\|$ and $\|QP'\|$ are arbitrarily small. One may also suppose that the above $U \in C(\mathbf{T}) \otimes (\mathcal{J}_n + \mathbf{C}1)$ and $b \in C(\mathbf{T}) \otimes \mathcal{J}_n$ are chosen to satisfy $(1 \otimes (\delta_\alpha + \text{ad } ib))(UQU^*) = 0$. This follows by keeping the original relation $P' = UPU^*$. Note, for each $s \in \mathbf{T}$, that $U(s)V(s)q$ is a partial isometry from $q\mathcal{H}_0$ onto $U(s)V(s)q\mathcal{H}_0$ and that

$$\max_{|t| \leq 1} \|e^{it(H_0 + b(s))}U(s)V(s)qe^{-itH_0} - U(s)V(s)q\| < \delta + 3\varepsilon',$$

where we have used $\|b\| < \varepsilon'$, $\|U - 1\| < \varepsilon'$, and

$$\|e^{itH_0}V(s)e^{-itH_0}q - V(s)q\| < \delta, \quad t \in [-1, 1].$$

It follows from the previous lemma that if (λ_i) is the increasing sequence of eigenvalues of $H_0q = H_0q_0$ and (μ_i) is the increasing sequence of eigenvalues of $(H_0 + b(s))q_s$ with $q_s = U(s)V(s)qV(s)^*U(s)^*$ then, by choosing sufficiently small δ and ε' , one has $|\lambda_i - \mu_i| < \varepsilon$ for all i .

If S_r denotes the set of eigenvalues of $H_0r = rH_0$ with r a projection, then $S_{p+q} = S_p \cup S_q$ (which is a disjoint union since we are counting multiplicity). We have assumed that the rank of q is much larger than the rank of p . Hence the cardinality of S_q is much larger than that of S_p . We also established that S_q overwhelms S_p in the sense that if S_{p+q} is arranged in increasing order then the difference of a pair of values which differ by the rank of p in this order is at most ε .

Note that $S_{p+q}(s) = S_p(s) \cup S_q(s)$ for each $s \in \mathbf{T}$, where $S_r(s)$ is the set of eigenvalues of $(H_0 + b(s))U(s)V(s)rV(s)^*U(s)^*$. Since $S_q(s)$ is close to $S_q = S_q(0)$ (as $\|b(s)\| < \varepsilon' \ll \varepsilon$) and S_q overwhelms $S_p(s)$, we can conclude the following. If (λ_i) is the increasing sequence of eigenvalues of $H_0(p + q)$ and $(\lambda_i^{(s)})$ is the increasing sequence of eigenvalues of $(H_0 + b(s))U(s)V(s)(p + q)V(s)^*U(s)^*$ then $|\lambda_i - \lambda_i^{(s)}| < 2\varepsilon$ for all i .

There are continuous functions f_i on $\mathbf{T}$ such that

$$f_i(s) \in \text{Spec}((1 \otimes H_0 + b(s))\text{Ad}(U(s)V(s))(p + q))$$

for all $s \in \mathbf{T}$, $f_i \leq f_{i+1}$, and $f_i(s) = \lambda_i^{(s)}$. By perturbing $1 \otimes H_0 + b$ restricted to $UV(1 \otimes (p + q))V^*U^*$ slightly, we may suppose that there are projections $p_i \in C(\mathbf{T}) \otimes \mathcal{J}_n$ such that $(1 \otimes H_0 + b)(UV(1 \otimes (p + q))V^*U^*) = \sum_i f_i p_i$ (see [1]). By a further perturbation up to 2ε , we may suppose that $f_i(s) = \lambda_i$, $s \in \mathbf{T}$ for all i . By using this, we can construct a partial isometry $W \in C(\mathbf{T}) \otimes \mathcal{J}_n$ such that $W^*W = 1 \otimes (p + q)$, $WW^* = UV(1 \otimes (p + q))V^*U^*$, and $e^{it(1 \otimes H_0 + b)}We^{-itH_0} = W$ for all t , where b now satisfies the estimate $\|b\| < 3\varepsilon$. Since $U(0)V(0)(p + q)V(0)^*U(0)^* = p + q$ one has $W(0) = p + q$. Then it follows that U^*W is a partial isometry from $1 \otimes (p + q)$ to $P + Q = V(1 \otimes (p + q))V^*$ with $(U^*W)(0) = p + q$ and that

$$\max_{|t| \leq 1} \|e^{it(1 \otimes H_0)}U^*We^{-it(1 \otimes H_0)} - U^*W\| \leq 2\|U - 1\| + 2\|b\|.$$

Thus we can set $Z = V(1 - 1 \otimes (p + q)) + U^*W$, which is the desired unitary in $C(\mathbf{T}) \otimes M(\mathcal{J}_n)_\alpha$. $\square$

Lemma 4.8. *If e is a projection in $M(\mathcal{J}_n)_\alpha / \mathcal{J}_n \cap D(\delta_\alpha)$ then there is a projection $E \in M(\mathcal{J}_n)_\alpha \cap D(\delta_\alpha)$ such that $Q(E) = e$.*

Proof. From the proof of Lemma 3.1 we can choose $B \in M(\mathcal{J}_n)_\alpha \cap D(\delta_\alpha)$ such that $Q(B) = e$ and $B^* = B$. Since $Q(B)$ is a projection, the essential spectrum of B is $\{0, 1\}$ (except for the trivial cases), i.e., $\text{Spec}(B)$ has a gap between 0 and 1. We can define a projection E from B by a C^∞ -functional calculus such that $E \in D(\delta_\alpha)$ and $Q(E) = e$. $\square$

When $e_0 \in M(\mathcal{J}_n)_\alpha / \mathcal{J}_n \cap D(\delta_\alpha)$ is a projection such that $\|\delta_\alpha(e_0)\|$ is small we set $h = -i(\delta_\alpha(e_0)e_0 - e_0\delta_\alpha(e_0)) \in M(\mathcal{J}_n)_\alpha / \mathcal{J}_n$. There is an $H \in M(\mathcal{J}_n)_\alpha$ such that $H^* = H$,

$Q(H) = h$, and $\|H\| = \|h\| = \|\delta_\alpha(e_0)\|$. If $E_0 \in M(\mathcal{J}_n)_\alpha \cap D(\delta_\alpha)$ is a projection such that $Q(E_0) = e_0$ as in the above lemma then we have $[H_0 - H, E_0] \in \mathcal{J}_n$. Hence, setting $K = -i((\delta_\alpha - \text{ad } iH)(E_0)E_0 - E_0(\delta_\alpha - \text{ad } iH)(E_0)) \in \mathcal{J}_n$, we have $[H_0 - H - K, E_0] = 0$. Thus we can talk about the essential spectrum of $(H_0 - H - K)E_0$, which is independent of particular choice of H and E_0 , and may be obtained as follows.

We define a homomorphism ϕ of $C_0(\mathbf{R})$ into $e_0(M(\mathcal{J}_n)_\alpha/\mathcal{J}_n)e_0$ by

$$\phi(f) = Q(f(H_0 - H - K)E_0), \quad f \in C_0(\mathbf{R}).$$

Then the kernel of ϕ is identified with $\{f \in C_0(\mathbf{R}): f|_S = 0\}$ for some closed subset S of $\mathbf{R}$. The essential spectrum of $(H_0 - H - K)E_0$ is equal to S .

If the above e_0 belongs to $\mathcal{O}_n$, then $h \in \mathcal{O}_n$ and $H \in \mathcal{E}_n$. The above map ϕ can be regarded as mapping $C_0(\mathbf{R})$ into the crossed product $\mathcal{O}_n \times_\alpha \mathbf{R}$, which is known to be simple. (The C^* -algebra generated by $\mathcal{E}_n$ and $f(H_0)$, $f \in C_0(\mathbf{R})$, is a quotient of $\mathcal{E}_n \times_\alpha \mathbf{R}$. The quotient by the maximal ideal $\mathcal{J}_n \times_\alpha \mathbf{R} \cong \mathcal{J}_n \otimes C_0(\mathbf{R})$ is isomorphic to $\mathcal{O}_n \times_\alpha \mathbf{R}$.) More precisely, ϕ is a homomorphism of $C_0(\mathbf{R})$ into $e_0(\mathcal{O}_n \times \mathbf{R})e_0 = e_0\mathcal{O}_ne_0 \times_{\alpha'} \mathbf{R}$ sending f into $\lambda(f)$ where α' is the flow on $e_0\mathcal{O}_ne_0$ generated by $(\delta_\alpha - \text{ad } ih)|_{e_0\mathcal{O}_ne_0}$ and λ is the canonical unitary group implementing α' in $M(e_0\mathcal{O}_ne_0 \times_{\alpha'} \mathbf{R})$. Hence if e_0 is a non-zero projection in $\mathcal{O}_n$, the essential spectrum of $(H_0 - H - K)E_0$ is always $\mathbf{R}$.

The following is a version of Lemma 4.4 for $M(\mathcal{J}_n)_\alpha/\mathcal{J}_n$ in place of $M(\mathcal{J}_n)_\alpha$.

Lemma 4.9. *For any $\varepsilon > 0$ there exists a $\delta > 0$ satisfying the following condition. If $e \in \mathcal{O}_n \cap D(\delta_\alpha) \subset M(\mathcal{J}_n)_\alpha/\mathcal{J}_n$ is a projection and $u \in (C(\mathbf{T}) \otimes M(\mathcal{J}_n)_\alpha/\mathcal{J}_n) \cap D(\text{id} \otimes \delta_\alpha)$ is a partial unitary such that $u^*u = 1 \otimes e = uu^*$, $\|(\text{id} \otimes \delta_\alpha)u\| < \delta/2$ and $u(0) = e$ then there is a rectifiable path w_s , $s \in [0, 1]$, of partial unitaries in $C(\mathbf{T}) \otimes M(\mathcal{J}_n)_\alpha/\mathcal{J}_n$ such that $w_s^*w_s = 1 \otimes e = w_sw_s^*$, $w_0 = 1 \otimes e$, $w_1 = u$, $w_s(0) = e$, $\max_{|t| \leq 1} \|\alpha_t(w_s) - w_s\| < \varepsilon$, and the length of w_s is less than $4\pi + \varepsilon$.*

Proof. Fix some $\delta > 0$ and choose e and u satisfying the hypotheses of the lemma.

Let $h = -i(\delta_\alpha(e)e - e\delta_\alpha(e)) \in M(\mathcal{J}_n)_\alpha/\mathcal{J}_n$. Then $\delta_\alpha(e) = \text{ad } ih(e)$ and $\|\delta_\alpha(e)\| < \delta$ because of the properties of e and u . We find a self-adjoint $H \in \mathcal{E}_n \subset M(\mathcal{J}_n)_\alpha$ such that $Q(H) = h$ and $\|H\| = \|h\|$.

Let $E \in \mathcal{E}_n \cap D(\delta_\alpha)$ be a projection such that $Q(E) = e$. Note that $(\delta_\alpha - \text{ad } iH)(E) \in \mathcal{J}_n$. Hence there is a self-adjoint $K \in \mathcal{J}_n$ such that

$$(\delta_\alpha - \text{ad } i(H + K))(E) = 0.$$

Since $u(0) = e$, we can find a continuous path $U(s)$, $s \in [0, 1]$, in $\mathcal{U}(EM(\mathcal{J}_n)_\alpha E)$ such that $U(0) = E$ and $Q(U(s)) = u(s)$, where $\mathbf{T}$ is regarded as the quotient space $[0, 1]/\{0, 1\}$. Since $Q(U(1)) = e$ or $U(1) \in E(\mathcal{J}_n + 1)E$ and the unitary group of $E(\mathcal{J}_n + 1)E$ is connected, we may suppose that $U(1) = E$, i.e., we have a lifting of u to the unitary U in $C(\mathbf{T}) \otimes EM(\mathcal{J}_n)_\alpha E$ such that $U(0) = E$. Let β be the flow on $M(\mathcal{J}_n)_\alpha$ generated by $\delta_\alpha - \text{ad } iH - \text{ad } iK$ which leaves E invariant, i.e., β_t is implemented by $e^{it(H_0 - H - K)}$. Since $Q(\beta_t(U(s)) - U(s)) = \alpha_t^{(-h)}(u(s)) - u(s)$ it follows that

$$\max_{|t| \leq 1} \|\beta_t(U(s)) - U(s) + \mathcal{J}_n\| < 2\|h\| + \delta < 3\delta,$$

where $\alpha^{(-h)}$ is the flow on $M(\mathcal{J}_n)_\alpha$ generated by $\delta_\alpha - \text{ad } ih$. We have remarked that the essential spectrum of $(H_0 - H - K)E$ is $\mathbf{R}$, where we may suppose that $(H_0 - H - K)E$ is diagonal by changing K if necessary. We apply Lemmas 4.7 and 4.4 to the flow $\beta|_{M(E\mathcal{J}_nE)_\beta}$ in place of $\alpha|_{M(\mathcal{J}_n)_\alpha}$ where $E\mathcal{J}_nE$ is the compact operators on $E\mathcal{H}_0$ since $\mathcal{J}_n$ is the compact operators on $\mathcal{H}_0$ and $M(E\mathcal{J}_nE)_\beta = EM(\mathcal{J}_n)_\beta E = EM(\mathcal{J}_n)_\alpha E$. This is possible because the properties we needed for $\alpha_t = \text{Ad } e^{itH_0}$ on $M(\mathcal{J}_n)_\alpha$ in these lemmas are that $\text{Spec}(H_0) = \mathbf{R}$ and that H_0 is diagonal. The same properties hold for $\beta_t = \text{Ad } e^{it(H_0 - H - K)E}$ on $M(E\mathcal{J}_nE)_\beta$ as asserted above.

We apply Lemma 4.7 to the unitary U in $C(\mathbf{T}) \otimes EM(\mathcal{J}_n)_\alpha E$ (or rather to $Q(U)$) with the Hilbert space $E\mathcal{H}_0$ and the flow $\beta|_{EM(\mathcal{J}_n)_\alpha E}$. Thus we can choose a unitary $V \in C(\mathbf{T}) \otimes EM(\mathcal{J}_n)_\alpha E$ such that $V(0) = E$, $U(s) - V(s) \in \mathcal{J}_n$ for $s \in \mathbf{T}$ and $\max_{|t| \leq 1} \|\beta_t(V) - V\| < \varepsilon'$, where the value of ε' depends on δ and $\varepsilon' \rightarrow 0$ as $\delta \rightarrow 0$. Then we can invoke Lemma 4.4 to find a rectifiable path W_s , $s \in [0, 1]$, in $\mathcal{U}(C(\mathbf{T}) \otimes EM(\mathcal{J}_n)_\alpha E)$ such that $W_0 = E$, $W_1 = V$, $W_s(0) = E$, $\max_{|t| \leq 1} \|\beta_t(W_s) - W_s\| < \varepsilon$, and the length of W_s is less than $4\pi + \varepsilon$. Again the value of ε depends on δ and $\varepsilon \rightarrow 0$ as $\delta \rightarrow 0$. Now we can set $w_s = Q(W_s)$ and this satisfies all the required conditions. In particular, since $\max_{|t| \leq 1} \|\alpha_t^{(-h)}(w_s) - w_s\| < \varepsilon$, it follows that $\max_{|t| \leq 1} \|\alpha_t(w_s) - w_s\| < \varepsilon + 2\delta$. $\square$

Lemma 4.10. *For any $\varepsilon > 0$ there exists a $\delta > 0$ satisfying the following condition. If $V \in C(\mathbf{T}) \otimes M(\mathcal{J}_n)_\alpha$ is a unitary such that $V(0) = 1$ at $0 \in \mathbf{T}$ and*

$$\max_{|t| \leq 1} \|(\text{id} \otimes \alpha_t)(V) - V\| < \delta,$$

then there is a unitary $Z \in C(\mathbf{T}) \otimes M(\mathcal{J}_n)_\alpha$ such that $Z(0) = 1$, $\|Q(V) - Q(Z)\lambda(Q(Z))^*\| < \varepsilon$ and

$$\max_{|t| \leq 1} \|(\text{id} \otimes \alpha_t)(Z) - Z\| < \varepsilon.$$

Proof. Let $N \in \mathbf{N}$ and $M = n^N$. We assume that $M^{-1} < \varepsilon$. By Lemma 4.5 there are projections e_i , $i = 0, \dots, M-1$, in $\mathcal{O}_n \cap D(\delta_\alpha) \subset M(\mathcal{J}_n)_\alpha/\mathcal{J}_n$ such that $\sum_i e_i = 1$, $\|\delta_\alpha(e_i)\| < \varepsilon'$, and $\|\lambda(e_i) - e_i\| < \varepsilon'$ for an arbitrarily small $\varepsilon' > 0$.

For each $s \in \mathbf{T}$ there is a natural isomorphism ϕ_s of $\mathcal{O}_n = C^*(Q(s_1), \dots, Q(s_n))$ onto $C^*(Q(V(s)s_1), \dots, Q(V(s)s_n)) \subset M(\mathcal{J}_n)_\alpha$. Note that $s \mapsto \phi_s$ is continuous in the sense that $s \mapsto \phi_s(x)$ is continuous for each $x \in \mathcal{O}_n$. Since $\phi_s \lambda|_{\mathcal{O}_n} = \text{Ad } Q(V(s))\lambda\phi_s|_{\mathcal{O}_n}$ the projections $f_i(s) = \phi_s(e_i)$ satisfy $\sum_i f_i(s) = 1$ and $\|\text{Ad } Q(V(s))\lambda(f_i(s)) - f_{i+1}(s)\| < \varepsilon'$. Since $\alpha_t(V(s)) \approx V(s)$ or $\phi_s \alpha_t|_{\mathcal{O}_n} \approx \alpha_t \phi_s|_{\mathcal{O}_n}$ depending on δ , we can also ensure that

$$\max_{|t| \leq 1} \|\alpha_t(f_i(s)) - f_i(s)\| < \varepsilon',$$

by choosing $\delta > 0$ sufficiently small. Note that $f_i : s \mapsto f_i(s)$ defines a projection in $C(\mathbf{T}) \otimes M(\mathcal{J}_n)_\alpha$.

We assert that there is a partial isometry $z \in C(\mathbf{T}) \otimes M(\mathcal{J}_n)_\alpha/\mathcal{J}_n$ such that $z^*z = 1 \otimes e_0$, $zz^* = f_0$, and $\max_{|t| \leq 1} \|(\text{id} \otimes \alpha_t)z - z\|$ is small.

Let $h = -i(\delta_\alpha(e_0)e_0 - e_0\delta_\alpha(e_0)) \in (\mathcal{O}_n)_{sa}$. Then $(\delta_\alpha - \text{ad } ih)(e_0) = 0$.

From the proof of Lemma 4.5 e_0 is defined as w^*p_0w , where p_0 is a projection in $\mathcal{O}_n^\gamma \subset \text{UHF}_n$ and w is an isometry in $\mathcal{O}_n$ with $ww^* \in \mathcal{O}_n^\gamma$ and $ww^* \geq p_0$. We may suppose that p_0 and ww^* are in the linear span of $Q(s_I s_J^*)$, $|I| = |J| = L$ for a large $L \in \mathbf{N}$. Since $\phi_s(s_I s_J^*) = Q(V(s))_L s_I s_J^* Q(V(s))_L^*$ for $I, J \in \{1, 2, \dots, n\}^*$ with $|I| = |J| \leq L$ it follows that

$$f_0(s) = \phi_s(e_0) = \phi_s(w^*)Q(V(s))_L p_0 Q(V(s))_L^* \phi_s(w),$$

where $Q(V(s))_L$ is defined inductively by $Q(V(s))_0 = 1$ and

$$Q(V(s))_L = Q(V(s))\lambda(Q(V(s))_{L-1}).$$

Thus

$$z(s) = \phi_s(w^*)Q(V(s))_L we_0, \quad s \in \mathbf{T},$$

has the required property as a partial isometry from $1 \otimes e_0$ onto f_0 if we choose δ sufficiently small. For example, $z(s)^*z(s) = e_0 w^* Q(V(s))_L^* \phi_s(w w^*) Q(V(s))_L we_0 = e_0 w^* \cdot w w^* \cdot we_0 = e_0$. Since $V(0) = 1$ and $\phi_0 = \text{id}$ one has $z(0) = e_0$. (Given $\varepsilon > 0$ we choose $N \in \mathbf{N}$ and the projections e_i for $i = 0, 1, \dots, n^N - 1$; in particular, the above w , p_0 , and L . Then making $\delta > 0$ sufficiently small depending on w and L we deduce that $\phi_s(w^*)$ and $Q(V)_L$ are almost α -invariant which ensures that z is almost $\text{id} \otimes \alpha$ -invariant.)

Now we proceed as follows. Choose unitaries ζ, η in $C(\mathbf{T}) \otimes M(\mathcal{J}_n)_\alpha / \mathcal{J}_n$ such that $\zeta(s) = \zeta(0) = \eta(0)$, $\zeta \approx 1$, $\text{Ad } \zeta \lambda(1 \otimes e_i) = 1 \otimes e_{i+1}$, $\eta \approx 1$, and $\text{Ad}(\eta Q(V))\lambda(f_i) = f_{i+1}$, where λ denotes the endomorphism $\text{id} \otimes \lambda$ on $C(\mathbf{T}) \otimes M(\mathcal{J}_n)_\alpha / \mathcal{J}_n$. Define a map μ on $C(\mathbf{T}) \otimes M(\mathcal{J}_n)_\alpha / \mathcal{J}_n$ by $\mu = L_{\eta Q(V)} R_{\zeta^* \lambda}$, where L_x (respectively R_x) is the left (respectively right) multiplication of x . Note that $\mu(z) = \eta Q(V)\lambda(z)\zeta^*$ is a partial isometry from $1 \otimes e_1$ to f_1 , etc. Set

$$y = z^* \mu^M(z) \approx z^* Q(V)\lambda(Q(V)) \cdots \lambda^{M-1}(Q(V))\lambda^M(z).$$

This is a unitary in $C(\mathbf{T}) \otimes e_0(M(\mathcal{J}_n)_\alpha / \mathcal{J}_n)e_0$ such that $(\text{id} \otimes \alpha_t)(y) \approx y$ for $t \in [-1, 1]$ depending on $\delta > 0$. Since $V(0) = 1$ and $z(0) = e_0$ we deduce that $y(0) = e_0$.

By assuming $\delta > 0$ sufficiently small we can now invoke Lemma 4.9, i.e., we can find a rectifiable path y_s , $s \in [0, 1]$, in $C(\mathbf{T}) \otimes M(\mathcal{J}_n)_\alpha / \mathcal{J}_n$ such that $y_s y_s^* = 1 \otimes e_0 = y_s y_s^*$, $y_0 = 1 \otimes e_0$, $y_1 = y$, $y_s(0) = e_0$, $\|\alpha_t(y_s) - y_s\| \approx 0$ for $t \in [-1, 1]$ and the length of y_s is less than 5π .

We are now in a familiar situation. Define a unitary $u \in C(\mathbf{T}) \otimes M(\mathcal{J}_n)_\alpha / \mathcal{J}_n$ by

$$u = \sum_{k=0}^{M-1} \mu^k(z y_k),$$

where (y_k) is a finite sequence of unitaries in $C(\mathbf{T}) \otimes e_0(M(\mathcal{J}_n)_\alpha / \mathcal{J}_n)e_0$ chosen from the above path y_s as follows: $y_0 = y$, $\|y_k - y_{k+1}\| < 5\pi/M$, $y_k(0) = e_0$, and $y_{M-1} = e_0$. Note that $\mu^M(z) = zy$. Then u satisfies $\|\mu(u) - u\| < 5\pi/M$, i.e., $Q(V)\lambda(u) \approx u$. It also follows that $u(0) = 1$ and $(\text{id} \otimes \alpha_t)(u) \approx u$ for $t \in [-1, 1]$. Then we can use Lemma 4.7 to lift u to a unitary Z in $C(\mathbf{T}) \otimes M(\mathcal{J}_n)_\alpha$ such that $Q(Z) \approx u$, $Z(0) = 1$, and $(\text{id} \otimes \alpha_t)Z \approx Z$ for $t \in [-1, 1]$. Then Z satisfies the required conditions. $\square$

Proof of Lemma 3.4. Let δ be a positive number whose value will be decided in the course of the proof. Suppose that we are given a unitary $V \in M(\mathcal{J}_n)_\alpha$ such that $Q(V) \in (\mathcal{O}_n)'$ and $\max_{|t| \leq 1} \|\alpha_t(V) - V\| < \delta$. Then by applying Lemma 4.1 there is a rectifiable path V_s , $s \in [0, 1]$, in $\mathcal{U}(M(\mathcal{J}_n)_\alpha)$ such that $V_0 = 1$, $V_1 = V$, and

$$\varepsilon \equiv \max_{s \in [0, 1]} \max_{|t| \leq 1} \|\alpha_t(V_s) - V_s\|$$

is very small for small δ .

Since $Q(V\lambda(V)^*) = 1$, the map $s \in [0, 1] \mapsto Q(V_s)\lambda(Q(V_s)^*)$ defines a unitary $w \in C(\mathbf{T}) \otimes M(\mathcal{J}_n)_\alpha/\mathcal{J}_n$ such that $w(0) = 1$ and $\max_{|t| \leq 1} \|(\text{id} \otimes \alpha_t)(w) - w\| \leq 2\varepsilon$ for $t \in [-1, 1]$. By 4.7 we obtain a unitary $W \in C(\mathbf{T}) \otimes M(\mathcal{J}_n)_\alpha$ such that $Q(W) = w$, $W(0) = 1$, and $\max_{|t| \leq 1} \|(\text{id} \otimes \alpha_t)(W) - W\| \equiv \varepsilon'$ is small for small ε . Then by Lemma 4.10 we obtain a unitary $Z \in C(\mathbf{T}) \otimes M(\mathcal{J}_n)_\alpha$ such that $Z(0) = 1$, $w = Q(W) \approx Q(Z^*)\lambda(Q(Z))$, and $\max_{|t| \leq 1} \|(\text{id} \otimes \alpha_t)(Z) - Z\|$ is small for small ε' . Set $\tilde{V}_s = Z(s)V_s$, $s \in [0, 1]$. Then $\tilde{V}_0 = 1$, $\tilde{V}_1 = V$, $\lambda(Q(\tilde{V}_s)) \approx Q(\tilde{V}_s)$, and $\max_{s \in [0, 1]} \max_{|t| \leq 1} \|\alpha_t(\tilde{V}_s) - \tilde{V}_s\| \approx 0$. Thus we find the desired path $\tilde{V}_s$, $s \in [0, 1]$, connecting 1 with V in $\mathcal{U}(M(\mathcal{J}_n)_\alpha)$. This concludes the proof. $\square$

5. Cocycle conjugacy

A C^* -subalgebra C of A is a masa if and only if $A \cap C' = C$. If C is a masa of A and $u \in \mathcal{U}(A)$ then u is said to normalize C if $uCu^* = C$. We denote by $\mathcal{N}(C)$ the set of those $u \in \mathcal{U}(A)$ normalizing C . Note that $\mathcal{N}(C) \supset \mathcal{U}(C)$ and $\mathcal{N}(C)$ is a closed subgroup of $\mathcal{U}(A)$. When $\mathcal{N}(C)$ generates A as a C^* -algebra (or equivalently the closed linear span of $\mathcal{N}(C)$ is A), C is called a regular masa of A .

If a regular masa C of A satisfies (1) there is a norm-one projection of A onto C and (2) there is a character of C which uniquely extends to a state of A then we call C a Cartan masa of A .

Recall that if α is a flow on a C^* -algebra A then δ_α denotes its generator and the domain $D(\delta_\alpha)$ of δ_α is a dense $*$ -subalgebra of A .

Theorem 5.1. *Let A be a unital simple C^* -algebra and let C be a Cartan masa of A .*

Let α be a flow on A such that $D(\delta_\alpha) \supset C$. Then the following conditions are equivalent:

1. $\sup\{\|\alpha_t\delta_\alpha(x) - \delta_\alpha(x)\| : x \in C, \|x\| \leq 1\} \rightarrow 0$ as $t \rightarrow 0$.
2. $\delta_\alpha|_C$ is inner; i.e., there is an $h = h^* \in A$ such that $\delta_\alpha(x) = \text{ad } ih(x)$, $x \in C$.

Remark 5.2. The first condition of the above theorem follows if $D(\delta_\alpha^2) \supset C$. This follows since $\delta_\alpha^2|_C$ is bounded (see, e.g., [2]) and

$$\|\alpha_t\delta_\alpha(x) - \delta_\alpha(x)\| \leq \|\delta_\alpha^2|_C\| |t| \|x\|$$

for $x \in C$.

In the above theorem it is obvious that condition 2 implies condition 1 because using h in 2 one has the estimate

$$\|\alpha_t\delta_\alpha(x) - \delta_\alpha(x)\| \leq 2(\|\alpha_t(h) - h\| \|x\| + \|h\| \|\delta_\alpha|_C\| |t|), \quad x \in C.$$

Now assume 1. To derive condition 2 we establish the following lemmas, the first of which is proved in a more general context.

Lemma 5.3. *Let α be a flow on a C^* -algebra A and C an abelian C^* -subalgebra of A such that $C \subset D(\delta_\alpha)$. Then for each character ω of C there is a state φ of A such that $\|\varphi\alpha_t - \varphi\| \rightarrow 0$ as $t \rightarrow 0$ and $\varphi|_C = \omega$.*

Proof. Let ω be a character of C . Then there is a decreasing net (z_ν) in $\{x \in C: \|x\| \leq 1\}$ such that $\omega(z_\nu) = 1$ and (z_ν) converges to the minimal projection $p \in C^{**}$ with $\omega(p) = 1$. We regard $C^{**} \subset A^{**}$ and set $c = \|\delta_\alpha|_C\|$. Since $\|\alpha_t(z_\nu) - z_\nu\| \leq c|t|$ one deduces that $\|\alpha_t^{**}(p) - p\| \leq c|t|$, i.e., $t \mapsto \alpha_t^{**}(p)$ is norm-continuous. Let B be the C^* -subalgebra of A^{**} generated by A and $\alpha_t^{**}(p)$, $t \in \mathbf{R}$. Then B is left invariant under α^{**} and the restriction $\beta = \alpha^{**}|_B$ is a flow, i.e., is strongly continuous. There is a β -cocycle u such that $\text{Ad } u_t \beta_t(p) = p$. Hence there is an $\text{Ad } u\beta$ -invariant pure state $\hat{\varphi}$ of B such that $\hat{\varphi}(p) = 1$. Let $\varphi = \hat{\varphi}|_A$. Since $\alpha = \beta|_A = \alpha^{**}|_A$ one has $\|\varphi\alpha_t - \varphi\| \leq \|\hat{\varphi}\beta_t - \hat{\varphi}\|$. Since $\|\hat{\varphi}\beta_t - \hat{\varphi}\| \leq \|(\hat{\varphi} - \hat{\varphi}\text{Ad } u_t)\beta_t\| \leq 2\|u_t - 1\|$, one then has $\|\varphi\alpha_t - \varphi\| \leq 2\|u_t - 1\|$. Since $\varphi|_C = \omega$, this concludes the proof. $\square$

Lemma 5.4. *Let ω be a character of C such that ω uniquely extends to a state φ of A , which is necessarily a pure state. Then $\pi_\varphi(C)''$ is a completely atomic masa of $\pi_\varphi(A)'' = B(\mathcal{H}_\omega)$, i.e., there is a family $\{p_i\}$ of one-dimensional projections on $\mathcal{H}_\varphi$ such that $\pi_\varphi(C)''$ is generated by the p_i . If Φ is a norm-one projection of A onto C then $\pi_\varphi\Phi(x) = \sum_i p_i\pi_\varphi(x)p_i$ for $x \in A$.*

Proof. The support projection p of ω in C^{**} is minimal in A^{**} . Hence $\pi_\varphi^{**}(p) \equiv p$ is a one-dimensional projection belonging to $\pi_\varphi(C)''$. Since C is regular the supremum of $\pi_\varphi(u)p\pi_\varphi(u)^* \in \pi_\varphi(C)''$ over $u \in \mathcal{N}(C)$ commutes with $\pi_\varphi(A)''$. Hence it equals 1. Thus we conclude that there is a family $\{p_i\}$ of one-dimensional projections in $\pi_\varphi(C)''$ such that $\sum_i p_i = 1$.

Let $x \in A$. If $z \in C$ satisfies that $0 \leq z \leq 1$ and $\pi_\varphi(z)\Omega_\varphi = \Omega_\varphi$, then

$$\pi_\varphi\Phi(x)\Omega_\varphi = \pi_\varphi\Phi(x)\pi_\varphi(z^2)\Omega_\varphi = \pi_\varphi\Phi(zxz)\Omega_\varphi.$$

Since $zxz \approx \varphi(x)z^2$ for a suitably chosen z one deduces that

$$\pi_\varphi\Phi(x)\Omega_\varphi = \varphi(x)\Omega_\varphi = p\pi_\varphi(x)p\Omega_\varphi$$

for $x \in A$, where p is the projection onto $\mathbf{C}\Omega_\varphi$. Let $u \in \mathcal{N}(C)$. Since $x \in C \mapsto \langle \pi_\varphi(x)\pi_\varphi(u)\Omega_\varphi, \pi_\varphi(u)\Omega_\varphi \rangle = \omega(u^*xu)$ is also a character one may apply the same argument and deduce that if $u \in \mathcal{N}(C)$ then

$$\pi_\varphi\Phi(x)\pi_\varphi(u)\Omega_\varphi = \varphi(u^*xu)\pi_\varphi(u)\Omega_\varphi = p_u\pi_\varphi(x)p_u\pi_\varphi(u)\Omega_\varphi = \left(\sum_i p_i\pi_\varphi(x)p_i\right)\pi_\varphi(u)\Omega_\varphi,$$

for $x \in A$, where $p_u = \pi_\varphi(u)p\pi_\varphi(u)^* \in \pi_\varphi(C)''$ is the projection onto $\mathbf{C}\pi_\varphi(u)\Omega_\varphi$ and $\{p_i\}$ is chosen in the first paragraph. This concludes the proof. $\square$

Lemma 5.5. *Under condition 1 of the theorem it follows that $\mathcal{N}(C) \subset D(\delta_\alpha)$.*

Proof. Let ω be a character of C which uniquely extends to a state, say φ , of A . Note that φ is a pure state of A . From Lemma 5.3 it follows that $\|\varphi\alpha_t - \varphi\| \rightarrow 0$ as $t \rightarrow 0$. Hence there is a unitary flow U on $\mathcal{H}_\varphi$ such that $\text{Ad } U_t \pi_\varphi(x) = \pi_\varphi(\alpha_t(x))$, $x \in A$. Let H denote the self-adjoint generator of U .

Let h be an invariant mean of the function on the abelian group $\mathcal{U}(C)$ defined by $u \mapsto -i\pi_\varphi(\delta_\alpha(u)u^*) \in B(\mathcal{H}_\varphi)$. Then it follows that $h^* = h \in B(\mathcal{H}_\varphi)$ and $\text{ad } ih(\pi_\varphi(x)) = \pi_\varphi(\delta_\alpha(x))$, $x \in C$. We argue that $t \mapsto \text{Ad } U_t(h)$ is norm-continuous.

Let $c = \|\delta_\alpha|_C\|$. One has

$$\|\alpha_t(\delta_\alpha(u)u^*) - \delta_\alpha(u)u^*\| \leq \|\delta_\alpha(\alpha_t(u) - u)\| + c^2|t|$$

for $u \in \mathcal{U}(C)$. Since $\text{Ad } U_t(ih) - ih$ is the invariant mean of $u \mapsto \pi_\varphi(\alpha_t(\delta_\alpha(u)u^*) - \delta_\alpha(u)u^*)$, it follows that

$$\|\text{Ad } U_t(h) - h\| \leq \sup_{u \in \mathcal{U}(C)} \|\delta_\alpha(\alpha_t(u) - u)\| + c^2|t|.$$

Thus $t \mapsto \text{Ad } U_t(h)$ is norm-continuous.

Let B be the C^* -algebra generated by $\pi_\varphi(A)$ and $\text{Ad } U_t(h)$, $t \in \mathbf{R}$. Then $\text{Ad } U_t$ leaves B invariant and $x \mapsto \text{Ad } U_t(x)$ is norm-continuous for $x \in B$. Hence $t \mapsto \text{Ad } U_t|_B$ defines a flow on B , which we denote by β . Since $h \in B$, $\text{Ad } e^{it(H-h)}$ leaves B invariant and defines a perturbed flow $\beta^{(-h)}$, which fixes each element of $\pi_\varphi(C)$, i.e., $e^{it(H-h)} \in \pi_\varphi(C)'$.

Let $u \in \mathcal{N}(C)$ and define

$$W_t = \pi_\varphi(u)\beta_t^{(-h)}(\pi_\varphi(u^*)), \quad t \in \mathbf{R}.$$

Then W_t is a unitary in B and $t \mapsto W_t$ is a $\beta^{(-h)}$ -cocycle. For $z \in \mathcal{U}(C)$ one has $[\pi_\varphi(z), W_t] = 0$ which implies that $W_t \in B \cap \pi_\varphi(C)' \subset \pi_\varphi(C)''$. Since $\beta_t^{(-h)}|_{B \cap \pi_\varphi(C)'}$ is the identity map it follows that $t \mapsto W_t$ is a norm-continuous unitary flow, i.e., there is a $k = k^* \in B \cap \pi_\varphi(C)''$ such that $W_t = e^{ikt}$. Thus it follows that $\pi_\varphi(u)$ is in the domain of the generator of $\beta^{(-h)}$ or β . Since $\beta_t \pi_\varphi(u) = \pi_\varphi \alpha_t(u)$ and π_φ is isometric we deduce that $u \in D(\delta_\alpha)$. This completes the proof of the inclusion $\mathcal{N}(C) \subset D(\delta_\alpha)$. $\square$

It follows from the above proof that for $u \in \mathcal{N}(C)$,

$$\pi_\varphi(u\delta_\alpha(u^*)) - \pi_\varphi(u) \text{ad } ih(\pi_\varphi(u^*)) = ik,$$

where $k = k^* \in \pi_\varphi(C)''$.

Let $\bar{\Phi}$ be the norm-one projection of $B(\mathcal{H}_\varphi)$ onto $\pi_\varphi(C)''$. For $z \in C$ one has

$$\pi_\varphi \Phi \delta_\alpha(z) = \bar{\Phi} \pi_\varphi \delta_\alpha(z) = [i\bar{\Phi}(h), \pi_\varphi(z)] = 0.$$

Since h is the invariant mean of $-i\pi_\varphi(\delta_\alpha(z)z^*)$, $z \in \mathcal{U}(C)$, and

$$\bar{\Phi}(-i\pi_\varphi(\delta_\alpha(z)z^*)) = -i\pi_\varphi(\bar{\Phi}(\delta_\alpha(z)z^*)) = 0,$$

we deduce that $\bar{\Phi}(h) = 0$. Using $\bar{\Phi}(\pi_\varphi(u)h\pi_\varphi(u^*)) = \pi_\varphi(u)\bar{\Phi}(h)\pi_\varphi(u^*) = 0$ for $u \in \mathcal{N}(C)$ we deduce from the previous paragraph that

$$\pi_\varphi(\Phi(u\delta_\alpha(u^*))) = ik,$$

i.e., $k \in \pi_\varphi(C)$. Hence

$$\text{ad } ih(\pi_\varphi(u^*)) = \pi_\varphi(\delta_\alpha(u^*)) - \pi_\varphi(u^*)ik \in \pi_\varphi(A).$$

Since this is the case for all $u \in \mathcal{N}(C)$ we can conclude that $\text{ad } ih$ defines a bounded derivation on $\pi_\varphi(A)$. Since $\pi_\varphi(A)$ is unital and simple and $\pi_\varphi(A)'' = B(\mathcal{H}_\varphi)$ this implies that $h \in \pi_\varphi(A)$ (because any derivation of a unital simple C^* -algebra is inner). Since $\text{ad } ih \pi_\varphi(z) = \pi_\varphi(\delta_\alpha(z))$, $z \in C$, this concludes the proof of Theorem 5.1.

In the above theorem, the regularity of the masa C is not really needed. We call a masa C of A weakly regular if

$$\mathcal{N}_w(C) = \{u \in \mathcal{PI}(A) : uu^*, u^*u \in C, uCu^* = Cuu^*\}$$

generates A , where $\mathcal{PI}(A)$ is the set of partial isometries of A . If a weakly regular masa C satisfies the conditions: (1) there is a norm-one projection of A onto C and (2) there is a character of C which uniquely extends to a state of A , then we call C a weak Cartan masa. We deduce the following from the above proof.

Corollary 5.6. *Let A be a unital simple C^* -algebra and C a weak Cartan masa of A .*

Let α be a flow on A such that $D(\delta_\alpha) \supset C$. Then the following conditions are equivalent.

1. $\sup\{\|\alpha_t \delta_\alpha(x) - \delta_\alpha(x)\| : x \in C, \|x\| \leq 1\} \rightarrow 0$ as $t \rightarrow 0$.
2. δ_α is inner, i.e., there is an $h = h^* \in A$ such that $\delta_\alpha(x) = \text{ad } ih(x)$, $x \in C$.

Remark 5.7. Let A be a unital simple AF C^* -algebra. Let C be a masa of A . We call C a canonical AF masa if there is an increasing sequence (A_n) of finite-dimensional C^* -subalgebras of A such that the union is dense in A and C is generated by $C \cap A_n \cap A'_{n-1}$, $n = 1, 2, \dots$, with $A_0 = 0$. If C is a canonical AF masa of A then the pair (A, C) satisfies the assumptions of Theorem 5.1. Moreover, if α is a flow on A such that $D(\delta_\alpha) \supset C$, then $\delta_\alpha|_C$ is inner [16]. Note also that a canonical AF masa of A is unique up to transformation by automorphisms. Hence the assumption $\gamma(C) = C$ below can be attained for a given automorphism γ_* of $K_0(B)$.

Proposition 5.8. *Let B be a stable AF C^* -algebra, C a canonical AF masa, and γ an automorphism of B such that $\gamma(C) = C$ and γ fixes no non-trivial ideals of B . Suppose that $\gamma_*(g)$ is strictly smaller than g for any $g \in K_0(B)_+$, i.e., if ϕ is a positive homomorphism of $K_0(B)$ into $\mathbf{R}$ such that $\phi(g) > 0$, then $\phi(g) > \phi\gamma_*(g)$. Let e be a non-zero projection in C such that $\gamma(e) \leq e$. Let $A = e(B \times_\gamma \mathbf{Z})e$ and regard $C = Ce$ as a C^* -subalgebra of A . Then it follows that*

1. A is a unital simple C^* -algebra.
2. There exists a norm-one projection of A onto C and C is weakly regular.
3. There exists a character of C which extends uniquely to a state of A .

If, in addition, B is simple then C is a regular masa of A .

Proof. Since the Connes spectrum of γ is full (otherwise some power of γ would be universally weakly inner on some non-zero ideal of B), it is well known that $B \times_\gamma \mathbf{Z}$ is simple. Thus assertion 1 follows.

Using the dual action of $\mathbf{T}$ on $B \times_\gamma \mathbf{Z}$ one obtains a norm-one projection of $A = e(B \times_\gamma \mathbf{Z})e$ onto eBe . One also has a norm-one projection of eBe onto $C = Ce$. Composing these projections gives the desired norm-one projection from A onto C . Let U be the canonical unitary multiplier of $B \times_\gamma \mathbf{Z}$ implementing γ . Then $U^n e \in e(B \times_\gamma \mathbf{Z})e$ for $n \geq 1$ is an isometry normalizing C . In

combination with the fact $\mathcal{N}_w(\mathcal{C}) \cap eBe$ generates B , this implies that $\mathcal{C}$ is weakly regular. Thus assertion 2 follows.

To prove 3 we assert that there is a decreasing sequence (p_k) of non-zero projections in $\mathcal{C}$ such that $\|p_k u \gamma^n(p_k)\| \rightarrow 0$ for all $u \in \mathcal{N}_w(\mathcal{C}) \cap eBe$ for all $n \in \mathbf{N}$. Then we take a character ω of $\mathcal{C}$ such that $\omega(p_k) = 1$ for all k . If φ is an extension of ω to a state of A then $\varphi(xU^n) = \lim_k \varphi(p_k x \gamma^n(p_k) U^n) = 0$ for all $x \in eBe$ and all $n = 1, 2, \dots$, where we have used that $\mathcal{C}$ is a regular masa of eBe . This implies that φ is uniquely determined by ω .

To prove the above assertion on (p_k) , let (u_m, n_m) , $m = 1, 2, \dots$, be a dense sequence in $(\mathcal{N}_w(\mathcal{C}) \cap eBe) \times \mathbf{N}$. Suppose that we have chosen a non-zero projection $p \in \mathcal{C}$ such that $p u_m \gamma^{n_m}(p) = 0$ for $m < \ell$. Then we have to find a non-zero projection $p' \in \mathcal{C}$ such that $p' \leq p$ and $p' u_\ell \gamma^{n_\ell}(p') = 0$. Since $q = \text{Ad } u_\ell \gamma^{n_\ell}(p)$ is a projection in $\mathcal{C}$ whose equivalence class is strictly smaller than $[p]$ we deduce that $p(1 - q) \neq 0$. Thus we may set $p' = p(1 - q)$. One obtains the desired sequence of projections by repeating this procedure.

Recall that we have set $\mathcal{N}(\mathcal{C}) = \{u \in \mathcal{U}(A) : u\mathcal{C}u^* = \mathcal{C}\}$. To prove the last statement we have to show that $\mathcal{N}(\mathcal{C})$ generates A . Note that $\lim_{k \rightarrow \infty} \phi([\gamma^k(e)]) = 0$ for any positive homomorphism ϕ of $K_0(B)$ into $\mathbf{R}$ (otherwise $\lim_k \phi \gamma^k$ would define a non-zero γ -invariant positive homomorphism ψ of $K_0(B)$ into $\mathbf{R}$ such that $\psi([e]) > 0$). We choose $k > 1$ such that $[\gamma^{k-1}(e)] + [\gamma^k(e)] \leq [e]$ and then choose $v, w \in \mathcal{N}(\mathcal{C}) \cap eBe$ such that

$$\gamma^k(e) \perp \text{Ad } v \gamma^{k-1}(e), \quad \gamma^{k-1}(e) \perp \text{Ad } w \gamma^k(e),$$

and

$$\gamma^k(e) + \text{Ad } v \gamma^{k-1}(e) = \gamma^{k-1}(e) + \text{Ad } w \gamma^k(e) \equiv f.$$

Then it follows that

$$S^k S^{*k-1} + v S^{k-1} S^{*k} w^* + e - f \in \mathcal{N}(\mathcal{C}),$$

where $S = Ue$. Hence by multiplying $\gamma^{k-1}(e) \in \mathcal{C}$ from the right we deduce that $S^k S^{*k-1} = S \gamma^{k-1}(e)$ belongs to the C^* -algebra $C^*(\mathcal{N}(\mathcal{C}))$ generated by $\mathcal{N}(\mathcal{C})$. For $u \in \mathcal{N}(\mathcal{C}) \cap eBe$ we have that $\gamma(u) + e - \gamma(e) \in \mathcal{N}(\mathcal{C})$ and that $\gamma(u) S \gamma^{k-1}(e) u^* = S u \gamma^{k-1}(e) u^*$ belongs to $C^*(\mathcal{N}(\mathcal{C}))$. Since eBe is simple, it follows that $S \in C^*(\mathcal{N}(\mathcal{C}))$. Since $C^*(\mathcal{N}(\mathcal{C})) \supset eBe$ as follows easily, this concludes the proof. $\square$

Given flows α and β , we say that α is an approximate cocycle perturbation of β if there is a sequence (u_n) of β -cocycles such that $\max_{|t| \leq 1} \|\alpha_t(x) - \text{Ad } u_n(t) \beta_t(x)\|$ converges to zero as $n \rightarrow \infty$ for all $x \in A$. If α is an approximate cocycle perturbation of β then the converse does not follow, in general.

We recall that the above automorphism γ of B may have the Rohlin property if B is simple (see [3,7]). In this case we can show:

Lemma 5.9. *Let A be a unital nuclear simple purely infinite C^* -algebra and $\mathcal{C} = Ce$ be a Cartan masa of A as in Proposition 5.8. Suppose also that B is simple and the automorphism γ of B has the Rohlin property.*

Let α be a Rohlin flow on A and let β be a flow on A such that $\beta_t|_{\mathcal{C}} = \text{id}$. Then β is an approximate cocycle perturbation of α .

Proof. We may assume the following: there exists an increasing sequence (B_n) of finite-dimensional C^* -subalgebras of B such that $B = \overline{\bigcup_n B_n}$, $e, \gamma(e) \in B_1$, the central support $\gamma(e)$ in eB_1e is e , $\gamma^{\pm 1}(B_n) \subset B_{n+1}$, and C is generated by $C \cap B_n \cap B'_{n-1}$, $n = 1, 2, \dots$, where $B_0 = 0$. To prove this suppose we are given an automorphism γ of B such that $\gamma(C) = C$ and $\gamma(e) \leq e$. Since C is a canonical AF masa there is an increasing sequence (B_n) of finite-dimensional C^* -subalgebras of B such that $B = \overline{\bigcup_n B_n}$ and C is generated by $C \cap B_n \cap B'_{n-1}$, $n = 1, 2, \dots$, where $B_0 = 0$. As B is simple we may suppose that the central support of $\gamma(e)$ in eB_1e is e . We assert that there is a unitary $U \in \mathcal{U}(C + \mathbf{C}1)$ such that $(\text{Ad } U \cdot \gamma)^{\pm 1}(B_{k_n}) \subset B_{k_n+1}$ for some subsequence (k_n) with $k_1 = 1$. Since $\text{Ad } U \circ \gamma(C) = C$ and C is generated by $C \cap B_{k_n} \cap B'_{k_n-1}$, $n = 1, 2, \dots$, with $k_0 = 0$, we can take $\text{Ad } U \circ \gamma$ for γ and (B_{k_n}) for (B_n) .

Set $k_1 = 1$. We choose k_2 such that $\gamma(B_1)$ is almost contained in B_{k_2} ; in particular, $\gamma(C \cap B_1) \subset C \cap B_{k_2}$. We also choose a generating family $\{e_{ij}^k\}$ of matrix units of B_1 such that $e_{ii}^k \in C$ and $e_{ij}^k \in \mathcal{N}_w(C)$. Since $\gamma(e_{ii}^k) \in C \cap B_{k_2}$ and $\gamma(e_{ij}^k) \in \mathcal{N}_w(C)$, there exists a family $\{f_{ij}^k\}$ of matrix units of B_{k_2} such that $\gamma(e_{ij}^k) \approx f_{ij}^k$, $\gamma(e_{ii}^k) = f_{ii}^k$ and $f_{ij}^k \in \mathcal{N}_w(C)$. (We express $\gamma(e_{11}^k)$ as the sum of minimal projections $p_l \in C \cap B_{k_2}$; then $\gamma(e_{11}^k)p_l$ is close to a partial isometry $w_l \in B_{k_2}$. Set $f_i^k = \sum w_l \in B_{k_2}$. Since $w_l \in \mathcal{N}_w(C)$ it follows that $f_i^k \in \mathcal{N}_w(C)$ and that $\|f_i^k - \gamma(e_{11}^k)\| = \max_l \|\gamma(e_{11}^k)p_l - w_l\|$. We set $f_{ij}^k = f_i^k(f_j^k)^*$.) Then $\text{Ad}(\gamma(e_{11}^k)f_{ii}^k)$ defines an automorphism of Cf_{ii}^k . Since it is close to the identity, it follows that $\gamma(e_{11}^k)f_{ii}^k \in C$. Let

$$U_1 = \sum_k \sum_i f_{ii}^k \gamma(e_{11}^k) + 1 - \sum_k \sum_i f_{ii}^k.$$

Then $U_1 \in \mathcal{U}(C + \mathbf{C}1)$ and $U_1 \approx 1$. Since $\text{Ad } U_1 \gamma(e_{ij}^k) = f_{ij}^k$, one has $\text{Ad } U_1 \gamma(B_1) \subset B_{k_2}$.

Next we find B_{k_3} such that $\text{Ad } U_1 \gamma(B_{k_3})$ almost contains B_{k_2} . Then in a similar way we obtain $V_1 \in \mathcal{U}(C + \mathbf{C}1)$ and $V_1 \approx 1$ and $B_{k_2} \subset \text{Ad}(V_1 U_1) \gamma(B_{k_3})$. Let $D_1 = \text{Ad } U_1 \gamma(B_1)$. Since $D_1 \subset B_{k_2} \subset \text{Ad}(V_1 U_1) \gamma(B_{k_3})$ and $\text{Ad } V_1(D_1) = \text{Ad}(V_1 U_1) \gamma(B_1) \subset \text{Ad}(V_1 U_1) \gamma(B_{k_3})$, there is a unitary $W_1 \in C \cap \text{Ad}(V_1 U_1) \gamma(B_{k_3}) + \mathbf{C}(1 - \gamma(1_{B_{k_3}}))$ such that $\text{Ad}(W_1 V_1)|_{D_1} = \text{id}$ and $\|W_1 - 1\| \leq \|(\text{Ad } V_1 - \text{id})|_{D_1}\| \leq 2\|V_1 - 1\|$. Set $U_2 = W_1 V_1$. Then $U_2 \in \mathcal{U}(C + \mathbf{C}1)$, $U_1 \approx 1$, $\text{Ad } U_2|_{\text{Ad } U_1 \gamma(B_1)} = \text{id}$ and

$$\text{Ad}(U_2 U_1) \gamma(B_1) \subset B_{k_2} \subset \text{Ad}(U_2 U_1) \gamma(B_{k_3}).$$

We can repeat this process. Namely we obtain a unitary $U_3 \in \mathcal{U}(C + \mathbf{C}1)$ and $k_4 > k_3$ such that $U_3 \approx 1$, $\text{Ad } U_3|_{B_{k_2}} = \text{id}$ and

$$\text{Ad}(U_3 U_2 U_1) \gamma(B_{k_3}) \subset B_{k_4}.$$

In this way we obtain a sequence (U_n) in $\mathcal{U}(C + \mathbf{C}1)$ and a sequence (k_n) such that $U = \lim_{n \rightarrow \infty} U_n U_{n-1} \cdots U_1$ converges, $\text{Ad}(U_{2n-1} U_{2n-2} \cdots U_1) \gamma(B_{k_{2n-1}}) \subset B_{k_{2n}}$, $B_{k_{2n}} \subset \text{Ad}(U_{2n} \times U_{2n-1} \cdots U_1) \gamma(B_{k_{2n+1}})$, $\text{Ad } U_{2n}|_{\text{Ad}(U_{2n-1} \cdots U_1) \gamma(B_{k_{2n-1}})} = \text{id}$ and $\text{Ad } U_{2n+1}|_{B_{k_{2n}}} = \text{id}$. Hence we conclude that $\text{Ad } U \gamma(B_{k_1}) \subset B_{k_2} \subset \text{Ad } U \gamma(B_{k_3}) \subset B_{k_4} \subset \cdots$, which implies that

$$(\text{Ad } U \gamma)^{\pm 1}(B_{k_{2n-1}}) \subset B_{k_{2n+1}}.$$

This concludes the proof of the above assertion.

Let $u \in \mathcal{N}(\mathcal{C}) \cap eB_ne$. Then $\beta_t(u) = e^{itk}u$ for some $k = k^* \in \mathcal{C}$. Since $\mathcal{N}(\mathcal{C}) \cap eB_ne$ generates eB_ne we deduce that β fixes eBe and that $D(\delta_\beta)$ contains the union of eB_ne , $n = 1, 2, \dots$

For each n there exists a $z_n = z_n^* \in \mathcal{C}$ so that $\delta_\beta|_{eB_ne} = \text{ad } iz_n|_{eB_ne}$. We make a bounded perturbation of δ_β and pass to a subsequence of (B_n) such that $z_n \in Ce \cap eB_{n+1}e$, i.e., the C^* -subalgebra D_n of $eB_{n+1}e$ generated by eB_ne and $e(B_{n+1} \cap B'_n)e \cap \mathcal{C}$ is left invariant under β . Note that $eB_ne \subset D_n \subset eB_{n+1}e$.

The domain $D(\delta_\alpha)$ may not contain $\bigcup_n eB_ne$, but we can find a $u \in \mathcal{U}(A)$ such that $D(\delta_\alpha) \supset ueB_neu^*$ for all n . Thus replacing α_t by a cocycle perturbation $\text{Ad}(u^*\alpha_t(u))\alpha_t$, we may suppose that $D(\delta_\alpha) \supset \bigcup_n eB_ne = \bigcup_n D_n$.

There exists $y_n = y_n^* \in A$ such that $\delta_\alpha|_{D_n} = \text{ad } iy_n|_{D_n}$. Then it follows that $(\delta_\alpha + \text{ad } i(z_n - y_n))|_{D_n} = \delta_\beta|_{D_n}$, which implies that $\alpha_t^{(z_n - y_n)}(x) = \beta_t(x)$ for $x \in D_n$. Thus, for any n , by a bounded perturbation on α one may always assume that $\alpha = \beta$ on D_n .

Let $S = Ue$ and define $w_t = S^*\beta_t(S)$, which is a β -cocycle. We know that $w_t \in \mathcal{C}$. Hence there is a $k = k^* \in \mathcal{C}$ such that $w_t = e^{itk}$. We approximate k by $k_n = k_n^* \in Ce \cap eB_ne = Ce \cap D_{n-1}$ such that $\|k - k_n\| \rightarrow 0$ as $n \rightarrow \infty$.

Let T be a large constant and n be a large integer such that $1/T \approx 0$ and $T\|k - k_n\| \approx 0$. Let N be also a large integer. We suppose that $\alpha_t(x) = \beta_t(x)$, $x \in D_{n+N}$.

Let $v_t = S^*\alpha_t(S)$, which is a unitary since $\alpha_t(SS^*) = SS^*$ and forms an α -cocycle. Let $x \in Ce \cap eB_{n+N}e = Ce \cap D_{n+N-1}$. Then since $SxS^* \in Ce \cap B_{n+N+1} \subset D_{n+N}$, one has $xv_t = S^*SxS^*\alpha_t(S) = S^*\alpha_t(SxS^*) = v_tx$, which implies, in particular, that $e^{-itk_n}v_t = v_te^{-itk_n}$ (as $k_n \in Ce \cap D_{n-1}$). Since $\alpha_t(e^{-itk_n}) = e^{-itk_n}$, it follows that $t \mapsto e^{-itk_n}v_t$ is an α -cocycle.

Let $x \in D_{n+N-1}$. Since $S\beta_{-t}(x)S^* = S\alpha_{-t}(x)S^* \in D_{n+N}$ one has $x\beta_t(S^*)\alpha_t(S) = \beta_t(S^*S\beta_{-t}(x)S^*)\alpha_t(S) = \beta_t(S^*)\alpha_t(S)x$, i.e., $e^{-itk}v_t \in A \cap D'_{n+N-1}$. Hence, for $x \in D_{n+N-1}$ and $t \in [0, T]$ we deduce that $\|[e^{-itk_n}v_t, x]\| = \|[e^{-itk_n}e^{itk}, x]e^{-itk}v_t\| \leq 2T\|k - k_n\|\|x\|$.

Since α has the Rohlin property and $t \mapsto e^{-itk_n}v_t$ is an α -cocycle one obtains a unitary w in $\mathcal{U}(A)$ such that $\|w\alpha_t(w^*) - e^{-itk_n}v_t\|$ is of the order of $|t|/T$. Specifically let us assume that $\|w\alpha_t(w^*) - e^{-itk_n}v_t\| \leq \varepsilon$ for $t \in [-1, 1]$ with $\varepsilon \approx 1/T$. To define such a unitary w we use the path $(e^{-itk_n}v_t)_{0 \leq t \leq T}$ which connects 1 with $e^{-iT k_n}v_T$, along which any element of the unit ball of D_{n+N-1} almost commutes up to the order of $T\|k - k_n\|$, which insures that $\|[w, x]\| \leq \varepsilon'\|x\|$ for $x \in D_{n+N-1}$ with ε' of the order of $T\|k - k_n\|$. Since $\|[w, e^{-itk_n}]\| \leq \varepsilon'$ we deduce that

$$\|\alpha_t(Sw) - Sw e^{itk_n}\| = \|Sv_t\alpha_t(w)e^{-itk_n}w^* - S\|$$

is less than $\varepsilon + \varepsilon'$ for $t \in [-1, 1]$. Note that by construction w is connected to 1 by a path which almost commutes with D_{n+N-1} . We may just as well assume that $w \in A \cap D'_{n+N-1}$ with $\max_{|t| \leq 1} \|\alpha_t(Sw) - Sw e^{itk_n}\| \leq \varepsilon + 3\varepsilon'$.

Since γ has the Rohlin property and satisfies $\gamma^{\pm 1}(D_k) \subset D_{k+1}$ we may choose a unitary $u \in A \cap (eB_2e)'$ such that $\|w - u\lambda(u^*)\|$ is of order of $1/N$ and u belongs to $A \cap D'_n$, where λ is a unital homomorphism of $A \cap (eB_2e)'$ into $A \cap (eB_1e)'$ such that $\lambda(x)\gamma(e) = SxS^*$, $x \in A \cap (eB_2e)'$. (To construct u we use a path of unitaries which connects 1 with $w\lambda(w)\lambda^2(u) \cdots \lambda^M(w)$ with $M \approx N$ and lies in the commutant of D_n . This is why $u \in A \cap D'_n$ while $w \in A \cap D_{n+N-1}$; see, e.g., [7].) Since $\lambda(u)S\lambda(u)^* = Su\lambda(u)^* \approx Sw$, $\alpha_t(\lambda(u)S\lambda(u)^*) \approx Sw e^{itk_n}$, and $[e^{itk_n}, \lambda(u)] = 0$ one has

$$\text{Ad } v^*\alpha_t \text{Ad } v(S) \approx v^*Sw e^{itk_n}v = Su^*w e^{itk_n}\lambda(u) \approx Se^{itk_n} \approx \beta_t(S)$$

for $v = \lambda(u)$. Moreover, for $x \in D_{n-1}$

$$\operatorname{Ad} v^* \alpha_t \operatorname{Ad} v(x) = \alpha_t(x) = \beta_t(x).$$

This concludes the proof. $\square$

Recall that a flow α on A is defined to be $C^{1+\varepsilon}$ if $D(\delta_\alpha) \supset C$ and

$$\sup_{x \in C, \|x\| \leq 1} \|(\alpha_t - \operatorname{id})\delta_\alpha(x)\|$$

converges to 0 as $t \rightarrow 0$.

Further recall the following result from [17]. If A is a unital separable nuclear purely infinite simple C^* -algebra, and if each of two Rohlin flows on A is an approximate cocycle perturbation of the other, then they are cocycle-conjugate with each other. By Theorem 5.1 and Lemma 5.9 one then obtains the following.

Corollary 5.10. *Let A be a unital separable nuclear simple purely infinite C^* -algebra and $\mathcal{C} = Ce$ be a Cartan masa of A as in Lemma 5.9. Suppose, in particular that γ has the Rohlin property.*

Let α and β be Rohlin flows on A such that both α and β are $C^{1+\varepsilon}$ on $\mathcal{C}$. Then α and β are cocycle-conjugate.

Proof. It follows from Corollary 5.6 that we may suppose $\alpha|_{\mathcal{C}} = \operatorname{id} = \beta|_{\mathcal{C}}$ by inner perturbation. Then by Lemma 5.9 α is a cocycle perturbation of β and vice versa. By using the result of [17] quoted above we obtain the desired conclusion. $\square$

Let $\mathcal{K}$ denote the C^* -algebra of compact operators on an infinite-dimensional separable Hilbert space and let $\tilde{\mathcal{K}} = \mathcal{K} + \mathbf{C}1$. Fix a minimal projection p in $\mathcal{K}$. For a bounded interval I of $\mathbf{Z}$ define $\tilde{\mathcal{K}}(I) = \bigotimes_I \tilde{\mathcal{K}}$ and $\mathcal{K}(I) = \bigotimes_I \mathcal{K} \subset \tilde{\mathcal{K}}(I)$. If $I = [a, b]$, we embed $\tilde{\mathcal{K}}([a, b])$ into $\tilde{\mathcal{K}}([a, b+1])$ by $x \mapsto x \otimes 1$ and $\tilde{\mathcal{K}}([a, b])$ into $\tilde{\mathcal{K}}([a-1, b])$ by $x \mapsto p \otimes x$. Let $\tilde{B}$ denote the inductive limit of the system $(\tilde{\mathcal{K}}(I))_I$ with these embeddings and let B be the C^* -subalgebra of $\tilde{B}$ generated by $(\mathcal{K}(I))_I$. Further let σ denote the automorphism of $\tilde{B}$ induced by the shift on $\mathbf{Z}$ to right. In particular, $1 \in \tilde{\mathcal{K}}(\{0\})$ maps into $p \in \tilde{\mathcal{K}}(\{0\})$. We denote by $e \in \tilde{B}$ the projection corresponding to $1 \in \tilde{\mathcal{K}}(\{0\})$ or equivalently to $p \in \mathcal{K}(\{-1\})$.

Note that σ leaves B invariant and $e \in B$. Further note that B is an AF algebra.

It is known [6] that the crossed product $B \times_\sigma \mathbf{Z}$ is isomorphic to $\mathcal{K} \otimes \mathcal{O}_\infty$ and that $e(B \times_\sigma \mathbf{Z})e \cong \mathcal{O}_\infty$.

Let $(e_{ij})_{i,j=1}^\infty$ be a family of matrix units in $\mathcal{K}$ such that $e_{1,1} = p$ and $\mathcal{K}$ is the closed linear span of (e_{ij}) . Let C be the abelian C^* -subalgebra of $\tilde{\mathcal{K}}$ generated by the $e_{i,i}$ and 1. It follows that C is a regular masa of $\tilde{\mathcal{K}}$.

Let C_∞ be the C^* -subalgebra generated by $\mathcal{K}(I) \cap \bigotimes_I C$ with all bounded intervals I . It follows that $e \in C_\infty$ and that C_∞ is a regular masa of B which is left invariant by σ . Let U denote the unitary multiplier of $B \times_\sigma \mathbf{Z}$ which implements σ and set $S = Ue \in e(B \times_\sigma \mathbf{Z})e$. Then S is an isometry. For each $x \in \tilde{\mathcal{K}}(\{0\})$ we use the same symbol x to denote the corresponding element in B .

Under the isomorphism $e(B \times_\sigma \mathbf{Z})e \cong \mathcal{O}_\infty$ which sends $e_{k1}S$ onto s_k , eC_∞ is the abelian C^* -subalgebra $\mathcal{C}_\infty$ generated by $s_I s_I^*$, $I \in \mathbf{N}^*$, where $\mathbf{N}^*$ is the set of finite sequences in $\mathbf{N}$. It is immediate that $\mathcal{C}_\infty$ is a weak Cartan masa of $\mathcal{O}_\infty$.

Hence Corollary 5.6 is applicable to the pair $A = \mathcal{O}_\infty$ and $\mathcal{C}_\infty = e\mathcal{C}_\infty$. Since B is not simple and σ does not have the Rohlin property, we cannot apply Lemma 5.9 to this pair $(\mathcal{O}_\infty, \mathcal{C}_\infty)$. (We used simplicity of B at the beginning of the proof, but this is required to define the unital partial endomorphism λ of eBe out of γ , whose Rohlin property we would also need.) Instead, however, we can use the fact that any unital endomorphism of $\mathcal{O}_\infty$ is approximately inner [23], in the proof of Lemma 5.9.

More precisely we define a unital endomorphism ϕ of $\mathcal{O}_\infty$ by $\phi(s_k) = s_k w$, or $\phi(e_{k1}S) = e_{k1}Sw$, for $k = 1, 2, \dots$, where w is the unitary described in the proof of Lemma 5.9. In particular, $w \in \mathcal{O}_\infty \cap D'_{n+N-1}$. Note that $\phi(e_{ij}) = \phi(s_i s_j^*) = s_i w w^* s_j^* = s_i s_j^* = e_{ij}$. For $e_{i_1, j_1} \otimes e_{i_2, j_2} \in \mathcal{K}([0, 1])$, if e_{i_2, j_2} commutes with w , we have that

$$\phi(e_{i_1, j_1} \otimes e_{i_2, j_2}) = \phi(s_{i_1} s_{i_2} s_{j_2}^* s_{j_1}^*) = s_{i_1} w e_{i_2, j_2} w^* s_{j_1}^* = e_{i_1, j_1} \otimes e_{i_2, j_2}.$$

In this way, we can show that if $e_k = \sum_{i=1}^k e_{ii}$ is a projection in $\mathcal{K}$ such that w is in the commutant of $D(k) = Ce + e_k \mathcal{K} e_k + e_k \mathcal{K} e_k \otimes e_k \mathcal{K} e_k + \dots + e_k \mathcal{K} e_k \otimes e_k \mathcal{K} e_k \otimes \dots \otimes e_k \mathcal{K} e_k$ ($k+1$ terms), then $\phi|_{D(k)} = \text{id}$. Hence we can find a unitary $v \in \mathcal{O}_\infty$ such that $vSv^* \approx Sw$ and $vxv^* \approx x$ for $x \in D(k)$. Since $\bigcup_k D(k)$ is dense in eBe , this leads us to the same conclusion of Lemma 5.9.

Thus we have proved Corollary 1.3 just as we did Corollary 5.10. Since there are quite a few Rohlin flows on $\mathcal{O}_\infty$ which are trivial on $\mathcal{C}_\infty$ this result is certainly non-empty.

We can define a Cartan masa $\mathcal{C}_n$ of $\mathcal{O}_n$ in the same way as $\mathcal{C}_\infty$. Since Corollary 5.10 is applicable to the pair $(\mathcal{O}_n, \mathcal{C}_n)$, let us state:

Corollary 5.11. *Let $n = 2, 3, \dots$ or $n = \infty$ and let $\mathcal{C}_n \subset \mathcal{O}_n$ be as above. Then there are Rohlin flows which are $C^{1+\varepsilon}$ on $\mathcal{C}_n$ and any two Rohlin flows of this type are cocycle conjugate with each other.*

Let $\{s_k\}$ be a generating family of isometries in $\mathcal{O}_\infty$ and let $\mathcal{H}$ be the closed linear span of $\{s_k\}$. This is a Hilbert space with inner product given by $\langle \xi, \eta \rangle = \eta^* \xi$. If a flow α on $\mathcal{O}_\infty$ leaves $\mathcal{H}$ invariant then α is quasi-free in a wider sense and is determined by the unitary flow U on $\mathcal{H}$ defined by $\alpha_t(\xi) = U_t \xi$, $\xi \in \mathcal{H}$. If U is diagonal or its self-adjoint generator H_U is diagonal then α is conjugate to a quasi-free flow that we have discussed.

Corollary 5.12. *Let U be a unitary flow on $\mathcal{H} \subset \mathcal{O}_\infty$ and let α be the corresponding flow on $\mathcal{O}_\infty$. Suppose that $\text{Spec}(H_U)$ generates $\mathbf{R}$ as a closed subsemigroup. Define a flow β on $\mathcal{O}_\infty$ by $\beta_t(s_1) = e^{it}s_1$, $\beta_t(s_2) = e^{-i\sqrt{2}t}s_2$ and $\beta_t(s_k) = s_k$ for $k \geq 3$. Then α has the Rohlin property if and only if α and β are cocycle conjugate with each other.*

Proof. It follows from Theorem 1.1 that β has the Rohlin property. Hence the if part is obvious. So we suppose that α has the Rohlin property.

It follows from the remark preceding Corollary 5.11 that β is an approximate cocycle perturbation of α . For $H = H_U$ there is a sequence (H_n) of diagonal self-adjoint operators on $\mathcal{H}$ such that $H - H_n$ has a bounded extension, $\|H - H_n\| \rightarrow 0$ and H_n has two eigenvalues which generate $\mathbf{R}$ as a closed subsemigroup. Then the flow $\alpha^{(n)}$ on $\mathcal{O}_\infty$ determined by $t \mapsto e^{itH_n}$ has the Rohlin property and converges to α as $n \rightarrow \infty$, i.e., $\max_{|t| \leq 1} \|\alpha_t(x) - \alpha_t^{(n)}(x)\| \rightarrow 0$ for $x \in \mathcal{O}_\infty$. Since $\alpha^{(n)}$ is quasi-free in our restrictive sense Corollary 5.11 tells us that $\alpha^{(n)}$ and β are cocycle conjugate, or $\alpha^{(n)}$ is an approximate cocycle perturbation of β . Hence we deduce that α is an

approximate cocycle perturbation of β . Since α and β are approximate cocycle perturbations of each other we conclude by [17] that α and β are cocycle conjugate. $\square$

Let $\lambda \in (0, 1)$ and let G_λ be the subgroup of $\mathbf{R}$ generated by λ^n , $n \in \mathbf{Z}$. Then G_λ , as an ordered subgroup of $\mathbf{R}$, is a simple dimension group and there is a stable simple AF algebra B_λ such that $K_0(B_\lambda) \cong G_\lambda$. Let γ be an automorphism of B_λ such that γ induces the multiplication by λ on $K_0(B_\lambda) = G_\lambda$. We may suppose that γ leaves a canonical AF masa C of B_λ invariant. By [7] γ has the Rohlin property and by [26] the crossed product $B_\lambda \times_\gamma \mathbf{Z}$ is purely infinite and simple. (We can obtain more examples of this kind from [3].)

If $\{f \in \mathbf{Z}[t]: f(\lambda) = 0\} = p(t)\mathbf{Z}[t]$ for some non-zero $p(t) \in \mathbf{Z}[t]$, then $A_\lambda \equiv B_\lambda \times_\gamma \mathbf{Z}$ is isomorphic to $\mathcal{O}_n \otimes \mathcal{K}$ where $n = |p(1)| + 1$; otherwise A_λ is isomorphic to $\mathcal{O}_\infty \otimes \mathcal{K}$. By cutting off A_λ by a projection $e \in C$ with $[e]$ a generator of $K_0(A_\lambda)$ one obtains a Cartan masa $\mathcal{C} = Ce$ in $eA_\lambda e$ which is isomorphic to $\mathcal{O}_n$ with n depending on λ as above. Thus there are many ways to construct a Cartan masa of $\mathcal{O}_n$ as in Proposition 5.8 but we do not know whether we obtain a new Cartan masa (in case $n < \infty$) which is not obtained as an image of the above $\mathcal{C}_n$ by an automorphism, and if we can, whether we have a Rohlin flow which is trivial on this Cartan masa.

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References

- [1] B. Blackadar, O. Bratteli, G.A. Elliott, A. Kumjian, Reduction of real rank in inductive limits of C^* -algebras, *Math. Ann.* 292 (1992) 111–126.
- [2] O. Bratteli, Derivations, Dissipations and Group Actions on C^* -algebras, *Lecture Notes in Math.*, vol. 1229, Springer, 1986.
- [3] O. Bratteli, A. Kishimoto, Trace scaling automorphisms of certain stable AF algebras, II, *Q. J. Math.* 51 (2000) 131–154.
- [4] O. Bratteli, D.W. Robinson, *Operator Algebras and Quantum Statistical Mechanics, I*, Springer, 1979.
- [5] O. Bratteli, D.E. Evans, G.A. Elliott, A. Kishimoto, Homotopy of a pair of approximately commuting unitaries in a simple purely infinite unital C^* -algebra, *J. Funct. Anal.* 160 (1998) 466–523.
- [6] J. Cuntz, Simple C^* -algebras generated by isometries, *Comm. Math. Phys.* 57 (1977) 173–185.
- [7] D.E. Evans, A. Kishimoto, Trace scaling automorphisms of certain stable AF algebras, *Hokkaido Math. J.* 26 (1997) 211–224.
- [8] U. Haagerup, All nuclear C^* -algebras are amenable, *Invent. Math.* 74 (1983) 305–319.
- [9] E. Kirchberg, N.C. Phillips, Embedding of exact C^* -algebras in the Cuntz algebra $\mathcal{O}_2$, *J. Reine Angew. Math.* 525 (2000) 17–53.
- [10] E. Kirchberg, N.C. Phillips, Embedding of continuous fields of C^* -algebras in the Cuntz algebra $\mathcal{O}_2$, *J. Reine Angew. Math.* 525 (2000) 55–94.
- [11] A. Kishimoto, Simple crossed products of C^* -algebras by locally compact abelian groups, *Yokohama Math. J.* 28 (1980) 69–85.
- [12] A. Kishimoto, The Rohlin property for automorphisms of UHF algebras, *J. Reine Angew. Math.* 465 (1995) 183–196.
- [13] A. Kishimoto, The Rohlin property for shifts on UHF algebras and automorphisms of Cuntz algebras, *J. Funct. Anal.* 140 (1996) 100–123.

- [14] A. Kishimoto, A Rohlin property for one-parameter automorphism groups, *Comm. Math. Phys.* 179 (1996) 599–622.
- [15] A. Kishimoto, Locally representable one-parameter automorphism groups of AF algebras and KMS states, *Rep. Math. Phys.* 45 (2000) 333–356.
- [16] A. Kishimoto, Examples of one-parameter automorphism groups of UHF algebras, *Comm. Math. Phys.* 216 (2001) 395–408.
- [17] A. Kishimoto, Rohlin flows on the Cuntz algebra $\mathcal{O}_2$, *Internat. J. Math.* 13 (2002) 1065–1094.
- [18] A. Kishimoto, Rohlin property for flows, in: *Contemp. Math.*, vol. 335, Amer. Math. Soc., 2003, pp. 195–207.
- [19] A. Kishimoto, Central sequence algebras of a purely infinite simple C^* -algebra, *Canad. J. Math.* 56 (2004) 1237–1258.
- [20] A. Kishimoto, The one-cocycle property for shifts, *Ergodic Theory Dynam. Systems* 25 (2005) 823–859.
- [21] A. Kishimoto, A. Kumjian, Crossed products of Cuntz algebras by quasi-free automorphisms, *Fields Inst. Commun.* 13 (1997) 173–192.
- [22] A. Kishimoto, N. Ozawa, S. Sakai, Homogeneity of the pure state space of a separable C^* -algebra, *Canad. Math. Bull.* 46 (2003) 365–372.
- [23] H. Lin, N.C. Phillips, Approximate unitary equivalence of homomorphisms from $\mathcal{O}_\infty$, *J. Reine Angew. Math.* 464 (1995) 173–186.
- [24] H. Nakamura, Aperiodic automorphisms of nuclear purely infinite simple C^* -algebras, *Ergodic Theory Dynam. Systems* 20 (2000) 1749–1765.
- [25] N.C. Phillips, A classification theorem for nuclear purely infinite simple C^* -algebras, *Doc. Math.* 5 (2000) 49–114 (electronic).
- [26] M. Rørdam, Classification of certain infinite simple C^* -algebras, *J. Funct. Anal.* 131 (1995) 415–458.
- [27] S. Sakai, *Operator Algebras in Dynamical Systems*, Cambridge Univ. Press, 1991.