Finite trees as ordinals

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University of Oslo

Honouring Wilfried
München April 5, 2008

Typical trees

The natural numbers:

$0 = \ .$

Typical trees

The natural numbers:

$0 =$

$1 =$

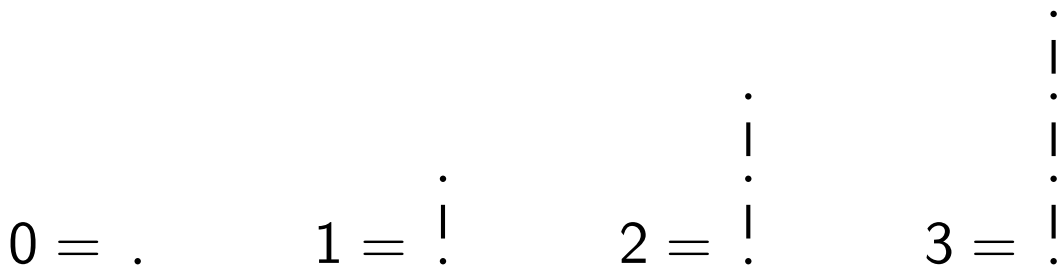
Typical trees

The natural numbers:

$0 =$ $1 =$ $2 =$

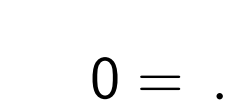
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Typical trees

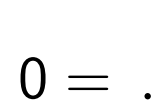
The natural numbers:

$0 =$  $1 =$ $2 =$ $3 =$

And some ordinals:

Typical trees

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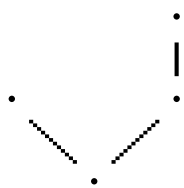
$\omega =$ 

Typical trees

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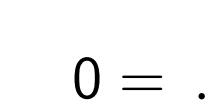
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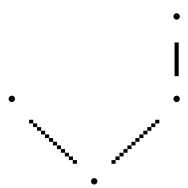
$\omega =$  $\omega^\omega =$ 

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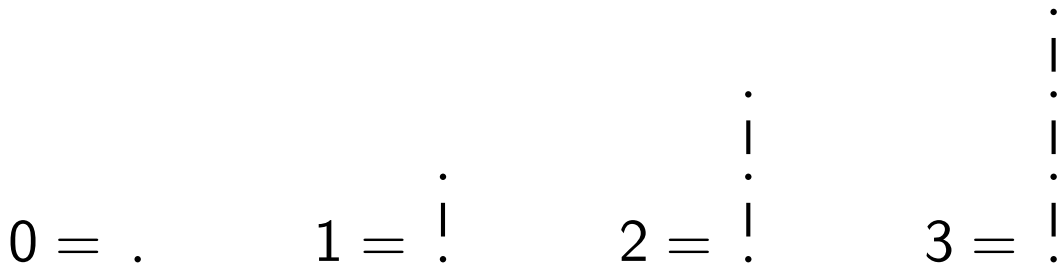
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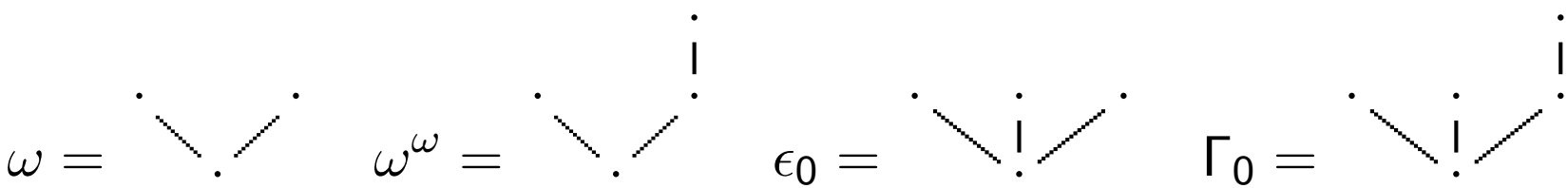
$\omega =$  $\omega^\omega =$  $\epsilon_0 =$ 

Typical trees

The natural numbers:



And some ordinals:



The ordering

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  - Length of sequences
  - Rightmost element where they differ

Elementary properties

$$\boxed{\mathbf{A} < \mathbf{B} \;\Leftrightarrow\; \mathbf{A} \leq \langle \mathbf{B} \rangle \;\vee\; (\langle \mathbf{A} \rangle < \mathbf{B} \wedge \langle \mathbf{A} \rangle < \langle \mathbf{B} \rangle)}$$

- Decidable

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- Decidable
- Transitive
- Linear
- Equality is the usual tree equality

Some ordinal functions

Zero:

$$. = 0$$

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$$\dot{.} = 0$$

Successor:

$$\overset{\alpha}{\dot{!}} = \alpha + 1$$

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Exponentiation:

$$\begin{array}{c} \cdot \\ \diagdown \; \diagup^{\alpha} \\ . \end{array} \sim \omega^{\omega^{\alpha}}$$

where $\sim$ means we jump over fix points.

Some ordinal functions

Zero:

$$\cdot = 0$$

Successor:

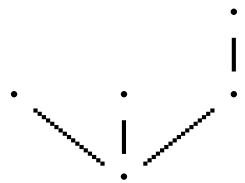
$$\begin{matrix}\alpha \\ ! \\ \cdot\end{matrix} = \alpha + 1$$

Exponentiation:

$$\begin{matrix}\cdot & & \\ & \ddots & \\ & \cdot & \alpha\end{matrix} \sim \omega^{\omega^{\alpha}}$$

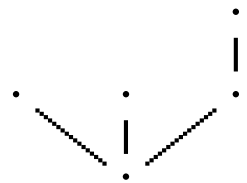
where $\sim$ means we jump over fix points.
In general we get the fix point free n-ary Veblen functions.

Approximating from below 1

$\Gamma_0 =$ 

Start with immediate subtrees:

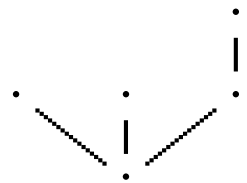
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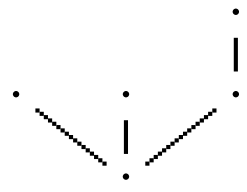
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Use function with smaller arity:

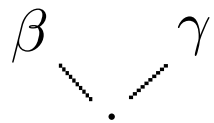
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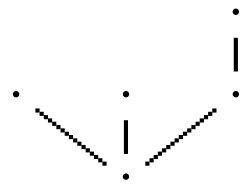
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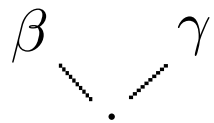
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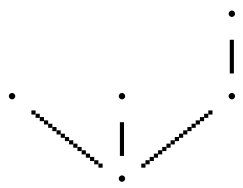
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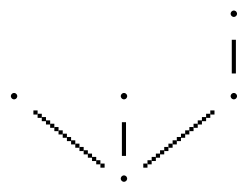
 

Approximating from below 2

$\Gamma_0 =$ 

Less in lexicographical ordering:

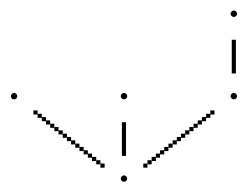
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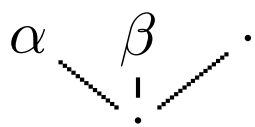
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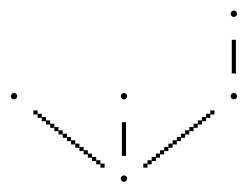
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This gives all trees less than Γ_0.

Approximating from below 2

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This gives all trees less than Γ_0. To get a cofinal set we only need



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Both arguments are straightforward.

Further work

Linear extensions of embeddings

- Diana Schmidt

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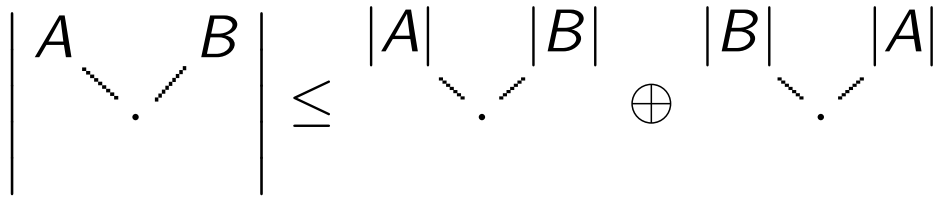
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This gives Higmans lemma. Further work gives Kruskals theorem.

Further work

Finite trees with labels

- Wellordered set of labels

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- Each node has a label

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