

PRELIMINARY VERSION

Finite elements for the Laplace-Beltrami equation on parametric surfaces

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Abstract

In this paper we make a thorough study of the use of the finite element method to numerically compute harmonic maps from parametric surfaces to the plane. There are essentially two choices to be made in the FE method: (1) which elements and (2) which quadrature. We show that by using linear elements and point-based linear quadrature the method reduces to the cotangent method studied by Dziuk, Polthier and others. We are thus able to give a new convergence analysis of this method.

1 Introduction

In recent years many methods have been developed for parameterizing piecewise linear surfaces, i.e., triangular meshes, because surfaces of this kind often arise from digital geometry and no parameterization is available. A survey of such methods can be found in [5]. One popular method for parameterizing meshes is to use a discrete analog of a harmonic map. Harmonic maps are closely related to conformal maps which are maps that preserve angles. In fact for a fixed boundary map, the harmonic map minimizes a certain measure of angle distortion, known as conformal energy. A discrete form of a harmonic map was derived by Pinkall and Polthier [8] and Eck et al. [3] by minimizing Dirichlet energy over the mesh.

Our goal is to carry out a thorough analysis of the use of the finite element method to compute discrete harmonic maps. Instead of letting the minimization of Dirichlet energy be the connection between the continuous and the discrete case, we start from the weak form of the defining PDE. We will assume that we are given a parametric surface, so that we can formulate the map in terms of the given coordinates of the surface as the solution of a second order elliptic differential equation. This is the Laplace-Beltrami equation, and when we apply the finite element method there are two choices to make: (1) choice of elements and (2) choice of quadrature. One of our observations is that by making a special choice of quadrature: using only surface points, rather than derivatives, we reproduce the discrete harmonic map of Pinkall and Polthier and Eck et al., which is independent of the given parameterization. This observation enables us to show further that the discrete harmonic map does indeed converge in the L^2 norm to the true harmonic map, and the convergence is second order. This is a well-known fact in the special case that the surface is planar, but seems only to have been conjectured for general surfaces.

2 The Laplace-Beltrami equation

Suppose a parametric surface is given, $\mathbf{f} : \Omega \rightarrow \mathbb{R}^3$, where Ω is an open, convex and bounded set in $\mathbb{R}^2$. We will assume that its boundary $\partial\Omega$ is piecewise C^∞ . The surface, when viewed as a subset of $\mathbb{R}^3$ is then the set of points

$$S = \{\mathbf{f}(x_1, x_2) : (x_1, x_2) \in \Omega\}.$$

We will assume that $\mathbf{f}$ is *uniformly regular*, by which we mean that $\mathbf{f}$ is continuously differentiable, and that there exists a constant $\alpha > 0$ such that for every point $(x_1, x_2) \in \Omega$,

$$|D_1\mathbf{f} \times D_2\mathbf{f}| \geq \alpha. \tag{1}$$

We now seek to compute a harmonic function on S , that is a function $\phi : S \rightarrow \mathbb{R}$ satisfying

$$\Delta_S \phi = 0, \quad \text{in the interior of } S \tag{2}$$

$$\phi = g_S, \quad \text{on the boundary of } S \tag{3}$$

where Δ_S is the Laplace-Beltrami operator and g_S is a function defined on the boundary of S , giving Dirichlet-type boundary conditions.

However, we will solve (2) in the given coordinates of S , which means that we will be looking for a function $u := \phi \circ \mathbf{f} : \Omega \rightarrow \mathbb{R}$ satisfying the PDE

$$Lu(x_1, x_2) = 0, \quad (x_1, x_2) \in \Omega \quad (4)$$

$$u(x_1, x_2) = \bar{g}, \quad (x_1, x_2) \in \partial\Omega \quad (5)$$

where $\bar{g} := g_S \circ \mathbf{f}$ and L is the second order elliptic operator

$$L := \frac{1}{\sqrt{\det G}} \sum_{i,j=1}^2 D_i(\sqrt{\det G} g^{ij} D_j).$$

In the definition above, the terms g^{ij} are the elements of the inverse metric tensor, the 2×2 matrix given by

$$G^{-1} = (g^{ij}) = \frac{1}{\det G} \begin{pmatrix} g_{22} & -g_{12} \\ -g_{21} & g_{11} \end{pmatrix},$$

$$G = (g_{ij}) = (D_i \mathbf{f} \cdot D_j \mathbf{f}),$$

where $\cdot$ denotes the scalar product in $\mathbb{R}^3$.

Note that since the right hand side of (4) is zero, the scaling factor $1/\sqrt{\det G}$ in the operator L plays no role in the equation. Further, we can simplify the equation a little if we define

$$A = (a_{ij}) = \sqrt{\det G} (g^{ij}) = \frac{1}{\sqrt{\det G}} \begin{pmatrix} g_{22} & -g_{12} \\ -g_{12} & g_{11} \end{pmatrix}.$$

Then the equation for u becomes

$$\sum_{i,j=1}^2 D_i(a_{ij} D_j u) = 0. \quad (6)$$

Unless the surface $\mathbf{f}$ is quite simple, such as a plane, sphere or cylinder, we cannot hope to find closed-form solutions. Worse, there might not even exist any solution u at all in $C^2(\Omega)$ satisfying the PDE (4). To ensure existence and uniqueness, we have to weaken our notion of solution.

We will now motivate the weak form of the equation. Assume temporarily that there exists a $u \in C^2(\Omega)$ solving (6). We may multiply (6) by any smooth function with compact support in Ω , $v \in C_c^\infty(\Omega)$, and integrate to obtain

$$\int_{\Omega} \sum_{i,j=1}^2 (D_i(a_{ij}(\mathbf{x})D_j u(\mathbf{x}))) v(\mathbf{x}) d\mathbf{x} = 0. \quad \text{for all } v \in C_c^\infty(\Omega).$$

Now we apply integration by parts, giving

$$\int_{\Omega} \sum_{i,j=1}^2 a_{ij}(\mathbf{x}) D_j u(\mathbf{x}) D_i v(\mathbf{x}) d\mathbf{x} = 0. \quad \text{for all } v \in C_c^\infty(\Omega). \quad (7)$$

The above we may write as $a(u, v) = 0$, where $a(\cdot, \cdot)$ is the symmetric bilinear form

$$a(u, v) = \int_{\Omega} \sum_{i,j=1}^2 a_{ij}(\mathbf{x}) D_i u(\mathbf{x}) D_j v(\mathbf{x}) d\mathbf{x}. \quad (8)$$

It is evident that (7) holds for all classical (C^2) solutions, yet the expression is valid even if u is in C^1 , since only one derivative appears. In fact, we will only require that u is in the Sobolev space $H^1(\Omega)$, the space of L^2 functions on Ω with weak derivatives in L^2 . The test function u will be in $H_0^1(\Omega)$, defined by

$$H_0^1(\Omega) := \{v \in H^1(\Omega) : Tv = 0\},$$

where T is the trace operator taking functions in $H^1(\Omega)$ to functions in $L^2(\partial\Omega)$. For more detailed definitions and explanations, see for instance [4] or [6]. From now on, we will usually write H^1 for $H^1(\Omega)$, L^2 for $L^2(\Omega)$ and so on.

Definition 1 (Weak solution) *We say that $u \in H^1$ is a weak solution of the Laplace-Beltrami equation with Dirichlet boundary conditions provided that*

1. $a(u, v) = 0$ for all $v \in H_0^1$
2. $Tu = \bar{g}$

This definition will enable us to prove existence and uniqueness.

3 Properties of the solution u

In this section, we show existence and uniqueness of the weak solution u , as well as investigate its regularity.

We will homogenize the problem in order to apply certain theorems. By homogenizing, we mean modifying the equation so that the Dirichlet boundary conditions are simply that the trace of the solution is 0 on the boundary.

Let g be a function in H^1 with $\bar{g}$ as its trace. Let $w := u - g$. Then $w \in H_0^1$ and

$$a(w, v) = -a(g, v) \quad \text{for all } v \in H_0^1. \quad (9)$$

A remark on g : for such a function to exist, we must make certain assumptions on $\bar{g}$. By Theorem 1.5.2.8 in [7], it is sufficient for $\bar{g}$ to be

1. $\bar{g} \in C^1$ on any side of $\partial\Omega$ and
2. continuous at the corners.

We will call such $\bar{g}$ *compatible* Dirichlet boundary conditions. Then in fact $g \in H^2(\Omega)$. Of course, then we may integrate by parts, so (9) becomes

$$a(w, v) = (Lg, v)_0 \quad \text{for all } v \in H_0^1. \quad (10)$$

where $Lg \in L^2$ and $(\cdot, \cdot)_0$ is the inner product in L^2 .

By Theorem 8.9 in [6], we need to satisfy the following criteria in order to have a unique solution $u \in H^1 \cap H_{\text{loc}}^2$:

1. a_{ij} must be Lipschitz continuous,
2. L must be uniformly elliptic, that is, there exists a constant $\theta > 0$ such that

$$\sum_{i,j=1}^2 a_{ij}(\mathbf{x}) \xi_i \xi_j \geq \theta |\xi|^2, \quad \forall \mathbf{x} \in \Omega, \xi \in \mathbb{R}^2. \quad (11)$$

Alternatively, there exists a constant $\alpha > 0$ such that

$$a(v, v) \geq \alpha \|v\|_{H^1}^2. \quad (12)$$

The first criterion holds for any regular C^2 parametrized surface, but may fail for C^1 surfaces if $D_i \mathbf{f}$ is not Lipschitz. For simplicity, we will assume that

the surface is C^2 from now on. This will include for instance cubic tensor-product spline surfaces.

The second criterion holds if the matrix $(a_{ij}(\mathbf{x}))$ is uniformly positive definite in Ω . This follows from (1), since both a_{11} and $\det(a_{ij})$ are bounded away from zero uniformly in Ω by the regularity of $\mathbf{f}$.

Now we have established that $u \in H_{\text{loc}}^2$, that is, for every $V \subset\subset \Omega$, $u \in H^2(V)$. We would like to have $u \in H^2$. We can deduce that w , the homogenized solution, has this property from Theorem 3.2.1.2 in [7], since our domain is convex and bounded and w satisfies (10). Since $u = w + g$, and we have already shown $g \in H^2$, we have $u \in H^2$.

Since $\Omega \in \mathbb{R}^2$, it follows from various Sobolev inequalities ([4], section 5.6) that u is in fact continuous on $\bar{\Omega}$.

4 Finite element method

Consider now a finite element approximation to the solution u . Much of the basic theory is well-documented; see for example Braess [1].

To derive a finite element approximation to u we start from the weak formulation, but replace the infinite-dimensional function spaces by finite-dimensional subspaces.

Let $\mathcal{T}$ be some triangulation of Ω and let $S_h \subset H^1$ denote the spline space consisting of continuous functions on Ω where the restriction of each function to any triangle in $\mathcal{T}$ is linear. The subscript h denotes the diameter of the largest triangle in $\mathcal{T}$. We also need to consider the subspace

$$S_{h,0} = \{v \in S_h : v|_{\partial\Omega} = 0\}.$$

Let $g_h \in S_h$ be some approximation to g . For the error analysis to follow in section 6, we will need to assume that $|g - g_h| \leq Ch^{3/2}$ on $\partial\Omega$. We now seek a function u_h in S_h such that $u_h - g_h \in S_{h,0}$ and

$$a(u_h, v) = 0, \quad \text{for all } v \in S_{h,0}. \quad (13)$$

To solve this we need a basis for S_h , and the set of hat functions is suitable. Let V be the set of vertices in $\mathcal{T}$ and for each vertex $\mathbf{v}$ of V let $B_{\mathbf{v}}$ be the associated hat function, i.e., the element of S_h such that $B_{\mathbf{v}}(\mathbf{v}) = 1$ and $B_{\mathbf{v}}(\mathbf{w}) = 0$ for $\mathbf{w} \in V$ with $\mathbf{w} \neq \mathbf{v}$. These hats form a basis of S_h and

so u_h can be expressed as

$$u_h(\mathbf{x}) = \sum_{\mathbf{v} \in V} u_{\mathbf{v}} B_{\mathbf{v}}(\mathbf{x}).$$

Further, let V_B denote the boundary vertices of $\mathcal{T}$, and V_I the inner vertices. Since the inner hat functions $B_{\mathbf{v}}$, $\mathbf{v} \in V_I$, form a basis for $S_{h,0}$ we take v in (13) to be $B_{\mathbf{w}}$ for some $w \in V_I$ yielding the linear system

$$\sum_{\mathbf{v} \in V} c_{\mathbf{vw}} u_{\mathbf{v}} = 0, \quad \mathbf{w} \in V_I, \quad (14)$$

where

$$c_{\mathbf{vw}} = a(B_{\mathbf{v}}, B_{\mathbf{w}}), \quad (15)$$

and $u_{\mathbf{v}} = g_h(\mathbf{v})$ for $\mathbf{v} \in V_B$. The coefficient $c_{\mathbf{vw}}$ is only non-zero when either $\mathbf{v} = \mathbf{w}$ or $\mathbf{v}$ and $\mathbf{w}$ are neighbours in the triangulation $\mathcal{T}$. In practice we can only approximate the integral defining $c_{\mathbf{vw}}$. The usual way to approximate it is simply to apply some known quadrature rule directly to the integral, a rule whose accuracy matches the accuracy of the approximation $u_h \approx u$. However, the functions a_{ij} involve derivatives (of first order) of $\mathbf{f}$ and so such a quadrature rule would require differentiating $\mathbf{f}$. We would like to avoid that and we propose an alternative approach. We let $\mathbf{f}_h$ be a piecewise linear interpolant to $\mathbf{f}$ over $\mathcal{T}$. For each triangle $T = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$, $\mathbf{f}_h$ interpolates $\mathbf{f}$ at the 3 corner points of T . We now use the approximation

$$\tilde{a}(u, v) = \int_{\Omega} \sum_{i,j=1}^2 \tilde{a}_{ij}(\mathbf{x}) D_i u(\mathbf{x}) D_j v(\mathbf{x}) d\mathbf{x}. \quad (16)$$

to $a(u, v)$ where

$$(\tilde{a}_{ij})_{ij} = \frac{1}{\sqrt{\det \tilde{G}}} \begin{pmatrix} \tilde{g}_{22} & -\tilde{g}_{12} \\ -\tilde{g}_{21} & \tilde{g}_{11} \end{pmatrix},$$

and

$$\tilde{G} = (\tilde{g}_{ij}) = (D_i \mathbf{f}_h \cdot D_j \mathbf{f}_h).$$

We now seek the function $\tilde{u}_h$ in S_h such that $\tilde{u}_h - g_h \in S_{h,0}$ and

$$\tilde{a}(\tilde{u}_h, v) = 0, \quad \text{for all } v \in S_{h,0}. \quad (17)$$

This means solving the linear system

$$\sum_{\mathbf{v} \in V} \tilde{c}_{\mathbf{v}\mathbf{w}} \tilde{u}_{\mathbf{v}} = 0, \quad \mathbf{w} \in V_I, \quad (18)$$

where

$$\tilde{c}_{\mathbf{v}\mathbf{w}} = \tilde{a}(B_{\mathbf{v}}, B_{\mathbf{w}}), \quad (19)$$

and $\tilde{u}_{\mathbf{v}} = g_h(\mathbf{v})$ for $\mathbf{v} \in V_B$.

5 Deducing the cotangent weights

We can split the integral in $\tilde{c}_{\mathbf{v}\mathbf{w}}$ into a sum of integrals over triangles,

$$\tilde{c}_{\mathbf{v}\mathbf{w}} = \sum_{T \in \mathcal{T}} \tilde{c}_{\mathbf{v}\mathbf{w}T},$$

where

$$\tilde{c}_{\mathbf{v}\mathbf{w}T} = \int_T \sum_{i,j=1}^2 \tilde{a}_{ij}(\mathbf{x}) D_i B_{\mathbf{v}}(\mathbf{x}) D_j B_{\mathbf{w}}(\mathbf{x}) d\mathbf{x}.$$

Since $\mathbf{f}_h$ is piecewise linear, the integrand in $\tilde{c}_{\mathbf{v}\mathbf{w}T}$ is then a constant in $\mathbf{x}$ and so this integral can be found exactly. In fact this gives a well-known method. Suppose $T = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$ and let $\mathbf{f}_i = \mathbf{f}(\mathbf{v}_i)$ for $i = 1, 2, 3$, and let α_i denote the angle in the triangle $\mathbf{f}_h(T) = [\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3]$ incident on the vertex $\mathbf{f}_i$. Let

$$\tilde{c}_{ij} = \tilde{c}_{\mathbf{v}_i \mathbf{v}_j T}.$$

Theorem 1 *For $\tilde{c}_{ij}$ and α_i as defined above,*

$$(\tilde{c}_{ij})_{i,j} = \frac{1}{2} \begin{pmatrix} \cot \alpha_2 + \cot \alpha_3 & -\cot \alpha_3 & -\cot \alpha_2 \\ -\cot \alpha_3 & \cot \alpha_3 + \cot \alpha_1 & -\cot \alpha_1 \\ -\cot \alpha_2 & -\cot \alpha_1 & \cot \alpha_1 + \cot \alpha_2 \end{pmatrix}. \quad (20)$$

It follows that the linear system (18) reduces to

$$\sum_{w \in N_{\mathbf{v}}} (\cot \alpha_{\mathbf{v}\mathbf{w}} + \cot \beta_{\mathbf{v}\mathbf{w}}) (\tilde{u}_{\mathbf{u}} - \tilde{u}_{\mathbf{w}}) = 0, \quad (21)$$

which is the method Pinkall and Polthier [8] and Eck et al [3] derived.

Proof. Let $\lambda_i : T \rightarrow \mathbb{R}$, $i = 1, 2, 3$ be the three barycentric coordinates associated with $\mathbf{v}_i$ in T , i.e, λ_i is the linear polynomial such that $\lambda_i(\mathbf{v}_j) = \delta_{ij}$ for $i, j = 1, 2, 3$. Then for $\mathbf{x} \in T$, $B_{\mathbf{v}_i}(\mathbf{x}) = \lambda_i(\mathbf{x})$, $i = 1, 2, 3$. By the definition of $\tilde{G}$,

$$\tilde{c}_{ij} = \int_T \frac{1}{\sqrt{\det \tilde{G}}} (-D_2 \lambda_i D_1 \mathbf{f}_h + D_1 \lambda_i D_2 \mathbf{f}_h) \cdot (-D_2 \lambda_j D_1 \mathbf{f}_h + D_1 \lambda_j D_2 \mathbf{f}_h) d\mathbf{x}.$$

Since

$$\mathbf{f}_h(\mathbf{x}) = \sum_{i=1}^3 \lambda_i(\mathbf{x}) \mathbf{f}_i,$$

and

$$\sum_{i=1}^3 \lambda_i(\mathbf{x}) = 1,$$

we have that for any i , interpreting subscripts modulo 3,

$$D_k \mathbf{f}_h = D_k \lambda_{i+1}(\mathbf{f}_{i+1} - \mathbf{f}_i) + D_k \lambda_{i+2}(\mathbf{f}_{i+2} - \mathbf{f}_i), \quad k = 1, 2.$$

So

$$\begin{aligned} -D_2 \lambda_i D_1 \mathbf{f}_h + D_1 \lambda_i D_2 \mathbf{f}_h &= (D_2 \lambda_{i+1} + D_2 \lambda_{i+2}) D_1 \mathbf{f}_h - (D_1 \lambda_{i+1} + D_1 \lambda_{i+2}) D_2 \mathbf{f}_h \\ &= (D_2 \lambda_{i+1} D_1 \lambda_{i+2} - D_1 \lambda_{i+1} D_2 \lambda_{i+2})(\mathbf{f}_{i+2} - \mathbf{f}_{i+1}) \\ &= \frac{-1}{2A}(\mathbf{f}_{i+2} - \mathbf{f}_{i+1}), \end{aligned}$$

where A is the area of the triangle T . Further, letting $\hat{A}$ be the area of the spatial triangle $\mathbf{f}_h(T)$, we have

$$\hat{A} = \int_{\mathbf{f}_h(T)} dA = \int_T \sqrt{\det \tilde{G}} dx_1 dx_2 = \sqrt{\det \tilde{G}} A,$$

from which it follows that $\sqrt{\det \tilde{G}} = \hat{A}/A$. Therefore

$$\tilde{c}_{ij} = \frac{1}{4\hat{A}}(\mathbf{f}_{i+2} - \mathbf{f}_{i+1}) \cdot (\mathbf{f}_{j+2} - \mathbf{f}_{j+1}). \quad (22)$$

Suppose $i \neq j$. Then since

$$(\mathbf{f}_{i+1} - \mathbf{f}_i) \cdot (\mathbf{f}_{i+2} - \mathbf{f}_i) = |\mathbf{f}_{i+1} - \mathbf{f}_i| |\mathbf{f}_{i+2} - \mathbf{f}_i| \cos \alpha_i,$$

and

$$\hat{A} = \frac{1}{2} |\mathbf{f}_{i+1} - \mathbf{f}_i| |\mathbf{f}_{i+2} - \mathbf{f}_i| \sin \alpha_i,$$

we find

$$\tilde{c}_{ij} = -\frac{1}{2} \cot \alpha_k,$$

where $k \neq i, j$. This gives the off-diagonal elements of the matrix (20). Finally, summing (22) over j implies

$$\sum_{j=1}^3 \tilde{c}_{ij} = 0,$$

and so the sum of the each row in (20) is zero, which gives the diagonal elements. $\square$

6 Error analysis

6.1 H^1 error

Theorem 2 (H^1 error of the cotangent method) *Let u be a weak solution of the Laplace-Beltrami equation on Ω with compatible Dirichlet boundary conditions $\bar{g}$. For a shape-regular and shape-uniform family of triangulations $\{\mathcal{T}_h\}$ of Ω , the solution $\tilde{u}_h$ of the linear system of equations (21) satisfies*

$$\|u - \tilde{u}_h\|_{H^1} \leq Ch$$

for sufficiently small h .

Proof.

The proof will be based on the first lemma of Strang, in the following form:

Lemma 1 (First lemma of Strang) *Let u be a weak solution of the Laplace-Beltrami equation as discussed in sections 2 and 3, and let the functions $\tilde{u}_h, g_h \in S_h$ and the bilinear form $\tilde{a} : S_h \times S_h \rightarrow \mathbb{R}$ satisfy the following:*

1. $\tilde{a}$ is uniformly elliptic, that is, there exists a constant β independent of h such that

$$\tilde{a}(v, v) \geq \beta \|v\|_{H^1}^2 \quad v \in S_h,$$

2. $\tilde{u}_h - g_h \in S_{h,0}$,
3. $\tilde{a}(\tilde{u}_h, v) = 0$ for all $v \in S_{h,0}$.

Then there is a constant c independent of h such that

$$\begin{aligned} \|u - \tilde{u}_h\|_{H^1} \leq & c \left(\inf_{v_h \in S_{h,0} + g_h} \left\{ \|u - v_h\|_{H^1} \right. \right. \\ & \left. \left. + \sup_{w_h \in S_{h,0}} \frac{|a(v_h, w_h) - \tilde{a}(v_h, w_h)|}{\|w_h\|_{H^1}} \right\} \right) \end{aligned} \quad (23)$$

The notation $S_{h,0} + g_h$ means the set of functions $v \in S_h$ such that $v - g_h \in S_{h,0}$. A proof of this lemma may be found for instance in [1], page 107.

Prerequisite 2 and 3 are satisfied by the definition of $\tilde{u}_h$, equation (17).

It will be shown later that $|\det G - \det \tilde{G}| \leq Ch$, so

$$\det \tilde{G} \geq \tilde{\alpha} > 0 \quad (24)$$

for sufficiently small h . Then the surface $\tilde{\mathbf{f}}$ is regular, and ellipticity follows for $\tilde{a}$ as it did for a in section 3. This gives prerequisite 1.

In order to compute the bound given in the Lemma, we may choose any arbitrary v_h , obtaining

$$\|u - \tilde{u}_h\|_{H^1} \leq c \left(\|u - v_h\|_{H^1} + \sup_{w_h \in S_{h,0}} \frac{|a(v_h, w_h) - \tilde{a}(v_h, w_h)|}{\|w_h\|_{H^1}} \right). \quad (25)$$

In the following, C and c will denote arbitrary constants independent of h , that may change from line to line. Since u is continuous, we may apply interpolation operators to it. Let I_h denotes interpolation on the triangulation $\mathcal{T}_h$ by a piecewise linear function. Then we let $v_h = I_h u$ on interior nodes, and $v_h = g_h$ on boundary nodes. Since $\{\mathcal{T}_h\}$ is shape-regular, by Theorem 6.4 in [1], page 79 we have

$$\|u - I_h u\|_{H^1} \leq ch|u|_{H^2} \leq Ch. \quad (26)$$

Here $|\cdot|_{H^2}$ denotes the H^2 semi-norm. Recall also that $u \in H^2$, so its H^2 semi-norm is bounded.

The first term may now be estimated.

$$\begin{aligned}
E_1 &:= \|u - v_h\|_{H^1} \leq \|u - I_h u\|_{H^1} + \|I_h u - v_h\|_{H^1} \\
&\leq Ch + \sqrt{\int_{\Omega} |I_h u(\mathbf{x}) - v_h(\mathbf{x})|^2 + |DI_h u(\mathbf{x}) - Dv_h(\mathbf{x})|^2 d\mathbf{x}} \\
&= Ch + \sqrt{\sum_{T \in \partial \mathcal{T}} \int_T |I_h u(\mathbf{x}) - v_h(\mathbf{x})|^2 + |DI_h u(\mathbf{x}) - Dv_h(\mathbf{x})|^2 d\mathbf{x}},
\end{aligned}$$

where $\partial \mathcal{T}$ contains all triangles in $\mathcal{T}$ with at least one vertex on the boundary. From the assumption that $|g - g_h| \leq Ch^{3/2}$ on the boundary, we may conclude that $|I_h u - v_h| \leq Ch^{3/2}$ and by shape-regularity of $\{\mathcal{T}_h\}$, also that $|DI_h u - Dv_h| \leq Ch^{1/2}$ on the boundary. So we have

$$\begin{aligned}
E_1 &\leq Ch + \sqrt{\sum_{T \in \partial \mathcal{T}} \int_T ch d\mathbf{x}}, \\
&\leq Ch + \sqrt{\sum_{T \in \partial \mathcal{T}} ch^3} \\
&\leq Ch + \sqrt{ch^2} \leq Ch.
\end{aligned}$$

The second inequality follows from the fact that we are integrating over triangles with area bounded by h^2 . The third inequality follows from summing over a number of triangles bounded by $1/h$. This requires the condition of shape-uniformity.

We now estimate the second term. Let

$$M := \sup_{\mathbf{x} \in \Omega} \max_{ij} |a_{ij}(\mathbf{x}) - \tilde{a}_{ij}(\mathbf{x})|.$$

Then

$$\begin{aligned}
E_2 &:= \sup_{w_h \in S_{h,0}} \frac{|a(v_h, w_h) - \tilde{a}(v_h, w_h)|}{\|w_h\|_{H^1}} \\
&= \sup_{w_h \in S_{h,0}} \frac{1}{\|w_h\|_{H^1}} \left| \int_{\Omega} \sum_{i,j=1}^2 (a_{ij}(\mathbf{x}) - \tilde{a}_{ij}(\mathbf{x})) D_i v_h D_j w_h \, d\mathbf{x} \right| \\
&\leq M \sup_{w_h \in S_{h,0}} \frac{1}{\|w_h\|_{H^1}} \sum_{i,j=1}^2 \int_{\Omega} |D_i v_h D_j w_h| \, d\mathbf{x} \\
&\leq 4M \sup_{w_h \in S_{h,0}} \frac{1}{\|w_h\|_{H^1}} |v_h|_{H^1} |w_h|_{H^1} \\
&= 4M |v_h|_{H^1}.
\end{aligned}$$

By the analysis of E_1 , $\|u - v_h\|_{H^1} \leq Ch$, so clearly $|v_h|_{H^1}$ is bounded for sufficiently small h . Writing g for $\det G$ and $\tilde{g}$ for $\det \tilde{G}$, we now show that $M \leq Ch$ for sufficiently small h ,

$$\begin{aligned}
\max_{ij} |a_{ij} - \tilde{a}_{ij}| &= \max_{ij} \left| \frac{g_{ij}}{\sqrt{g}} - \frac{\tilde{g}_{ij}}{\sqrt{\tilde{g}}} \right| \\
&= \max_{ij} \left| \frac{g_{ij}\sqrt{\tilde{g}} - \tilde{g}_{ij}\sqrt{g}}{\sqrt{g\tilde{g}}} \right| \\
&= \max_{ij} \left| \frac{g_{ij}(\tilde{g} - g)}{\sqrt{g\tilde{g}}(\sqrt{\tilde{g}} + \sqrt{g})} + \frac{g_{ij} - \tilde{g}_{ij}}{\sqrt{\tilde{g}}} \right| \\
&\leq \max_{ij} (C_1 |g_{ij}(\tilde{g} - g)| + C_2 |\tilde{g}_{ij} - g_{ij}|).
\end{aligned}$$

The last inequality follows from (1) and (24). We let $\mathbf{e} := \tilde{\mathbf{f}} - \mathbf{f}$, and single subscripts denote partial derivatives. Then

$$\begin{aligned}
\tilde{g}_{ij} &= \mathbf{f}_i \mathbf{f}_j + \mathbf{f}_i \mathbf{e}_j + \mathbf{f}_j \mathbf{e}_i + \mathbf{e}_i \mathbf{e}_j \\
&= g_{ij} + \mathbf{f}_i \mathbf{e}_j + \mathbf{f}_j \mathbf{e}_i + R_{ij}
\end{aligned} \tag{27}$$

$$\begin{aligned}
\tilde{g} &= \tilde{g}_{11} \tilde{g}_{22} - \tilde{g}_{12}^2 \\
&= g + 2(g_{22} \mathbf{f}_1 - g_{12} \mathbf{f}_2) \mathbf{e}_1 + 2(g_{11} \mathbf{f}_2 - g_{12} \mathbf{f}_1) \mathbf{e}_2 + R
\end{aligned} \tag{28}$$

where R_{ij} and R only contain terms with at least two factors of $\mathbf{e}_i$ or $\mathbf{e}_j$. Again using Theorem 6.4 of [1], we know that $|\mathbf{e}|_{H^1} \leq Ch$, so $|\mathbf{e}_i| \leq Ch$, and

obtain

$$\begin{aligned} |\tilde{g}_{ij} - g_{ij}| &\leq Ch \\ |\tilde{g} - g| &\leq Ch \end{aligned}$$

So $M \leq Ch$ and the theorem follows. $\square$

6.2 L^2 error

Theorem 3 (L^2 error of the cotangent method) *Under the same conditions as in Theorem 2,*

$$\|u - \tilde{u}_h\|_{L^2} \leq Ch^2.$$

Proof. To show this, we will apply a form of the Aubin-Nitsche Lemma (see 4.1.3, page 203 of [2]):

Lemma 2 (Aubin-Nitsche Lemma) *Under the same conditions as in Lemma 1, the finite element solution $\tilde{u}_h$ satisfies*

$$\begin{aligned} \|u - \tilde{u}_h\|_{L^2} \leq C \sup_{w \in L^2} \frac{1}{\|w\|_{L^2}} &\left(\inf_{v_h \in S_{h,0}} \{ \|u - \tilde{u}_h\|_{H^1} \|\phi_w - v_h\|_{H^1} \right. \\ &\left. + |a(\tilde{u}_h, v_h) - \tilde{a}(\tilde{u}_h, v_h)| \} \right), \end{aligned}$$

where for each w , ϕ_w is the unique function in H_0^1 satisfying

$$a(v, \phi_w) = (w, v)_0 \quad \text{for all } v \in H_0^1.$$

In this proof, any statement that a quantity is $\leq Ch$ is implicitly assumed to hold only for sufficiently small h . Let $v_h = I_h \phi_w$. Then as before, we may split the error into two parts. The first part is bounded by Ch^2 because as we have already shown, $\|u - \tilde{u}_h\|_{H^1} \leq Ch$, and by (26),

$$\|\phi_w - I_h \phi_w\|_{H^1} \leq ch |\phi_w|_2 \leq Ch \|w\|_{L^2}.$$

The last inequality follows from Theorem 6, page 306 of [4].

The second part will require a bit more effort. Going back to (27) and (28), we write

$$\begin{aligned}
\tilde{g}_{ij} - g_{ij} &= \mathbf{f}_i \mathbf{e}_j + \mathbf{f}_j \mathbf{e}_i + R_{ij} \\
&=: \mathbf{p}_{ij} \mathbf{e}_1 + \mathbf{q}_{ij} \mathbf{e}_2 + R_{ij}, \\
\tilde{g} - g &= 2(g_{22} \mathbf{f}_1 - g_{12} \mathbf{f}_2) \mathbf{e}_1 + 2(g_{11} \mathbf{f}_2 - g_{12} \mathbf{f}_1) \mathbf{e}_2 + R \\
&=: \mathbf{p} \mathbf{e}_1 + \mathbf{q} \mathbf{e}_2 + R.
\end{aligned}$$

By the assumption that $\mathbf{f} \in C^2(\Omega)$, $\mathbf{p}, \mathbf{q}, \mathbf{p}_{ij}, \mathbf{q}_{ij}$ are all in $C^1(\Omega)$. The coefficients of $a(\cdot, \cdot) - \tilde{a}(\cdot, \cdot)$ may be written

$$\begin{aligned}
E_{ij} &:= (-1)^{i+j} (a_{2-i,2-j} - \tilde{a}_{2-i,2-j}) \\
&= \frac{g_{ij}}{\sqrt{g}} - \frac{\tilde{g}_{ij}}{\sqrt{\tilde{g}}} \\
&= \frac{g_{ij}(\tilde{g} - g)}{\sqrt{g\tilde{g}}(\sqrt{\tilde{g}} + \sqrt{g})} + \frac{g_{ij} - \tilde{g}_{ij}}{\sqrt{\tilde{g}}} \\
&=: \tilde{\gamma}_{ij}(\tilde{g} - g) + \tilde{\gamma}(g_{ij} - \tilde{g}_{ij}). \\
&= (\tilde{\gamma}_{ij} \mathbf{p} + \tilde{\gamma} \mathbf{p}_{ij}) \mathbf{e}_1 + (\tilde{\gamma}_{ij} \mathbf{q} + \tilde{\gamma} \mathbf{q}_{ij}) \mathbf{e}_2 + \tilde{\gamma}_{ij} R + \tilde{\gamma} R_{ij} \\
&=: \tilde{\mathbf{r}}_{ij} \mathbf{e}_1 + \tilde{\mathbf{s}}_{ij} \mathbf{e}_2 + \tilde{Q}_{ij}.
\end{aligned}$$

In the above, all defined quantities written with a tilde ($\tilde{\cdot}$) are piecewise C^1 , but not continuous over Ω . With the above notation,

$$\begin{aligned}
|a(\tilde{u}_h, I_h \phi_w) - \tilde{a}(\tilde{u}_h, I_h \phi_w)| &= \left| \sum_{i,j=1}^2 (-1)^{i+j} \left(\int_{\Omega} \tilde{\mathbf{r}}_{2-i,2-j} D_i \tilde{u}_h D_j (I_h \phi_w) \mathbf{e}_1 d\mathbf{x} \right. \right. \\
&\quad + \int_{\Omega} \tilde{\mathbf{s}}_{2-i,2-j} D_i \tilde{u}_h D_j (I_h \phi_w) \mathbf{e}_2 d\mathbf{x} \\
&\quad \left. \left. + \int_{\Omega} \tilde{Q}_{2-i,2-j} D_i \tilde{u}_h D_j (I_h \phi_w) d\mathbf{x} \right) \right|.
\end{aligned}$$

Since all terms of $\tilde{Q}_{2-i,2-j}$ must contain at least two factors of $\mathbf{e}_1$ or $\mathbf{e}_j$, clearly the last term is bounded by $ch^2 |\tilde{u}_h|_{H^1} |I_h \phi_w|_{H^1} \leq Ch^2 \|w\|_{L^2}$.

To show that the same holds for the other two terms, we would use integration by parts to move the derivative from $\mathbf{e}$ to the other factors in the integral. However, those factors are not even continuous on Ω , so we need to replace them with smoother versions. $I_h \phi_w$ will replace ϕ_w , u will replace

$\tilde{u}_h$, and we will define below $\mathbf{r}_{ij}$ to replace $\tilde{\mathbf{r}}_{ij}$. We do the analysis for the first integral, the second will be similar.

Let

$$\begin{aligned}\gamma_{ij} &:= g_{ij}/(2g^{3/2}), \\ \gamma &:= 1/\sqrt{g}, \\ \mathbf{r}_{ij} &:= \gamma_{ij}\mathbf{p} + \gamma\mathbf{p}_{ij}.\end{aligned}$$

Then clearly $\mathbf{r}_{ij} \in C^1$.

We now show that $|\mathbf{r}_{ij} - \tilde{\mathbf{r}}_{ij}| \leq Ch$. This follows from

$$\begin{aligned}|\gamma - \tilde{\gamma}| &= \left| \frac{1}{\sqrt{g}} - \frac{1}{\sqrt{\tilde{g}}} \right| \\ &= \left| \frac{\tilde{g} - g}{\sqrt{g\tilde{g}}(\sqrt{\tilde{g}} + \sqrt{g})} \right| \\ &\leq c|\tilde{g} - g| \leq Ch,\end{aligned}$$

and from

$$\begin{aligned}|\gamma_{ij} - \tilde{\gamma}_{ij}| &= \left| \frac{g_{ij}}{2g\sqrt{g}} - \frac{g_{ij}}{\sqrt{g\tilde{g}}(\sqrt{\tilde{g}} + \sqrt{g})} \right| \\ &\leq c \left| \frac{1}{2g} - \frac{1}{\sqrt{\tilde{g}}(\sqrt{\tilde{g}} + \sqrt{g})} \right| \\ &\leq c \left| \frac{\sqrt{\tilde{g}}(\sqrt{\tilde{g}} + \sqrt{g}) - 2g}{2g\sqrt{\tilde{g}}(\sqrt{\tilde{g}} + \sqrt{g})} \right| \\ &\leq c \left| \tilde{g} + \sqrt{g\tilde{g}} - 2g \right| \\ &\leq c \left(|\tilde{g} - g| + \sqrt{g} \left| \frac{\tilde{g} - g}{\sqrt{\tilde{g}} + \sqrt{g}} \right| \right) \\ &\leq c|\tilde{g} - g| \leq Ch.\end{aligned}$$

Now the integral is bounded as follows:

$$\begin{aligned}I &:= \left| \int_{\Omega} \tilde{\mathbf{r}}_{2-i,2-j} D_i \tilde{u}_h D_j (I_h \phi_w) \mathbf{e}_1 d\mathbf{x} \right| \\ &= \left| \int_{\Omega} \mathbf{r}_{2-i,2-j} D_i \tilde{u}_h D_j (I_h \phi_w) \mathbf{e}_1 d\mathbf{x} + \int_{\Omega} (\tilde{\mathbf{r}}_{2-i,2-j} - \mathbf{r}_{2-i,2-j}) D_i \tilde{u}_h D_j (I_h \phi_w) \mathbf{e}_1 d\mathbf{x} \right| \\ &\leq \left| \int_{\Omega} \mathbf{r}_{2-i,2-j} D_i \tilde{u}_h D_j (I_h \phi_w) \mathbf{e}_1 d\mathbf{x} \right| + Ch^2 \|w\|_{L^2}.\end{aligned}$$

Similarly, since $\|u - \tilde{u}_h\|_{H^1} \leq Ch$ and $\|\phi_w - I_h \phi_w\|_{H^1} \leq Ch\|w\|_{L^2}$, we finally obtain an integrand with which we may perform integration by parts:

$$\begin{aligned} I &\leq \left| \int_{\Omega} \mathbf{r}_{2-i,2-j} D_i u D_j \phi_w \mathbf{e}_1 d\mathbf{x} \right| + Ch^2 \|w\|_{L^2} \\ &= \left| - \int_{\Omega} (\mathbf{r}_{2-i,2-j} D_i u D_j \phi_w)_1 \mathbf{e} d\mathbf{x} \right. \\ &\quad \left. + \int_{\partial\Omega} \mathbf{r}_{2-i,2-j} T(D_i u) T(D_j \phi_w) \mathbf{e} \nu_1 d\sigma \right| + Ch^2 \|w\|_{L^2}. \end{aligned}$$

Above, $T : H^1(\Omega) \rightarrow L^2(\partial\Omega)$ is the trace operator, ν_1 is the first component of the outward pointing normal of the boundary and $\int_{\partial\Omega} d\sigma$ denotes a path integral along the boundary.

Each integral above is bounded. For the first,

$$\begin{aligned} \left| \int_{\Omega} (\mathbf{r}_{2-i,2-j} D_i u D_j \phi_w)_1 \mathbf{e} d\mathbf{x} \right| &\leq Ch^2 \int_{\Omega} |(\mathbf{r}_{2-i,2-j} D_i u D_j \phi_w)_1| d\mathbf{x} \\ &\leq Ch^2 \|\mathbf{r}_{2-i,2-j}\|_{C^1} \|u\|_{H^2} \|\phi_w\|_{H^2} \\ &\leq Ch^2 \|w\|_{L^2}. \end{aligned}$$

And for the second,

$$\begin{aligned} \left| \int_{\partial\Omega} \mathbf{r}_{2-i,2-j} T(D_i u) T(D_j \phi_w) \mathbf{e} \nu_1 d\sigma \right| &\leq Ch^2 \int_{\partial\Omega} T(D_i u) T(D_j \phi_w) d\sigma \\ &\leq Ch^2 \|T(D_i u)\|_{L^2(\partial\Omega)} \|T(D_j \phi_w)\|_{L^2(\partial\Omega)} \\ &\leq Ch^2 \|D_i u\|_{H^1} \|D_j \phi_w\|_{H^1} \\ &\leq Ch^2 \|u\|_{H^2} \|\phi_w\|_{H^2} \\ &\leq Ch^2 \|w\|_{L^2}. \end{aligned}$$

So it follows that $I \leq Ch^2 \|w\|_{L^2}$. A similar analysis for the other integral terms show that $|a(\tilde{u}_h, I_h \phi_w) - \tilde{a}(\tilde{u}_h, I_h \phi_w)| \leq Ch^2 \|w\|_{L^2}$. The theorem then follows from the Aubin-Nitsche lemma. $\square$

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References

- [1] D. Braess. *Finite elements: theory, fast solvers, and applications in solid mechanics*. Cambridge University Press, 2001.
- [2] P. G. Ciarlet. *The Finite Element Method for Elliptic Problems*. North-Holland, 1978.
- [3] M. Eck, T. D. DeRose, T. Duchamp, H. Hoppe, M. Lounsbery, and W. Stuetzle. Multiresolution analysis of arbitrary meshes. In *Proceedings of SIGGRAPH '95*, pages 173–182, 1995.
- [4] L. C. Evans. *Partial differential equations*. AMS, 1998.
- [5] M. S. Floater and K. Hormann. Surface parameterization: a tutorial and survey. In *Advances in Multiresolution for Geometric Modelling*, pages 157–186, 2005.
- [6] D. Gilbarg and N. S. Trudinger. *Elliptic Partial Differential Equations of Second Order*. Springer, 1977.
- [7] P. Grisvard. *Elliptic Problems in Nonsmooth Domains*. Pitman, 1985.
- [8] U. Pinkall and K. Polthier. Computing discrete minimal surfaces and their conjugates. *Experimental Mathematics*, 2(1):15–36, 1993.