Lecture I: Models for Nuclear Interactions

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Outline

1. Introduction and Motivation to the Lectures

2. Nucler Interactions, from QCD to Effective Theories
   - Developing Models for the NN force
   - How to construct an Interaction: Partial Wave Analysis
   - From QCD to Effective Field Theories, a better Pion-Nucleon Model

Overall Motivation and Aims

1. To provide you with computational tools for nuclear structure analyses. Useful both for data analysis (finalizing articles, theses etc) as well as for writing proposals.
2. Programs in Fortran 90/95 for computing effective two-body interactions starting with free nucleon-nucleon interactions (proton-neutron formalism).
3. Shell-model code. Source code in C/C++ (portable to all systems) which allows you to address systems up to some 10^9 basic states. (`http://folk.uio.no/mhjensen/manybody`)
4. These shell-model codes use as input the above interactions.

Overall Motivation and Aims

1. At the end of these lectures you should be able to run codes for the effective interaction and perform test calculations for the tin isotopes with ^{100}Sn as closed shell core. Eventually one could use ^{132}Sn as closed shell core.
2. Compute the $B(E2 : 2_1^+ \rightarrow 0_1^+)$ value for ^{108}Sn.
3. Understand how nuclear spectra evolve from the underlying nuclear interactions.
4. Critical assessment of the methods, pros and cons and their limitations.

Lecture Plan, see also http://folk.uio.no/mhjensen/manybody/

1. Lecture set I: Nuclear interactions, models and calculational schemes, from one-boson exchange models to effective field theories.
2. Lecture set II: Methods for renormalizing the nucleon-nucleon interaction, G-matrix and similarity transformation methods.
3. Lecture set III: Nuclear many-body theories: Shell-model, coupled cluster and many-body perturbation theory. Construction of effective interactions, model spaces and wave functions.
4. Lecture set IV: Derivation of effective interactions and discussions of applications to stable nuclei.
5. Lecture set V: Effective interactions for weakly bound systems (optional).

45 mins lectures, 15 mins break.

Nuclear Many-Body Methods

1. Shell-model and No-core shell-model calculations; Large-scale diagonalization.
2. Perturbative many-body methods.
3. Coupled cluster theory
4. Extention to weakly bound systems. Complex scaling and complex shell model, Gamow shell model.
5. Density matrix renormalization group.
6. Variational, Diffusion and Path integral Monte Carlo methods.
7. Green's function theory, Unitary operator method....
8. FHNC, ab initio DFT,....

From Yukawa to Lattice QCD and Effective Field Theory

1930's

Chadwick (1932) disocvers the neutron and Heisenberg (1932) proposes the first Phenomenology (Isospin). Yukawa (1935) and his Meson Hypothesis

1940's

Discovery of the pion in cosmic ray (1947) and in the Berkeley Cyclotron Lab (1948). Nobelprize awarded to Yukawa (1949). Rabi (1948) measures quadrupole moment of the deuteron.

1950's

Taketani, Nakamura, Sasaki (1951): 3 ranges. One-Pion-Exchange (OPE): o.k. Multi-pion exchanges: Problems! Taketani, Machida, Onuma (1952); "Pion Theories" Brueckner, Watson (1953).

From Yukawa to Lattice QCD and Effective Field Theory

1960's

Many pions = multi-pion resonances: $\sigma(600)$, $\rho(770)$, $\omega(782)$ etc.One-Boson-Exchange Model. Refined Meson Theories

1970's

Sophisticated models for two-pion exchange: Paris Potential (Lacombe et al., Phys. Rev. C **21**, 861 (1980)) Bonn potential (Machleidt et al., Phys. Rep. **149**, 1 (1987))

1980's

Quark cluster models. Begin of effective field theory studies.

From Yukawa to Lattice QCD and Effective Field Theory

1990's

1993-2001: High-precision NN potentials: Nijmegen I, II, '93, Reid93 (Stoks et al. 1994), Argonne V18 (Wiringa et al, 1995), CD-Bonn (Machleidt et al. 1996 and 2001. Advances in effective field theory: Weinberg (1990); Ordonez, Ray, van Kolck and many more.

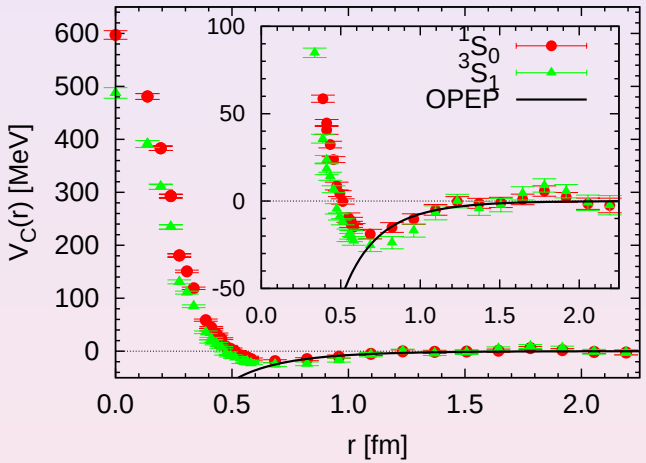
3rd Millenium

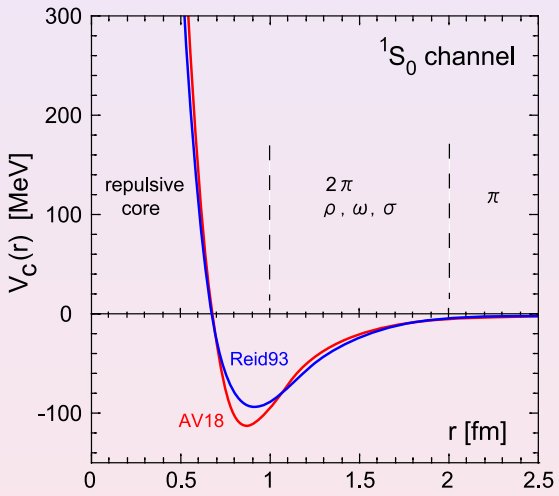
Another "pion theory"; but now right: constrained by chiral symmetry. Three-body and higher-body forces appear naturally at a given order of the chiral expansion.

2006

Nucleon-nucleon interaction from Lattice QCD, final confirmation of meson hypothesis of Yukawa?

Lattice QCD, Ishii *et al*, nucl-th/0611096





The nucleon-nucleon interaction, Phenomenology vs Lattice calculations.

Features of the Nucleon-Nucleon (NN) Force

The aim is to give you an overview over central features of the nucleon-nucleon interaction and how it is constructed, both technical and theoretical approaches.

1. The existence of the deuteron with $J^\pi = 1^+$ indicates that the force between protons and neutrons is attractive at least for the ${}^3S_1$ partial wave. Interference between Coulomb and nuclear scattering for the proton-proton partial wave ${}^1S_0$ shows that the NN force is attractive at least for the ${}^1S_0$ partial wave.
2. It has a short range and strong intermediate attraction.
3. Spin dependent, scattering lengths for triplet and singlet states are different,
4. Spin-orbit force. Observation of large polarizations of scattered nucleons perpendicular to the plane of scattering.

Features of the Nucleon-Nucleon (NN) Force, continued

1. Hard core. The s-wave phase shift becomes negative at ≈ 250 MeV implying that the singlet S has a hard core with range $0.4 - 0.5$ fm.
2. Charge independence (almost). Two nucleons in a given two-body state always (almost) experience the same force. Modern interactions break charge and isospin symmetry lightly. That means that the pp, neutron-neutron and pn parts of the interaction will be different for the same quantum numbers.
3. Non-central. There is a tensor force. First indications from the quadrupole moment of the deuteron pointing to an admixture in the ground state of both $l = 2$ ($^3D_1$) and $l = 0$ ($^3S_1$) orbital momenta.

Short Range Evidence

Comparison of the binding energies of ^{2}H (deuteron), ^{3}H (triton), ^{4}He (alpha - particle) show that the nuclear force is of finite range ($1 - 2$ fm) and very strong within that range. For nuclei with $A > 4$, the energy saturates: Volume and binding energies of nuclei are proportional to the mass number A.
Nuclei are also bound. The average distance between nucleons in nuclei is about 2 fm which must roughly correspond to the range of the attractive part.

Charge Dependence

- After correcting for the electromagnetic interaction, the forces between nucleons (pp, nn, or np) in the same state are almost the same.
- "Almost the same": Charge-independence is slightly broken.
- Equality between the pp and nn forces: Charge symmetry.
- Equality between pp/nn force and np force: Charge independence.
- Better notation: Isospin symmetry, invariance under rotations in isospin

Charge Dependence, $^1S_0$ Scattering Lengths

Charge-symmetry breaking (CSB), after electromagnetic effects have been removed:

- $a_{pp} = -17.3 \pm 0.4\text{fm}$
- $a_{nn} = -18.8 \pm 0.5\text{fm}$. Note however discrepancy from *nd* breakup reactions resulting in $a_{nn} = -18.72 \pm 0.13 \pm 0.65\text{fm}$ and $\pi^- + d \rightarrow \gamma + 2n$ reactions giving $a_{nn} = -18.93 \pm 0.27 \pm 0.3\text{fm}$.

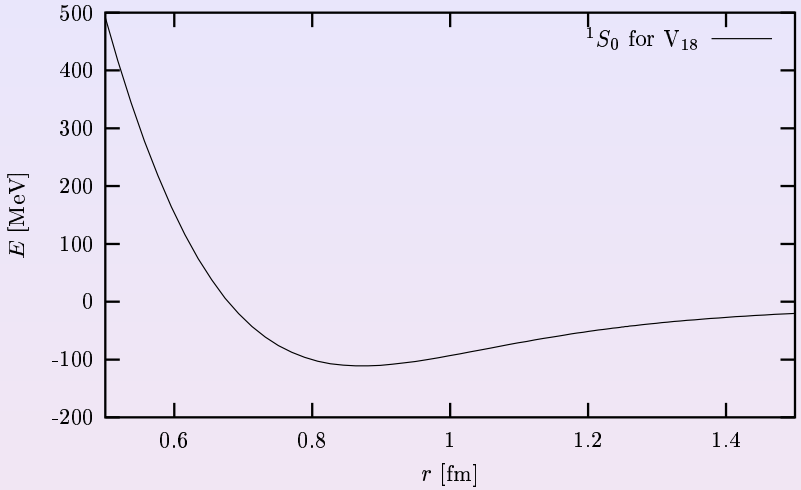
Charge-independence breaking (CIB)

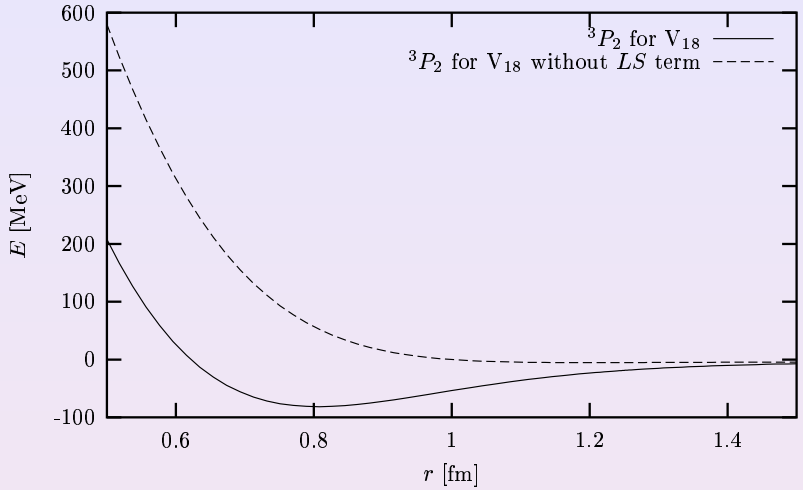
- $a_{pn} = -23.74 \pm 0.02\text{fm}$

Symmetries of the Nucleon-Nucleon (NN) Force

1. Translation invariance
2. Galilean invariance
3. Rotation invariance
4. Space reflection invariance
5. Time reversal invariance
6. Invariance under the interchange of particle 1 and 2
7. Almost isospin symmetry

A general two-body non-relativistic model under these symmetries was given by Okubo and Marshak, Ann. Phys. (NY) **4**, 166 (1958).





$$V(\mathbf{r}) = \left\{ C_c + C_\sigma \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 + C_T \left(1 + \frac{3}{m_\alpha r} + \frac{3}{(m_\alpha r)^2} \right) S_{12}(\hat{r}) \right.$$

$$\left. + C_{SL} \left(\frac{1}{m_\alpha r} + \frac{1}{(m_\alpha r)^2} \right) \mathbf{L} \cdot \mathbf{S} \right\} \frac{e^{-m_\alpha r}}{m_\alpha r}$$

How do we derive such terms?

References for Various Phenomenological Interactions

Potentials which are based upon the standard non-relativistic operator structure are called "Phenomenological Potentials" Some historically important examples are

- Gammel-Thaler potential (Phys. Rev. **107**, 291, 1339 (1957) and the · Hamada-Johnston potential, Nucl. Phys. **34**, 382 (1962)), bot with a hard core. core.
- Reid potential (Ann. Phys. (N.Y.) **50**, 411 (1968)), soft core.
- Argonne V_{14} potential (Wiringa et al., Phys. Rev. C **29**, 1207 (1984)) with 14 operators and the Argonne V_{18} potential (Wiringa et al., Phys. Rev. C **51**, 38 (1995)), uses 18 operators
- A good reference: R. Machleidt, Adv. Nucl. Phys **19**, 189 (1989).

Effective Degrees of Freedom

Since quantum chromodynamics (QCD) is commonly accepted as the theory of the strong interaction, the NN interaction V is completely determined by the underlying quark-quark dynamics in QCD. However, due to the non–perturbative character of QCD at low energies, one is still far from quantitative understanding of the NN interaction from the QCD point of view. Although there is no unique prescription for how to construct a free NN interaction, a description of the NN interaction in terms of various meson exchanges is presently the most quantitative representation of the NN interaction in the energy regime of low and medium energy nuclear physics, viz energies below 1 GeV.

Effective Degrees of Freedom, History

1. From 1950 till approximately 2000: One-Boson-Exchange (OBE) models dominate
2. Now: models based on chiral perturbation theory.

Dramatis Personae

Baryons	Mass (MeV)	Mesons	Mass (MeV)
p, n	938.926	π	138.03
Λ	1116.0	η	548.8
Σ	1197.3	σ	≈ 550.0
Δ	1232.0	ρ	770
		ω	782.6
		δ	983.0
		K	495.8
		$K^\star$	895.0

Lagrangians

To describe the interaction between the various baryons and mesons of the previous table we choose the following phenomenological lagrangians for spin 1/2 baryons

$$\mathcal{L}_{ps} = g^{ps}\overline{\Psi}\gamma^5\Psi\phi^{(ps)},$$

$$\mathcal{L}_{s} = g^{s}\overline{\Psi}\Psi\phi^{(s)},$$

and

$$\mathcal{L}_{v} = g^{v}\overline{\Psi}\gamma_\mu\Psi\phi^{(v)}_\mu + g^{t}\overline{\Psi}\sigma^{\mu\nu}\Psi\left(\partial_\mu\phi^{(v)}_\nu - \partial_\nu\phi^{(v)}_\mu\right),$$

for pseudoscalar (ps), scalar (s) and vector (v) coupling, respectively. The factors g^v and g^t are the vector and tensor coupling constants, respectively.

Lagrangians, contn.

Similarly the factor g^s is the phenomenological coupling coefficient for scalar mesons while g^{ps} is the corresponding coupling constant for pseudoscalar meson exchanges. These coupling constants may be constrained by e.g. the nucleon-nucleon scattering data. In the above equations, we have defined Ψ to be the baryon field for spin 1/2 baryons, while $\phi^{(ps)}$, $\phi^{(s)}$ and $\phi^{(v)}$ are the corresponding meson fields for pseudoscalar, scalar and vector mesons, respectively. Note that the above equations are for isoscalar mesons, however, for isovector mesons, the fields ϕ trivially modify to $\boldsymbol{\tau}\phi$ with $\boldsymbol{\tau}$ the familar isospinor Pauli matrices.

Spinors for Protons and Neutrons

For spin 1/2 baryons, the fields Ψ are expanded in terms of the Dirac spinors (positive energy solution shown here with $\bar{u}u = 1$)

$$u(k\sigma) = \sqrt{\frac{E(k)+m}{2m}} \begin{pmatrix} \chi \\ \\ \frac{\boldsymbol{\sigma}\mathbf{k}}{E(k)+m}\chi \end{pmatrix},$$

with χ the familiar Pauli spinor and $E(k) = \sqrt{m^2 + |\mathbf{k}|^2}$. The positive energy part of the field Ψ reads

$$\Psi(x) = \frac{1}{(2\pi)^{3/2}} \sum_{\mathbf{k}\sigma} u(k\sigma) e^{-ikx} a_{\mathbf{k}\sigma},$$

with a a fermion annihilation operator.

Standard Form for the NN-Force

The one-boson interaction then takes the form (omitting spin and isospin assignements)

$$V = \bar{u}(\mathbf{p}_1{}')\bar{u}(\mathbf{p}_2{}') \left(\sum_{j=1}^{5} v_j(\mathbf{k}) F_j \right) u(\mathbf{p}_1) u(\mathbf{p}_2).$$

Here F_j $(j = 1, ..., 5)$ denote the Fermi invariants defined as:

$$S = 1^{(1)}1^{(2)}, \quad V = \gamma_\mu^{(1)}\gamma_\mu^{(2)}, \quad T = \frac{1}{2}\sigma_{\mu\nu}^{(1)}\sigma_{\mu\nu}^{(2)},$$

$$A = i\gamma_5^{(1)}\gamma_\mu^{(1)} i\gamma_5^{(2)}\gamma_\mu^{(2)}, \quad P = \gamma_5^{(1)}\gamma_5^{(2)}.$$

Can have similar expressions with a pseudovector coupling for the pseudoscalar mesons. The labels $(1, 2)$ stand for the interacting baryons 1 and 2. The v_j term refers to the appropriate meson propagator including coupling constants.

Non-Relativistic Form for the NN-Force

Expanding the free Dirac spinors in terms of $1/m$ (m is here the mass of the relevant baryon) results, to lowest order, in the familiar non-relativistic expressions for baryon-baryon potentials. Explicitly, in momentum space, the non-relativistic version reads

$$V(k) = \sum_{j=1}^{5} \tilde{v}_j(k)\tilde{F}_j.$$

Here the operators $\tilde{F}_j$ are given by the familar expressions

$$\tilde{F}_1 = 1, \quad \tilde{F}_2 = \frac{1}{2} i(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)\cdot \mathbf{k} \times \mathbf{P}, \quad \tilde{F}_3 = \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 k^2 - 3\boldsymbol{\sigma}_1 \cdot \mathbf{k} \boldsymbol{\sigma}_2 \cdot \mathbf{k},$$

$$\tilde{F}_4 = \boldsymbol{\sigma}_1 \boldsymbol{\sigma}_2 \quad \tilde{F}_5 = \boldsymbol{\sigma}_1 \cdot \mathbf{k} \times \mathbf{P} \boldsymbol{\sigma}_2 \cdot \mathbf{k} \times \mathbf{P}.$$

The Classical Expression

In most models also operators like $\mathbf{L}^2$ or an antisymmetric spin-orbit term appear but these can be expressed as linear combinations of the above set on the energy shell. The configuration space version of the interaction can be obtained through Hankel transformations of the above momentum-space components resulting in

$$V(\mathbf{r}) = \left\{ C_C^0 + C_C^1 + C_\sigma \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 + C_T \left(1 + \frac{3}{m_\alpha r} + \frac{3}{(m_\alpha r)^2} \right) S_{12}(\hat{r}) \right.$$

$$\left. + C_{SL} \left(\frac{1}{m_\alpha r} + \frac{1}{(m_\alpha r)^2} \right) \mathbf{L} \cdot \mathbf{S} \right\} \frac{e^{-m_\alpha r}}{m_\alpha r},$$

where m_α is the mass of the relevant meson and S_{12} is the familiar tensor term.

OBE for Scalar Mesons

Employing the above lagrangians, it is then possible to construct a one-boson-exchange potential model. Typically, a contribution V_s^{OBE} arising from the exchange of a scalar meson between two spin 1/2 baryons with equal masses is given by

$$\left\langle p_1' p_2' \right| V_s^{OBE} \left| p_1 p_2 \right\rangle = g_s^2 \frac{\overline{u}(p_1')\overline{u}(p_2')u(p_1)u(p_2)}{(p_1' + p_2' - p_1 - p_2)^2 - m_s^2},$$

where m_s is the mass of the exchanged scalar meson. What does it look like in coordinate space and what about pion exchange and the spin-orbit force?

OBE for Pion Exchange

Employing the above lagrangians, it is then possible to construct a one-boson-exchange potential model. Here p_1, p_1', p_2, p_2' and $k = p_1 - p_1'$ denote four-momenta. The vertices are given by the pseudovector Lagrangian

$$\mathcal{L}_{pv} = \frac{f_\pi}{m_\pi}\overline{\psi}\gamma_5\gamma_\mu\psi\partial^\mu\phi_\pi.$$

From the Feynman rules we can write the two-body interaction as

$$V^{pv} = \frac{f_\pi^2}{m_\pi^2}\frac{\overline{u}(p_1')\gamma_5\gamma_\mu(p_1 - p_1')^\mu u(p_1)\overline{u}(p_2')\gamma_5\gamma_\nu(p_2' - p_2)^\nu u(p_2)}{(p_1 - p_1')^2 - m_\pi^2}.$$

OBE for Pion Exchange

The factors $p_1 - p_1' = p_2' - p_2$ are both the four-momentum of the exchanged meson and come from the derivative of the meson field in the interaction Lagrangian. The Dirac spinors obey

$$\begin{aligned} \gamma_\mu p^\mu u(p) &= m u(p) \\ \overline{u}(p)\gamma_\mu p^\mu &= m\overline{u}(p). \end{aligned}$$

Using these relations, together with $\{\gamma_5, \gamma_\mu\} = 0$, we find

$$\begin{aligned} \overline{u}(p_1')\gamma_5\gamma_\mu(p_1 - p_1')^\mu u(p_1) &= m\overline{u}(p_1')\gamma_5 u(p_1) + \overline{u}(p_1')\gamma_\mu p_1'^\mu \gamma_5 u(p_1) \\ &= 2m\overline{u}(p_1')\gamma_5 u(p_1) \end{aligned}$$

and

$$\overline{u}(p_2')\gamma_5\gamma_\mu(p_2' - p_2)^\mu = -2m\overline{u}(p_2')\gamma_5 u(p_1).$$

OBE for Pion Exchange

We get

$$V^{pv} = -\frac{f_\pi^2}{m_\pi^2} 4m^2 \frac{\overline{u}(p_1')\gamma_5 u(p_1)\overline{u}(p_2')\gamma_5 u(p_2)}{(p_1 - p_1')^2 - m_\pi^2}.$$

By inserting expressions for the Dirac spinors, we find

$$\begin{aligned}
\overline{u}(p_1')\gamma_5 u(p_1) &= \sqrt{\frac{(E_1'+m)(E_1+m)}{4m^2}} \begin{pmatrix} \chi^\dagger & -\frac{\sigma_1\cdot\mathbf{p_1}}{E_1'+m}\chi^\dagger \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
&= \times \begin{pmatrix} \chi \\ \frac{\sigma_1\cdot\mathbf{p_1}}{E_1+m}\chi \end{pmatrix} \\
&= \sqrt{\frac{(E_1'+m)(E_1+m)}{4m^2}} \left(\frac{\sigma_1\cdot\mathbf{p_1}}{E_1+m} - \frac{\sigma_1\cdot\mathbf{p_1'}}{E_1'+m} \right)
\end{aligned}$$

OBE for Pion Exchange

Similarly

$$\overline{u}(p_2')\gamma_5 u(p_1) = \sqrt{\frac{(E_2'+m)(E_2+m)}{4m^2}}\left(\frac{\sigma_2\cdot\mathbf{p}_2}{E_2+m} - \frac{\sigma_2\cdot\mathbf{p'}_2}{E_2'+m}\right).$$

In the CM system we have $\mathbf{p}_2 = -\mathbf{p}_1$, $\mathbf{p'}_2 = -\mathbf{p'}_1$ and so $E_2 = E_1$, $E_2' = E_1'$. We can then write down the relativistic contribution to the NN potential in the CM system:

$$\begin{aligned} V^{pv} &= \frac{f_\pi^2}{m_\pi^2}4m^2\frac{1}{(p_1-p_1')^2-m_\pi^2}\frac{(E_1+m)(E_1'+m)}{4m^2} \\ &\times \left(\frac{\sigma_1\cdot\mathbf{p}_1}{E_1+m} - \frac{\sigma_1\cdot\mathbf{p'}_1}{E_1'+m}\right)\left(\frac{\sigma_2\cdot\mathbf{p}_1}{E_1+m} - \frac{\sigma_2\cdot\mathbf{p'}_1}{E_1'+m}\right). \end{aligned}$$

OBE for Pion Exchange

In the non-relativistic limit we have to lowest order

$$E_1 = \sqrt{\mathbf{p}_1^2 + m^2} \approx m \approx E_1'$$

and then $(p_1 - p_1')^2 = -\mathbf{k}^2$, so we get for the contribution to the NN potential

$$\begin{aligned} V^{pv} &= -\frac{f_\pi^2}{m_\pi^2} 4m^2 \frac{1}{\mathbf{k}^2 + m^2} \frac{2m \cdot 2m}{4m^2} \frac{\sigma_1}{2m} \cdot (\mathbf{p}_1 - \mathbf{p'}_1) \frac{\sigma_2}{2m} \cdot (\mathbf{p}_1 - \mathbf{p'}_1) \\ &= -\frac{f_\pi^2}{m_\pi^2} \frac{(\sigma_1 \cdot \mathbf{k})(\sigma_2 \cdot \mathbf{k})}{\mathbf{k}^2 + m_\pi^2}. \end{aligned}$$

We have omitted exchange terms and isospin $\tau_1 \cdot \tau_2$.

OBE for Pion Exchange, from k-space to r-space

We have

$$V^{pv}(k) = -\frac{f_\pi^2}{m_\pi^2}\frac{(\sigma_1\cdot\mathbf{k})(\sigma_2\cdot\mathbf{k})}{\mathbf{k}^2+m_\pi^2}\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2.$$

In coordinate space we have

$$V^{pv}(r) = \int \frac{d^3k}{(2\pi)^3}e^{i\mathbf{kr}}V^{pv}(k)$$

resulting in

$$V^{pv}(r) = \frac{f_\pi^2}{m_\pi^2}\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2\sigma_1\cdot\nabla\sigma_2\cdot\nabla\int\frac{d^3k}{(2\pi)^3}e^{i\mathbf{kr}}\frac{1}{\mathbf{k}^2+m_\pi^2}.$$

We obtain

$$V^{pv}(r) = \frac{f_\pi^2}{m_\pi^2}\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2\sigma_1\cdot\nabla\sigma_2\cdot\nabla\frac{e^{-m_\pi r}}{r}.$$

OBE for Pion Exchange, really the last Step (I promise)

Carrying out the differentation of

$$V^{pv}(r) = \frac{f_\pi^2}{m_\pi^2}\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2 \sigma_1 \cdot \nabla \sigma_2 \cdot \nabla \frac{e^{-m_\pi r}}{r}.$$

we arrive at the famous one-pion exchange potential with central and tensor parts

$$V(\mathbf{r}) = \frac{f_\pi^2}{m_\pi^2}\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2 \left\{ \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 + C_T \left(1 + \frac{3}{m_\alpha r} + \frac{3}{(m_\alpha r)^2} \right) S_{12}(\hat{r}) \right\} \frac{e^{-m_\pi r}}{m_\pi r}.$$

For the full potential add the exchange part + isospin dependence as well.

Other Mesons: Collecting Terms, σ

When we perform similar non-relativistic expansions for scalar and vector mesons we obtain for the σ meson

$$V^{\sigma} = g^2_{\sigma NN} \frac{1}{\mathbf{k}^2 + m^2_{\sigma}} \left(-1 + \frac{\mathbf{q}^2}{2M^2_N} - \frac{\mathbf{k}^2}{8M^2_N} - \frac{\mathbf{LS}}{2M^2_N} \right).$$

We note an attractive central force and spin-orbit force. This term has an intermediate range. We have defined $1/2(p_1 + p'_1) = \mathbf{q}$.

Other Mesons: Collecting Terms, ω

We obtain for the ω meson

$$V^{\omega} = g^2_{\omega NN} \frac{1}{\mathbf{k}^2 + m^2_{\omega}} \left(1 - 3\frac{\mathbf{LS}}{2M^2_N}\right).$$

We note a repulsive central force and attractive spin-orbit force. This term has short range.

Other Mesons: Collecting Terms, ρ

Finally for the ρ meson

$$V^{\rho} = g^2_{\rho NN} \frac{\mathbf{k}^2}{\mathbf{k}^2 + m^2_{\rho}} \left(-2\sigma_1\sigma_2 + S_{12}(\hat{k}) \right) \tau_1\tau_2.$$

We note a tensor force with sign opposite to that of the pion. This term has short range.

Exercises Lecture I)

(a) List the quantum numbers $JSIT_z$ for partial waves of the nucleon-nucleon interaction (in the relative and center of mass frame of motion) up to angular momentum $J = 4$. The quantum numbers J, S, I and T_z represent the total angular momentum, the total spin, total orbital momentum and isospin projection, respectively. Use the Pauli principle for identical particles, expressed via the requirement that $I + S + T_z$ has to be an odd number. Repeat this for the nucleon-Λ and nucleon-Σ cases as well.

(b) Show that the two-body spin-orbit force **LS** gives a zero contribution for S-waves (orbital angular momentum $I = 0$).

Exercises Lecture I, optional)

(c) Show that the tensor operator of the nucleon-nucleon potential is given by

$$\begin{aligned}\langle ISJ|\, S_{12}\, |I'S'J\rangle = \quad & (-)^{S+J}\sqrt{30(2I+1)(2I'+1)(2S+1)(2S'+1)} \\ & \times \left\{ \begin{array}{ccc} J & S' & I' \\ 2 & I & S \end{array} \right\} \left(\begin{array}{ccc} I' & 2 & I \\ 0 & 0 & 0 \end{array} \right) \left\{ \begin{array}{ccc} s_1 & s_2 & S \\ s_3 & s_4 & S' \\ 1 & 1 & 2 \end{array} \right\} \\ & \times \langle s_1|\,|\sigma_1|\,|s_3\rangle \langle s_2|\,|\sigma_2|\,|s_4\rangle\,,\end{aligned}$$

and that it is zero for the ${}^1S_0$ partial wave.

(d) Calculate the contributions to $\left\langle {}^3S_1\right|\, S_{12}\, \left|{}^3S_1\right\rangle$, $\left\langle {}^3D_1\right|\, S_{12}\, \left|{}^3S_1\right\rangle$ and $\left\langle {}^3D_1\right|\, S_{12}\, \left|{}^3D_1\right\rangle$. Can we couple states with different total spins when we have only spin 1/2 fermions?

Bethe-Salpeter Equation

In order to obtain the parameters which define an NN potential derived from OBE or chiral perturbation theory models, the Bethe-Salpeter equation is used as the starting point for most calculations. This equation serves to define a two-particle interaction $\mathcal{T}$, meant to reproduce properties like low-energy scattering data. The fully covariant Bethe-Salpeter equation reads (suppressing spin and isospin) in an arbitrary frame

$$\begin{aligned}\left\langle p_1' p_2' \right| \mathcal{T} \left| p_1 p_2 \right\rangle &= \left\langle p_1' p_2' \right| \mathcal{V} \left| p_1 p_2 \right\rangle \\ &+ \frac{i}{(2\pi)^4} \int d^4k \left\langle p_1' p_2' \right| \mathcal{V} \left| P+k, P-k \right\rangle \\ &\times S_{(1)}(P+k) S_{(2)}(P-k) \left\langle P+k, P-k \right| \mathcal{T} \left| p_1 p_2 \right\rangle .\end{aligned}$$

Bethe-Salpeter Equation, contn.

Here we have defined P to be *half* the total four-momentum, i.e. $P = \frac{1}{2}(p_1 + p_2)$, and k to be the relative four-momentum. The term $S_{(i)}$ is the fermion propagator, which for e.g. positive energy spin 1/2 baryons reads

$$S_{(i)}(p) = (\not{p}_i - m_i + i\epsilon)^{-1} ,$$

with the subscript i referring to baryon i.

Bethe-Salpeter Equation, contn.

In principle $\mathcal{V}$ is supposed to represent all kinds of irreducible two-particle interactions, though it is commonly approximated by the lowest order two-particle diagram. With this prescription we obtain the familiar ladder approach to the Bethe-Salpeter equation, similar to the approach discussed in connection with the G-matrix. The schematic structure of the ladder equation is representative for both the scattering matrix and the reaction matrix G. It is a four-dimensional integral equation, which is rather tedious to solve numerically.

Bethe-Salpeter Equation, contn.

It is therefore commonly replaced by a three-dimensional quasi-potential equation, where the time components of the four-momenta of the incoming and outgoing particles have been fixed by some adequate choice. Still in an arbitrary frame we get

$$\begin{aligned}\left\langle \mathbf{p}_1' \mathbf{p}_2' \right| T \left| \mathbf{p}_1 \mathbf{p}_2 \right\rangle &= \left\langle \mathbf{p}_1' \mathbf{p}_2' \right| V \left| \mathbf{p}_1 \mathbf{p}_2 \right\rangle \\ &\quad + \frac{1}{(2\pi)^3} \int d^3k \left\langle \mathbf{p}_1' \mathbf{p}_2' \right| V \left| \mathbf{P} + \mathbf{k}, \mathbf{P} - \mathbf{k} \right\rangle \\ &\quad g(\mathbf{k}, s) \left\langle \mathbf{P} + \mathbf{k}, \mathbf{P} - \mathbf{k} \right| T \left| \mathbf{p}_1 \mathbf{p}_2 \right\rangle .\end{aligned}$$

V is now the so-called "quasi-potential", with fixed time components of the in- and outgoing particle momenta. As such, it is no longer an independent quantity. A much favored choice is to fix $p_1^0 = p_2^0 = \frac{1}{2}\sqrt{s}$, $s = (p_1 + p_2)^2$.

Bethe-Salpeter Equation, contn.

This prescription puts the two particles symmetrically off-shell, and is used in connection with the Blankenbecler-Sugar equation. The latter is one possibility of several three-dimensional reductions of the Bethe-Salpeter equation. The quantity g is related to the baryon propagators $S_{(1)}S_{(2)}$ through

$$g(\mathbf{k},s) = -i \int dk_0 S_{(1)}(k) S_{(2)}(-k).$$

The Blankenbecler-Sugar choice for g is (assuming $m_1 = m_2$ and spin 1/2 fermions)

$$g(\mathbf{k},s) = \frac{m^2}{E_k} \frac{\Lambda^+_{(1)}(\mathbf{k})\Lambda^+_{(2)}(-\mathbf{k})}{\frac{1}{4}s - E_k^2 + i\epsilon}.$$

All NN potentials reproduce essentially the same set of low-energy NN scattering data ($E_{lab} \leq 350$ MeV) and properties of the deuteron. These are referred to as the "on-shell" properties of an NN potential, since all potential models result in a roughly similar on-shell scattering matrix T. The crucial point is then the differing off-shell behavior of the NN potentials in nuclear structure studies. The Bethe-Salpeter equation reads in the center-of-mass system (omitting angular momentum, isospin, spin etc. assignments)

$$T(\mathbf{k},\mathbf{k}') = V(\mathbf{k},\mathbf{k}') + \int_0^\infty \frac{d^3q}{(2\pi)^3} V(\mathbf{k},\mathbf{q}) \frac{M_N^2}{E_q} \frac{\Lambda^+_{(1)}(\mathbf{q})\Lambda^+_{(2)}(-\mathbf{q})}{\mathbf{k}^2 - \mathbf{q}^2 + i\epsilon} T(\mathbf{q},\mathbf{k}'),$$

where $E_q = \sqrt{M_N^2 + \mathbf{q}^2}$.

Three-dimensional Reduction

For positive-energy spinors

$$T(\mathbf{k},\mathbf{k}') = V(\mathbf{k},\mathbf{k}') + \int_0^\infty \frac{d^3q}{(2\pi)^3} V(\mathbf{k},\mathbf{q}) \frac{M_N^2}{E_q} \frac{1}{\mathbf{k}^2 - \mathbf{q}^2 + i\epsilon} T(\mathbf{q},\mathbf{k}').$$

Using

$$\hat{T}(\mathbf{k},\mathbf{k}') = \sqrt{\frac{M_N}{E_{k'}}} T(\mathbf{k},\mathbf{k}') \sqrt{\frac{M_N}{E_k}},$$

and

$$\hat{V}(\mathbf{k},\mathbf{k}') = \sqrt{\frac{M_N}{E_{k'}}} V(\mathbf{k},\mathbf{k}') \sqrt{\frac{M_N}{E_k}},$$

gives

$$\hat{T}(\mathbf{k},\mathbf{k}') = \hat{V}(\mathbf{k},\mathbf{k}') + \int_0^\infty \frac{d^3q}{(2\pi)^3} \hat{V}(\mathbf{k},\mathbf{q}) \frac{1}{\mathbf{k}^2 - \mathbf{q}^2 + i\epsilon} \hat{T}(\mathbf{q},\mathbf{k}').$$

First Solution Step

In terms of the relative and center-of-mass momenta $\mathbf{k}$ and $\mathbf{K}$, the potential in momentum space is related to the nonlocal operator $V(\mathbf{r},\mathbf{r}')$ by

$$\langle \mathbf{k}'\mathbf{K}'| \, V \, |\mathbf{k}\mathbf{K}\rangle = \int d\mathbf{r} d\mathbf{r}' e^{-\imath \mathbf{k}'\mathbf{r}'} \, V(\mathbf{r}',\mathbf{r}) e^{\imath \mathbf{k}\mathbf{r}} \delta(\mathbf{K},\mathbf{K}').$$

We will assume that the interaction is spherically symmetric. Can separate the radial part of the wave function from its angular dependence. The wave function of the relative motion is described in terms of plane waves as

$$e^{\imath \mathbf{k}\mathbf{r}} = \langle \mathbf{r}| \, \mathbf{k}\rangle = 4\pi \sum_{lm} \imath^l j_l(kr) Y^*_{lm}(\hat{\mathbf{k}}) Y_{lm}(\hat{\mathbf{r}}),$$

where j_l is a spherical Bessel function and Y_{lm} the spherical harmonic.

Decomposing the NN Force

This partial wave basis is useful for defining the operator for the nucleon-nucleon interaction, which is symmetric with respect to rotations, parity and isospin transformations. These symmetries imply that the interaction is diagonal with respect to the quantum numbers of total angular momentum J, spin S and isospin T. Using the above plane wave expansion, and coupling to final J, S and T we get

$$\left\langle \mathbf{k}' \right| V \left| \mathbf{k} \right\rangle = (4\pi)^2 \sum_{STll'mJ} i^{l+l'} Y^*_{lm}(\hat{\mathbf{k}}) Y_{l'm'}(\hat{\mathbf{k}}')$$

$$\times \mathcal{C}^{l'SJ}_{m'M_SM} \mathcal{C}^{lSJ}_{mM_SM} \left\langle k'l'STJM \right| V \left| klSTJM \right\rangle ,$$

where we have defined

$$\left\langle k'l'STJM \right| V \left| klSTJM \right\rangle = \int j_{l'}(k'r') \left\langle l'STJM \right| V(r',r) \left| lSTJM \right\rangle j_l(kr) r'^2$$

The general structure of the T-matrix is

$$T^{\alpha}_{ll'}(kk'K\omega) = V^{\alpha}_{ll'}(kk')$$

$$+\frac{2}{\pi}\sum_{l''mM_S}\int_0^{\infty} d\mathbf{q}(C^{l''S\mathcal{J}}_{mM_SM})^2\frac{Y^*_{l''m}(\hat{\mathbf{q}})Y_{l''m}(\hat{\mathbf{q}})V^{\alpha}_{ll''}(kq)T^{\alpha}_{l''l'}(qk'K\omega)}{\omega - H_0},$$

The shorthand notation

$$T^{\alpha}_{ll'}(kk'K\omega) = \langle kKlL\mathcal{J}ST|\ T(\omega)\ \big|k'Kl'L\mathcal{J}ST\rangle\,,$$

denotes the T-matrix with momenta k and k' and orbital momenta l and l' of the relative motion, and K is the corresponding momentum of the center-of-mass motion. Further, L, $\mathcal{J}$, S and T are the orbital momentum of the center-of-mass motion, the total angular momentum, spin and isospin, respectively.

Using the orthogonality properties of the Clebsch-Gordan coefficients and the spherical harmonics, we obtain the well-known one-dimensional angle independent integral equation

$$T^{\alpha}_{ll'}(kk'K\omega) = V^{\alpha}_{ll'}(kk') + \frac{2}{\pi}\sum_{l''}\int_0^{\infty} dqq^2 \frac{V^{\alpha}_{ll''}(kq)\,T^{\alpha}_{l''l'}(qk'K\omega)}{\omega - H_0}.$$

Inserting the denominators for the Blankenbecler-Sugar of the full Bethe-Salpeter equation we arrive at

$$\hat{T}^{\alpha}_{ll'}(kk'K) = \hat{V}^{\alpha}_{ll'}(kk') + \frac{2}{\pi}\sum_{l''}\int_0^{\infty} dqq^2 \hat{V}^{\alpha}_{ll''}(kq)\frac{1}{k^2 - q^2 + i\epsilon}\hat{T}^{\alpha}_{l''l'}(qk'K).$$

Note that the OBE models, via the solution of the Bethe-Salpeter equation contain only a selected class of diagrams. However, there are also non-iterative diagrams which contribute to the nuclear force.

Numerical Solution, Simplifications

For scattering states, the energy is positive, $E > 0$. The Lippman-Schwinger equation, which is the non-relativistic version of the Bethe-Salpeter equation discussed above, is an integral equation where we have to deal with the amplitude $R(k, k')$ (reaction matrix, which is the real part of the full complex T-matrix discussed in the previous section) defined through the integral equation

$$R_l(k, k') = V_l(k, k') + \frac{2}{\pi}\mathcal{P}\int_0^\infty dq q^2 V_l(k, q)\frac{1}{E - q^2/m}R_l(q, k').$$

Numerical Solution, further Simplifications

The total kinetic energy of the two incoming particles in the center-of-mass system is

$$E = \frac{k_0^2}{m}.$$

The symbol $\mathcal{P}$ indicates that Cauchy's principal-value prescription is used in order to avoid the singularity arising from the zero of the denominator.
The matrix $R_l(k, k')$ relates to the the phase shifts through its diagonal elements as

$$R_l(k_0, k_0) = -\frac{tan\delta_l}{mk_0}.$$

Recipe I

From now on we will drop the subscript l in all equations. In order to solve the Lippman-Schwinger equation in momentum space, we need first to write a function which sets up the mesh points. We need to do that since we are going to approximate an integral through

$$\int_a^b f(x)dx \approx \sum_{i=1}^{N} w_i f(x_i),$$

where we have fixed N lattice points through the corresponding weights w_i and points x_i. Typically obtained via methods like Gaussian quadrature, see my course on computational physics (all material in english) `http://www.uio.no/studier/emner/matnat/fys/FYS3150/h06/`

Recipe II

If you use Gauss-Legendre the points are determined for the interval $x_i \in [-1, 1]$ You map these points over to the limits in your integral. You can then use the following mapping

$$k_i = const \times tan\left\{\frac{\pi}{4}(1 + x_i)\right\},$$

and

$$\omega_i = const\frac{\pi}{4}\frac{w_i}{cos^2\left(\frac{\pi}{4}(1 + x_i)\right)}.$$

If you choose units fm^{-1} for k, set $const = 1$. If you choose to work with MeV, set $const \sim 200$ ($\hbar c = 197$ MeVfm).

Recipe III

The principal value integral is rather tricky to evaluate numerically, mainly since computers have limited precision. We will here use a subtraction trick often used when dealing with singular integrals in numerical calculations. We introduce first the calculus relation

$$\int_{-\infty}^{\infty} \frac{dk}{k-k_0} = 0.$$

It means that the curve $1/(k-k_0)$ has equal and opposite areas on both sides of the singular point k_0. If we break the integral into one over positive k and one over negative k, a change of variable $k \longrightarrow -k$ allows us to rewrite the last equation as

$$\int_{0}^{\infty} \frac{dk}{k^2-k_0^2} = 0.$$

Recipe IV

We can then express a principal values integral as

$$\mathcal{P}\int_0^\infty \frac{f(k)dk}{k^2-k_0^2} = \int_0^\infty \frac{(f(k)-f(k_0))dk}{k^2-k_0^2},$$

where the right-hand side is no longer singular at $k = k_0$, it is proportional to the derivative df/dk, and can be evaluated numerically as any other integral.

Recipe V

We can then use this trick to obtain

$$R(k,k') = V(k,k') + \frac{2}{\pi}\int_0^\infty dq \frac{q^2 V(k,q)R(q,k') - k_0^2 V(k,k_0)R(k_0,k')}{(k_0^2 - q^2)/m}.$$

This is the equation has to solve numerically in order to calculate the phase shifts. We are interested in obtaining $R(k_0, k_0)$.

Recipe VI

How do we proceed?
Using the mesh points k_j and the weights ω_j, we reach

$$R(k,k') = V(k,k') + \frac{2}{\pi}\sum_{j=1}^{N}\frac{\omega_j k_j^2 V(k,k_j)R(k_j,k')}{(k_0^2-k_j^2)/m} - \frac{2}{\pi}k_0^2 V(k,k_0)R(k_0,k')$$

Recipe VII

This equation contains now the unknowns $R(k_i, k_j)$ (with dimension $N \times N$) and $R(k_0, k_0)$.
We can turn it into an equation with dimension $(N+1) \times (N+1)$ with a mesh which contains the original mesh points k_j for $j = 1, N$ and the point which corresponds to the energy k_0.
Consider the latter as the 'observable' point. The mesh points become then k_j for $j = 1, n$ and $k_{N+1} = k_0$.
With these new mesh points we define the matrix

$$A_{i,j} = \delta_{i,j} - V(k_i, k_j)u_j,$$

Recipe VIII

where δ is the Kronecker δ and

$$u_j = \frac{2}{\pi} \frac{\omega_j k_j^2}{(k_0^2 - k_j^2)/m} \qquad j = 1, N$$

and

$$u_{N+1} = -\frac{2}{\pi} \sum_{j=1}^{N} \frac{k_0^2 \omega_j}{(k_0^2 - k_j^2)/m}.$$

Recipe IX

The first task is then to set up the matrix A for a given k_0. This is an $(N+1)\times(N+1)$ matrix. It can be convenient to have an outer loop which runs over the chosen observable values for the energy k_0^2/m. *Note that all mesh points k_j for $j=1,N$ must be different from k_0. Note also that $V(k_i,k_j)$ is an $(N+1)\times(N+1)$ matrix.* With the matrix A we can rewrite the problem as a matrix problem of dimension $(N+1)\times(N+1)$. All matrices R, A and V have this dimension and we get

$$A_{i,l}R_{l,j}=V_{i,j},$$

or just

$$AR=V.$$

Recipe X

Since you already have defined A and V (these are stored as $(N+1)\times(N+1)$ matrices) The final equation involves only the unknown R. We obtain it by matrix inversion, i.e.,

$$R = A^{-1}V.$$

Thus, to obtain R, you will need to set up the matrices A and V and invert the matrix A. With the inverse A^{-1}, perform a matrix multiplication with V results in R.
With R you can then evaluate the phase shifts by noting that

$$R(k_{N+1}, k_{N+1}) = R(k_0, k_0) = -\frac{tan\delta}{mk_0},$$

where δ are the phase shifts.

Lippman-Schwinger Equation, Repeated

To parameterize the nucleon-nucleon interaction we solve the Lippman-Scwhinger equation

$$T^{\alpha}_{ll'}(kk'K) = V^{\alpha}_{ll'}(kk') + \frac{2}{\pi}\sum_{l''}\int_0^{\infty} dqq^2 V^{\alpha}_{ll''}(kq)\frac{1}{k^2-q^2+i\epsilon}T^{\alpha}_{l''l'}(qk'K).$$

The shorthand notation

$$T(\hat{V})^{\alpha}_{ll'}(kk'K\omega) = \langle kKlL\mathcal{J}ST|\, T(\omega)\,|k'Kl'L\mathcal{J}ST\rangle\,,$$

denotes the $T(V)$-matrix with momenta k and k' and orbital momenta l and l' of the relative motion, and K is the corresponding momentum of the center-of-mass motion. Further, L, $\mathcal{J}$, S and T are the orbital momentum of the center-of-mass motion, the total angular momentum, spin and isospin, respectively.

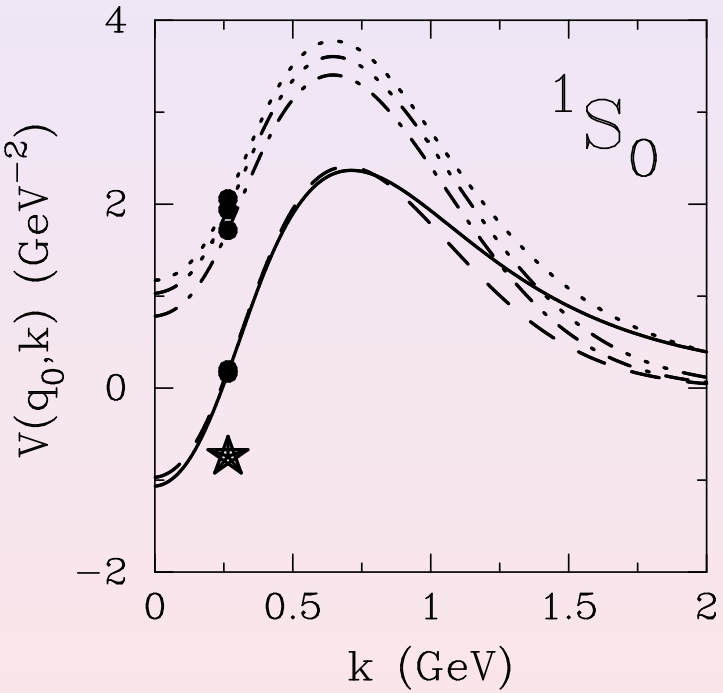
Numerical Solution, Repeated

For scattering states, the energy is positive, $E > 0$. The Lippman-Schwinger equation, which is the non-relativistic version of the Bethe-Salpeter equation discussed above, is an integral equation where we have to deal with the amplitude $R(k, k')$ (reaction matrix, which is the real part of the full complex T-matrix) defined through the integral equation for one partial wave (no coupled-channels)

$$R_l(k, k') = V_l(k, k') + \frac{2}{\pi}\mathcal{P}\int_0^\infty dq q^2 V_l(k, q)\frac{1}{E - q^2/m}R_l(q, k').$$

For negative energies (bound states) and intermediate states scattering states blocked by occupied states below the Fermi level, this expressions leads to the G-matrix to be discussed today.

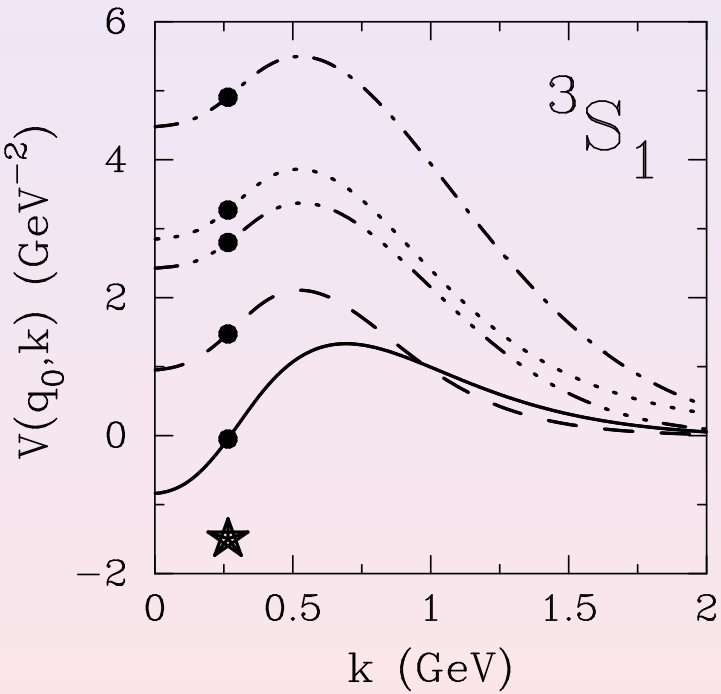
T-matrix and V



$^1S_0$

Matrix elements $V(q_0, k)$ for the $^1S_0$ partial wave for the CD-Bonn (solid line), Nijm-I (dashed), Nijm-II (dash-dot), Argonne V_{18} (dash-triple-dot) and Reid93 (dotted) potentials. The diagonal matrix elements with $k = q_0 = 265$ MeV/c (equivalent to $T_{lab} = 150$ MeV) are marked by a solid dot. The corresponding matrix element of the full scattering R-matrix is marked by the star.

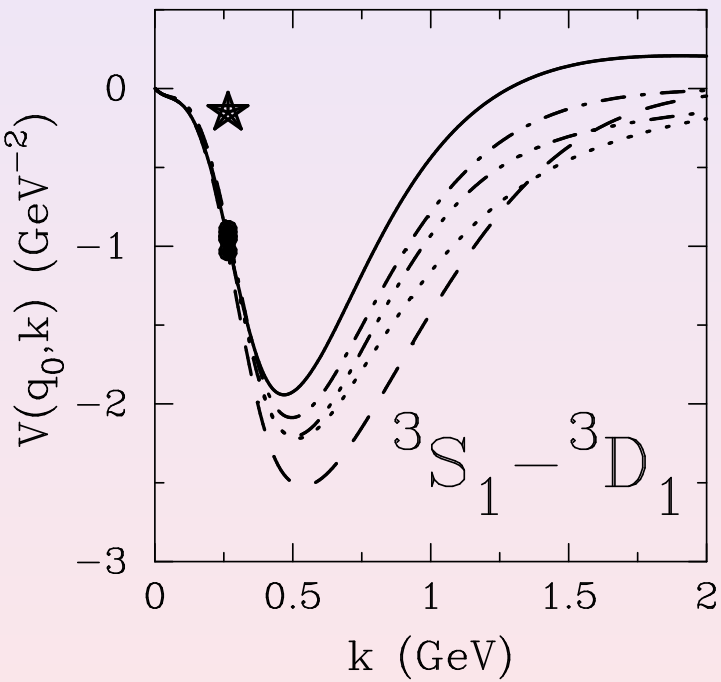
T-matrix and V



$^3S_1$

Matrix elements for the $^3S_1$ for the CD-Bonn (solid line), Nijm-I (dashed), Nijm-II (dash-dot), Argonne V_{18} (dash-triple-dot) and Reid93 (dotted) potentials. The diagonal matrix elements with $k = q_0 = 265$ MeV/c (equivalent to $T_{lab} = 150$ MeV) are marked by a solid dot. The corresponding matrix element of the full scattering R-matrix is marked by the star.

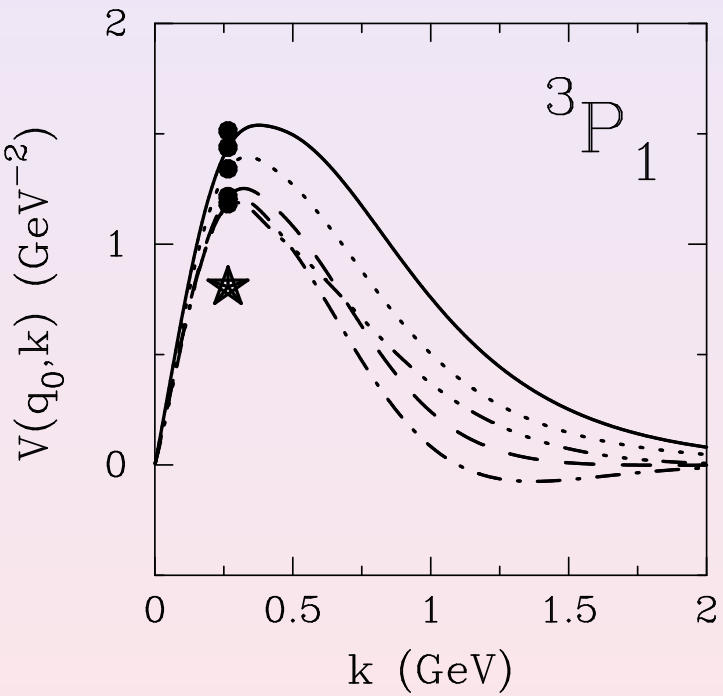
T-matrix and V



$^3S_1$–$^3D_1$

Matrix elements for $^3S_1$–$^3D_1$ for the CD-Bonn (solid line), Nijm-I (dashed), Nijm-II (dash-dot), Argonne V_{18} (dash-triple-dot) and Reid93 (dotted) potentials. The diagonal matrix elements with $k = q_0 = 265$ MeV/c (equivalent to $T_{lab} = 150$ MeV) are marked by a solid dot. The corresponding matrix element of the full scattering R-matrix is marked by the star.

T-matrix and V



$^3P_1$

Matrix elements for $^3P_1$ for the CD-Bonn (solid line), Nijm-I (dashed), Nijm-II (dash-dot), Argonne V_{18} (dash-triple-dot) and Reid93 (dotted) potentials.The diagonal matrix elements with $k = q_0 = 265$ MeV/c (equivalent to $T_{lab} = 150$ MeV) are marked by a solid dot. The corresponding matrix element of the full scattering R-matrix is marked by the star.

Does it matter?

The behavior seen here has important consequences for renormalizations in nuclear medium.
All interactions yield the same on-shell T-matrix, although V and its off-shell character can be very different from interaction model to interaction model. The off-shell part is not constrained by data.
Potentials with for example a weak tensor force (cancellation between the π and ρ meson contributions) can lead to important differences in for example nuclear binding energies.

OBE vs Chiral Perturbation Theory

Historically, the experimental discovery of heavy mesons in the early 1960s gave momentum to the one-boson-exchange (OBE) model. Prior to that it was pion physics which dominated the picture. The weak point of this model, however, is the scalar-isoscalar "sigma" or "epsilon" boson, for which the empirical evidence remains controversial. Since this boson is associated with the correlated (or resonant) exchange of two pions, a vast theoretical effort that occupied more than a decade (1970-1980) was launched to derive the 2π-exchange contribution of the nuclear force, which creates the intermediate range attraction.

The nuclear force problem appeared to be solved; however, with the discovery of quantum chromo-dynamics (QCD), all "meson theories" had to be relegated to models and the attempts to derive the nuclear force started all over again.

OBE vs Chiral Perturbation Theory

The problem with a derivation from QCD is that this theory is non-perturbative in the low-energy regime characteristic of nuclear physics, which makes direct solutions impossible. Therefore, during the first round of new attempts, QCD-inspired quark models became popular. These models were able to reproduce qualitatively some of the gross features of the nuclear force. But were useless for nuclear structure. Also, on a critical note, it has been pointed out that these quark-based approaches were nothing but another set of models and, thus, did not represent any fundamental progress. Equally well, one may then stay with the simpler and much more quantitative meson models.

Chiral Perturbation Theory

A major breakthrough occurred when the concept of an effective field theory (EFT) was introduced and applied to low-energy QCD. As outlined by Weinberg in 1979 one has to write down the most general Lagrangian consistent with the assumed symmetry principles, particularly the (broken) chiral symmetry of QCD. At low energy, the effective degrees of freedom are pions and nucleons rather than quarks and gluons; heavy mesons and nucleon resonances are "integrated out". So, in a certain sense we are back to the 1950s, except that we are smarter by 40 years of experience: broken chiral symmetry is a crucial constraint that generates and controls the dynamics and establishes a clear connection with the underlying theory, QCD.

Chiral Perturbation Theory

Chiral Perturbation Theory (CHPT) is the effective theory of QCD and, more generally, of the Standard Model, which was formulated by Weinberg and developed in to a systematic tool for analyzing low–energy QCD by Gasser and Leutwyler.
Consider the QCD Lagrangian in the two–flavor case of the light up and down quarks

$$\mathcal{L}_{\rm QCD} = \bar{q}\,(i\gamma_\mu D^\mu - \mathcal{M})q - \frac{1}{4}G^a_{\mu\nu}G^{a\,\mu\nu}\,,$$

where $D_\mu = \partial_\mu - ig_s G^a_\mu T^a$ with T^a, (with $a = 1\ldots 8$) are the $\mathrm{SU(3)_{color}}$ Gell–Mann matrices and q the quark fields. Further, $G^a_{\mu\nu}$ are the gluon field strength tensors, and the quark mass matrix is given by $\mathcal{M} = \mathrm{diag}(m_u,\, m_d)$.

Chiral Perturbation Theory

The left– and right–handed quark fields are defined by $q_{\rm L,R} = 1/2(1 \pm \gamma_5)q$. The chiral group G is a group of independent $\mathrm{SU(2)_{flavor}}$ transformations of the left– and right–handed quark fields, $G = \mathrm{SU(2)_L} \times \mathrm{SU(2)_R}$. Expressing the quark part in the QCD Lagrangian in terms of $q_{L,R}$, it is easy to see that the covariant derivative term is invariant with respect to global chiral rotations, while the quark mass term is not. The running quark masses at the renormalization scale $\mu = 1$ GeV are $m_u \sim 5$ MeV and $m_d \sim 9$ MeV. Given the fact that the masses of the up and down quarks are much smaller than the typical hadron scale of the order of 1 GeV, chiral $\mathrm{SU(2)_L} \times \mathrm{SU(2)_R}$ symmetry can be considered as a rather accurate symmetry of QCD.

Chiral Perturbation Theory

There is a strong evidence on both experimental and theoretical sides that chiral symmetry of QCD is spontaneously broken down to its vector subgroup (isospin group in the two–flavor case). Perhaps, the most striking evidence of the spontaneous breaking of the axial generators is provided by the nonexistence of degenerate parity doublets in the hadron spectrum and the presence of the triplet of unnaturally light pseudoscalar mesons (pions). The latter are natural candidates for the corresponding Nambu–Goldstone bosons which acquire a small nonzero mass due to the explicit chiral symmetry breaking by the nonvanishing quark masses.

Effective Field Theories and QCD

Two worlds

- At high energies: weak, asymptotic freedom; perturbative QCD.
- At low energies (= nuclear physics): strong QCD, non-perturbative; a totally different world.

The fact that the scenario at low- energy is so different from high-energy suggests that the effective description of the low-energy scenario should also be very different from high.

Effective Field Theories and QCD

QCD in the u/d sector has approximate chiral symmetry but this symmetry is broken in two ways:

- Explicitly broken, because the u and d quark masses are not exactly zero;
- Spontaneously broken

$$SU(2)_L \times SU(2)_R \approx SU(2)_V \times SU(2)_A \rightarrow SU(2)_V,$$

that is in the QCD ground state, axial symmetry is broken, while isospin symmetry is intact.

We obtain 3 Goldstone bosons: the pions!

Chiral Perturbation Theory

The chiral effective Lagrangian is given by an infinite series of terms with increasing number of derivatives and/or nucleon fields, with the dependence of each term on the pion field prescribed by the rules of broken chiral symmetry. Applying this Lagrangian to *NN* scattering generates an unlimited number of Feynman diagrams, which may suggest again an untractable problem. However, Weinberg showed that a systematic expansion of the nuclear amplitude exists in terms of $(Q/\Lambda_\chi)^\nu$, where Q denotes a momentum or pion mass, $\Lambda_\chi \approx 1$ GeV is the chiral symmetry breaking scale, and $\nu \geq 0$. For a given order ν, the number of contributing terms is finite and calculable; these terms are uniquely defined and the prediction at each order is model-independent. By going to higher orders, the amplitude can be calculated to any desired accuracy. The scheme just outlined has become known as chiral perturbation theory (χPT).

Effective Field Theories

Therefore, we want to describe the low-energy scenario of QCD by an Effective Field Theory (EFT). The steps to take:

- Write down the most general Lagrangian including all terms consistent with the assumed symmetries, particularly, spontaneously broken chiral symmetry.
- Calculate Feynman diagrams. Note: There will be infinitely many diagrams.
- Find a scheme for assessing the importance of the various diagrams, because we cannot calculate infinitely many diagrams.

Chiral Perturbation Theory

The starting point for the derivation of the *NN* interaction is an effective chiral Lagrangian

$$\mathcal{L} = \mathcal{L}_{\pi N} + \mathcal{L}_{\pi\pi} + \mathcal{L}_{NN},$$

which is given by a series of terms of increasing chiral dimension,

$$\mathcal{L}_{\pi N} = \mathcal{L}^{(1)}_{\pi N} + \mathcal{L}^{(2)}_{\pi N} + \mathcal{L}^{(3)}_{\pi N} + \ldots,$$

$$\mathcal{L}_{\pi\pi} = \mathcal{L}^{(2)}_{\pi\pi} + \ldots,$$

$$\mathcal{L}_{NN} = \mathcal{L}^{(0)}_{NN} + \mathcal{L}^{(2)}_{NN} + \mathcal{L}^{(4)}_{NN} + \ldots,$$

where the superscript refers to the number of derivatives or pion mass insertions (chiral dimension). Good review: Epelbaum, Prog. Part. Nucl. Phys. **57**, 654 (2006).

Chiral Perturbation Theory

Common to apply the heavy baryon (HB) formulation of chiral perturbation theory in which the relativistic Lagrangian is subjected to an expansion in terms of powers of $1/M_N$ (kind of a nonrelativistic expansion), the lowest order of which is

$$\begin{aligned}\widehat{\mathcal{L}}^{(1)}_{\pi N} &= \bar{N}\left(iD_0 - \frac{g_A}{2}\,\vec{\sigma}\cdot\vec{u}\right)N \\ &\approx \bar{N}\left[i\partial_0 - \frac{1}{4f_\pi^2}\,\boldsymbol{\tau}\cdot(\boldsymbol{\pi}\times\partial_0\boldsymbol{\pi}) - \frac{g_A}{2f_\pi}\,\boldsymbol{\tau}\cdot(\vec{\sigma}\cdot\vec{\nabla})\boldsymbol{\pi}\right]N + \ldots\end{aligned}$$

For the parameters that occur in the leading order Lagrangian, we apply $M_N = 938.919$ MeV, $m_\pi = 138.04$ MeV, $f_\pi = 92.4$ MeV, and $g_A = g_{\pi NN}\, f_\pi/M_N = 1.29$, which is equivalent to $g^2_{\pi NN}/4\pi = 13.67$.

NN Interaction

The chiral NN force has the general form

$$V_{2\mathrm{N}} = V_\pi + V_{\mathrm{cont}} \,,$$

where V_{cont} denotes the short–range terms represented by NN contact interactions and V_π corresponds to the long–range part associated with the pion–exchange contributions Both V_π and V_{cont} are determined within the low–momentum expansion.

Notice that the nucleon kinetic energy contributes to $\mathcal{L}^{(2)}$. The above terms determine the nuclear potential up to N^{2}LO (with the exception of the NN contact terms at NLO) in the limit of exact isospin symmetry.

NN Interaction

Consider now pion–exchange contributions to the potential

$$V_\pi = V_{1\pi} + V_{2\pi} + V_{3\pi} + \dots ,$$

where one–, two– and three–pion exchange (3PE) contributions $V_{1\pi}$, $V_{2\pi}$ and $V_{3\pi}$ can be written in the low–momentum expansion as

$$\begin{aligned} V_{1\pi} &= V_{1\pi}^{(0)} + V_{1\pi}^{(2)} + V_{1\pi}^{(3)} + V_{1\pi}^{(4)} + \dots , \\ V_{2\pi} &= V_{2\pi}^{(2)} + V_{2\pi}^{(3)} + V_{2\pi}^{(4)} + \dots , \\ V_{3\pi} &= V_{3\pi}^{(4)} + \dots . \end{aligned}$$

Here, the superscripts denote the corresponding chiral order and the ellipses refer to $(Q/\Lambda)^5$– and higher order terms. Contributions due to the exchange of four– and more pions are further suppressed: n–pion exchange diagrams start to contribute at the order $(Q/\Lambda)^{2n-2}$. Notice further that in addition to isopin–invariant contributions there are isospin–breaking corrections.

NN Interaction

The static 1PE potential at N^{3}LO has the form

$$V_{1\pi}^{(0)} + V_{1\pi}^{(2)} + V_{1\pi}^{(3)} + V_{1\pi}^{(4)} = -\left(\frac{g_A}{2F_\pi}\right)^2 (1+\delta)^2 \, \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2 \, \frac{\vec{\sigma}_1 \cdot \vec{q} \, \vec{\sigma}_2 \cdot \vec{q}}{\vec{q}^{\,2} + M_\pi^2} \, .$$

NN Interaction

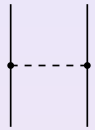
The 2PE contributions are convenient to express as $V_{2\pi}$ in the form:

$$\begin{aligned} V_{2\pi} &= V_C + \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2 \, W_C + [V_S + \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2 \, W_S] \, \vec{\sigma}_1 \cdot \vec{\sigma}_2 + [V_T + \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2 \, W_T] \, \vec{\sigma}_1 \cdot \vec{q} \, \vec{\sigma}_2 \cdot \vec{q} \\ &+ [V_{LS} + \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2 \, W_{LS}] \, i(\vec{\sigma}_1 + \vec{\sigma}_2) \cdot (\vec{q} \times \vec{k}) \\ &+ [V_{\sigma L} + \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2 \, W_{\sigma L}] \, \vec{\sigma}_1 \cdot (\vec{q} \times \vec{k}) \vec{\sigma}_2 \cdot (\vec{q} \times \vec{k}) \, , \end{aligned}$$

where the superscripts C, S, T, LS and σL of the scalar functions V_C, ..., $W_{\sigma L}$ refer to central, spin–spin, tensor, spin–orbit and quadratic spin–orbit components, respectively.

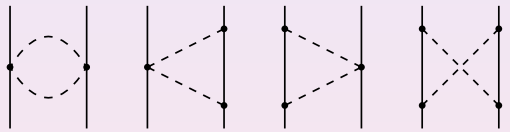
Chiral Perturbation Theory

Q^0



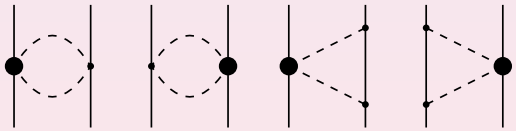
1π Exchange

Q^2



2π Exchange

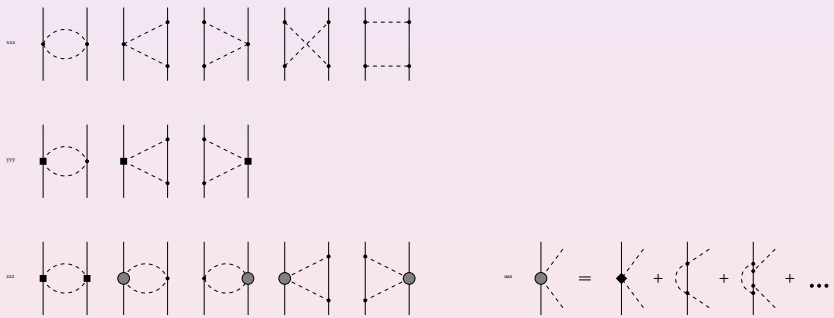
Q^3



χPT

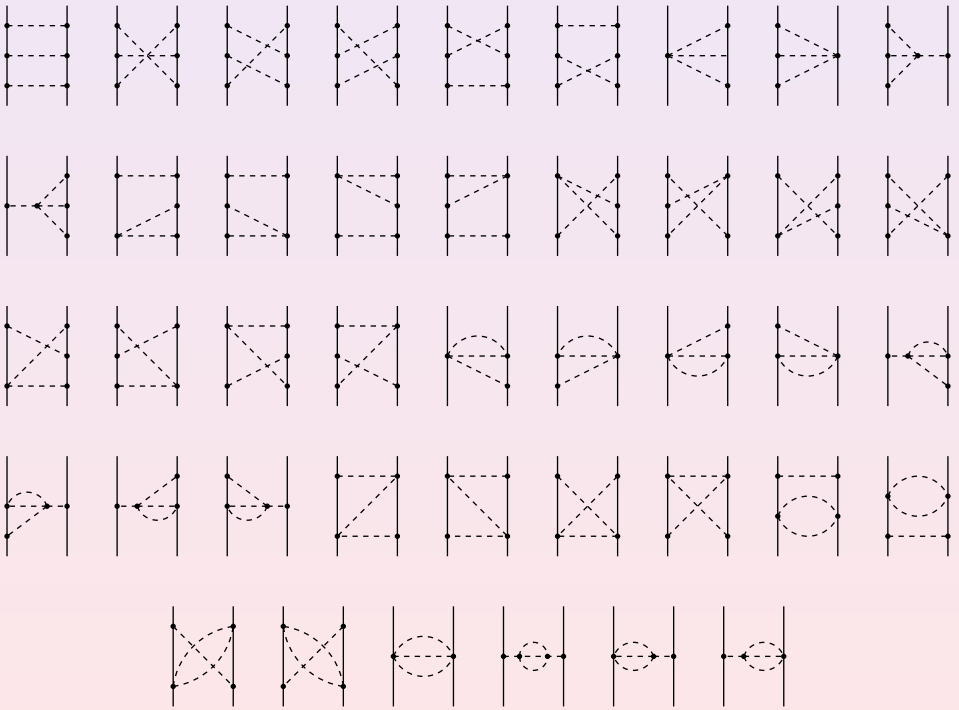
- The most important irreducible one- and two-pion exchange contributions to the NN interaction up to order Q^3.
- Vertices denoted by small dots are from $\widehat{\mathcal{L}}^{(1)}_{\pi N}$.
- Large dots refer to $\widehat{\mathcal{L}}^{(2)}_{\pi N,\,\mathrm{ct}}$

2π-Exchange

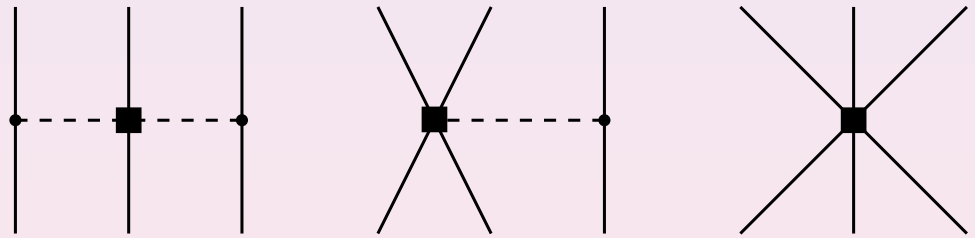


Leading $((\frac{Q}{\Lambda})^2)$, subleading $((\frac{Q}{\Lambda})^3)$ and sub-subleading $((\frac{Q}{\Lambda})^4)$ contributions to the chiral 2π–exchange potential. Solid (dashed) lines correspond to nucleons (pions). Solid dots, filled rectangles and filled diamonds represent vertices with $\Delta_i = 0$, 1 and 2, respectively. Shaded blob denotes the next–to–next–to–leading order contribution to the pion–nucleon scattering amplitude.

Leading contributions 3π-Exchange



Three-Nucleon Force at order $\nu = 3$.



3NF Interaction

The first non–vanishing 3NF contribution appears at order $\nu = 3$, i.e. at N^{2}LO. The contribution from graph (a)

$$V_{2\pi}^{(3)} = \sum_{i \neq j \neq k} \frac{1}{2} \left(\frac{g_A}{2F_\pi} \right)^2 \frac{(\vec{\sigma}_i \cdot \vec{q}_i)(\vec{\sigma}_j \cdot \vec{q}_j)}{(\vec{q}_i{}^2 + M_\pi^2)(\vec{q}_j{}^2 + M_\pi^2)} F_{ijk}^{\alpha\beta} \tau_i^\alpha \tau_j^\beta \,,$$

where $\vec{q}_i \equiv \vec{p}_i{}' - \vec{p}_i$; $\vec{p}_i$ ($\vec{p}_i{}'$) is the initial (final) momentum of the nucleon i and

$$F_{ijk}^{\alpha\beta} = \delta^{\alpha\beta} \left[-\frac{4c_1 M_\pi^2}{F_\pi^2} + \frac{2c_3}{F_\pi^2} \vec{q}_i \cdot \vec{q}_j \right] + \sum_\gamma \frac{c_4}{F_\pi^2} \epsilon^{\alpha\beta\gamma} \tau_k^\gamma \vec{\sigma}_k \cdot [\vec{q}_i \times \vec{q}_j] \,.$$

3NF Interaction

The contributions from the remaining graphs (b) and (c) take the form

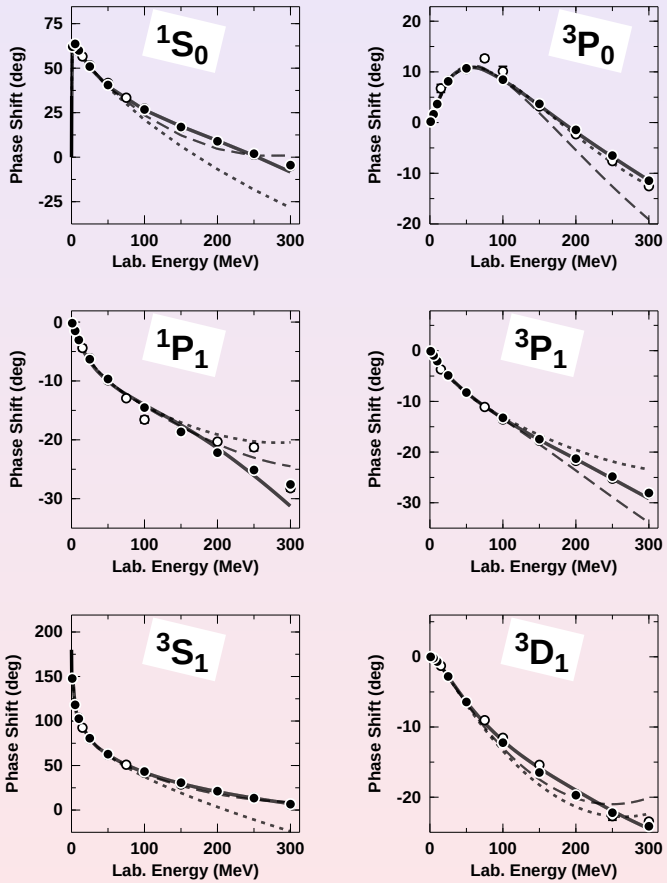
$$V_{1\pi,\,\mathrm{cont}}^{(3)} = -\sum_{i\neq j\neq k} \frac{g_A}{8F_\pi^2}\, D\, \frac{\vec{\sigma}_j\cdot\vec{q}_j}{\vec{q}_j{}^2 + M_\pi^2}\, (\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j)\,(\vec{\sigma}_i\cdot\vec{q}_j)\,, \qquad V_{\mathrm{cont}}^{(3)} = \frac{1}{2}\sum_{j\neq k} E\,(\boldsymbol{\tau}_j\cdot\boldsymbol{\tau}_k)\,,$$

where D and E are the corresponding low-energy constants from the Lagrangian of order $\nu = 1$.

NN and 3NF Interaction

Chiral order	2N force	3N force	4N force
$\nu = 0$	$V_{1\pi} + V_{\rm cont}$	–	–
$\nu = 1$	–	–	–
$\nu = 2$	$V_{1\pi} + V_{2\pi} + V_{\rm cont}$	–	–
$\nu = 3$	$V_{1\pi} + V_{2\pi}$	$V_{2\pi} + V_{1\pi,\,\rm cont} + V_{\rm cont}$	–
$\nu = 4$	$V_{1\pi} + V_{2\pi} + V_{3\pi} + V_{\rm cont}$	work in progress	work in progress

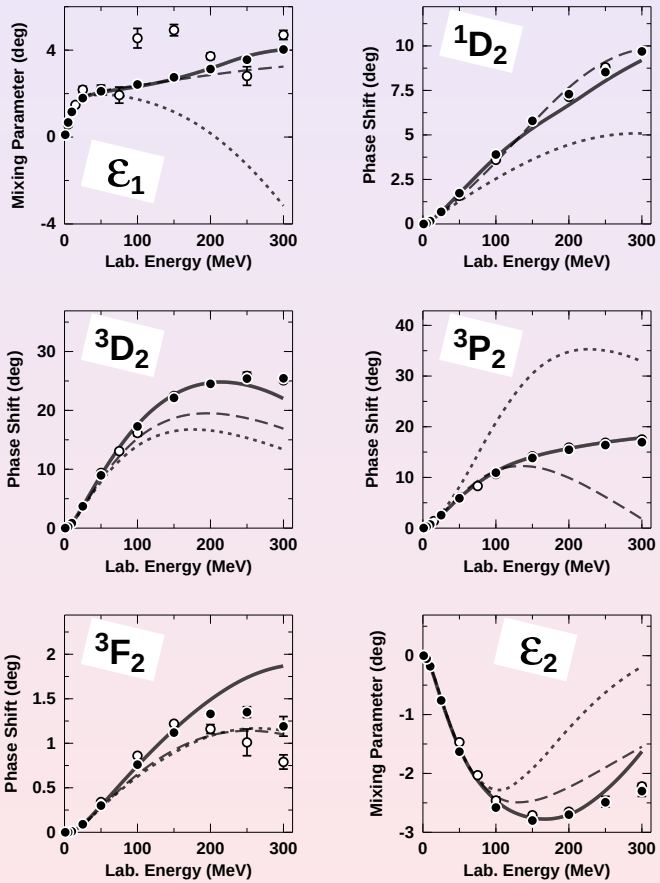
Results with NN interaction to $\nu = 4$



Phase Shifts

- *np* phase parameters below 300 MeV lab. energy for partial waves with $J \leq 2$. The solid line is the result at N^{3}LO.
- The dotted and dashed lines are the phase shifts at NLO and NNLO, respectively.
- The solid dots show the Nijmegen multi-energy *np* phase shift analysis

Results with NN interaction to $\nu = 4$



Phase Shifts

- *np* phase parameters below 300 MeV lab. energy for partial waves with $J \leq 2$. The solid line is the result at N^{3}LO.
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Appendix Outline

3 Appendix
- Hint for Exercises Lecture I, part c)

Exercises Lecture I, part c)

The tensor operator in the nucleon-nucleon potential can be written as (see for example Talmi's recent text)

$$V = \frac{3}{r^2}\left[\left[\sigma_1 \otimes \sigma_2\right]^{(2)} \otimes \left[\mathbf{r} \otimes \mathbf{r}\right]^{(2)}\right]_0^{(0)}$$

Since the irreducible tensor $\left[\mathbf{r} \otimes \mathbf{r}\right]^{(2)}$ operates only on the angular quantum numbers and $\left[\sigma_1 \otimes \sigma_2\right]^{(2)}$ operates only on the spin states we can write the matrix element

$$\begin{aligned}
\langle ISJ|\, V \,|ISJ\rangle &= \langle ISJ| \left[\left[\sigma_1 \otimes \sigma_2\right]^{(2)} \otimes \left[\mathbf{r} \otimes \mathbf{r}\right]^{(2)}\right]_0^{(0)} |I'S'J\rangle \\
&= (-1)^{J+I+S} \left\{ \begin{array}{ccc} I & S & J \\ I' & S' & 2 \end{array} \right\} \langle I|\,|\left[\mathbf{r} \otimes \mathbf{r}\right]^{(2)}|\,|I'\rangle \\
&\quad \times \langle S|\,|\left[\sigma_1 \otimes \sigma_2\right]^{(2)}|\,|S'\rangle
\end{aligned}$$

Exercises Lecture I, c)

You will need that the coordinate vector **r** can be written in terms of spherical components as

$$\mathbf{r}_\alpha = r\sqrt{\frac{4\pi}{3}}\,Y_{1\alpha}$$

Using this expression we get

$$[\mathbf{r}\otimes\mathbf{r}]^{(2)}_\mu \quad = \quad \frac{4\pi}{3}r^2\sum_{\alpha,\beta}\langle 1\alpha 1\beta|\,2\mu\rangle\,Y_{1\alpha}\,Y_{1\beta}$$

The product of two spherical harmonics can be written as

$$Y_{l_1 m_1}\,Y_{l_2 m_2} = \sum_{lm}\sqrt{\frac{(2l_1+1)(2l_2+1)(2l+1)}{4\pi}}\begin{pmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m\end{pmatrix}\begin{pmatrix} l_1 & l_2 & l \\ 0 & 0 & 0\end{pmatrix}$$

Exercises Lecture I, c)

Using this relation we get

$$
\begin{aligned}
[\mathbf{r} \otimes \mathbf{r}]^{(2)}_{\mu} &= \sqrt{4\pi} r^2 \sum_{lm} \sum_{\alpha,\beta} \langle 1\alpha 1\beta | \, 2\mu \rangle \, \langle 1\alpha 1\beta | \, l - m \rangle \frac{(-1)^{1-1-m}}{\sqrt{2l+1}} \begin{pmatrix} 1 & 1 & l \\ 0 & 0 & 0 \end{pmatrix} \\
&= \sqrt{4\pi} r^2 \begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} Y_{2-\mu} \\
&= \sqrt{4\pi} r^2 \sqrt{\frac{2}{15}} Y_{2-\mu}
\end{aligned}
$$

Exercises Lecture I, c)

Use then this relation to rewrite the reduced matrix element containing the position vector as

$$\begin{aligned}\langle l||[\mathbf{r}\otimes\mathbf{r}]^{(2)}||l'\rangle &= \sqrt{4\pi}\sqrt{\frac{2}{15}}r^2\langle l||Y_2||l'\rangle \\ &= \sqrt{4\pi}\sqrt{\frac{2}{15}}r^2(-1)^{l'}\sqrt{\frac{(2l+1)5(2l'+1)}{4\pi}}\begin{pmatrix} l & 2 & l' \\ 0 & 0 & 0\end{pmatrix}\end{aligned}$$

Exercises Lecture I, c)

Use the reduced matrix element of the spin operators defined as

$$\begin{aligned}\langle S||\left[\sigma_1 \otimes \sigma_2\right]^{(2)}||S'\rangle &= \sqrt{(2S+1)(2S'+1)5}\begin{Bmatrix} s_1 & s_2 & S \\ s_3 & s_4 & S' \\ 1 & 1 & 2 \end{Bmatrix} \\ &\times \langle s_1||\sigma_1||s_3\rangle\langle s_2||\sigma_2||s_4\rangle\end{aligned}$$

Exercises Lecture I, c)

Inserting these expressions for the two reduced matrix elements we get

$$\begin{aligned}\langle ISJ|\, V\, |I'S'J\rangle = \quad & (-1)^{S+J}\sqrt{30(2I+1)(2I'+1)(2S+1)(2S'+1)} \\ & \times \left\{ \begin{array}{ccc} I & S & J \\ I' & S & 2 \end{array} \right\} \left(\begin{array}{ccc} I & 2 & I' \\ 0 & 0 & 0 \end{array} \right) \left\{ \begin{array}{ccc} s_1 & s_2 & S \\ s_3 & s_4 & S' \\ 1 & 1 & 2 \end{array} \right\} \\ & \times \langle s_1||\sigma_1||s_3\rangle \langle s_2||\sigma_2||s_4\rangle \,.\end{aligned}$$