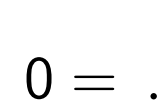
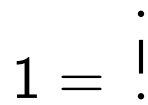
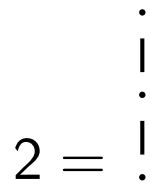
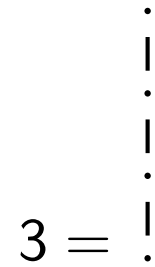
Finite trees as ordinals

Herman Ruge Jervell
University of Oslo

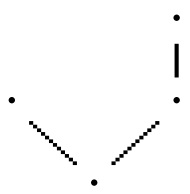
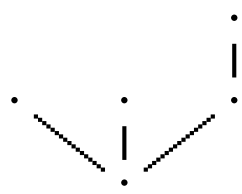
Honouring Wilfried
München April 5, 2008

Typical trees

The natural numbers:

$0 =$  $1 =$  $2 =$  $3 =$ 

And some ordinals:

$\omega =$  $\omega^\omega =$  $\epsilon_0 =$  $\Gamma_0 =$ 

The ordering

$$\boxed{\mathbf{A} < \mathbf{B} \;\Leftrightarrow\; \mathbf{A} \leq \langle \mathbf{B} \rangle \;\vee\; (\langle \mathbf{A} \rangle < \mathbf{B} \wedge \langle \mathbf{A} \rangle < \langle \mathbf{B} \rangle)}$$

- $\langle \mathbf{A} \rangle$ — sequence of immediate subtrees
- $\mathbf{A} \leq \langle \mathbf{B} \rangle$ $\mathbf{A} \leq$ some immediate subtree of $\mathbf{B}$
- $\langle \mathbf{A} \rangle < \mathbf{B}$ all immediate subtrees of $\mathbf{A}$ are $< \mathbf{B}$
- $\langle \mathbf{A} \rangle < \langle \mathbf{B} \rangle$ lexicographical ordering
  - Length of sequences
  - Rightmost element where they differ

Elementary properties

$$\mathbf{A} < \mathbf{B} \;\Leftrightarrow\; \mathbf{A} \leq \langle \mathbf{B} \rangle \;\vee\; (\langle \mathbf{A} \rangle < \mathbf{B} \wedge \langle \mathbf{A} \rangle < \langle \mathbf{B} \rangle)$$

- Decidable
- Transitive
- Linear
- Equality is the usual tree equality

Some ordinal functions

Zero:

$$. = 0$$

Successor:

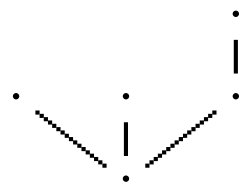
$$\overset{\alpha}{\underset{.}{!}} = \alpha + 1$$

Exponentiation:

$$\begin{array}{c} \cdot \\ \diagdown \\ \cdot \end{array} \begin{array}{c} \alpha \\ \diagup \end{array} \sim \omega^{\omega^\alpha}$$

where $\sim$ means we jump over fix points.
In general we get the fix point free n-ary Veblen functions.

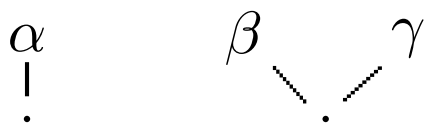
Approximating from below 1

$\Gamma_0 =$ 

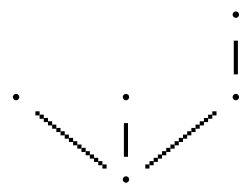
Start with immediate subtrees:

$0 =$ $0 =$  $1 =$

Use function with smaller arity:



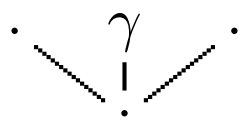
Approximating from below 2

$\Gamma_0 =$ 

Less in lexicographical ordering:



This gives all trees less than Γ_0. To get a cofinal set we only need



Wellfoundedness

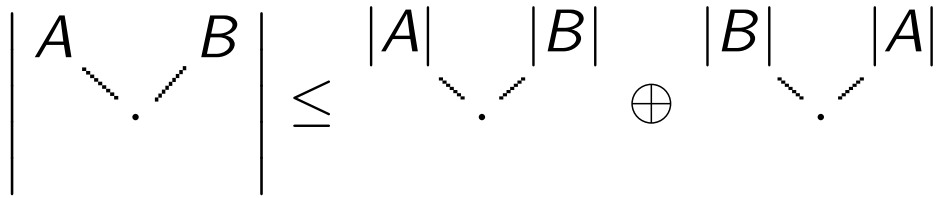
- Minimal bad argument
  - Minimal height
- Induction over wellfounded trees

Both arguments are straightforward.

Further work

Linear extensions of embeddings

- Diana Schmidt
- Linear extensions of topological embeddings of trees
- $|A|$ maximal ordertype



This gives Higmans lemma. Further work gives Kruskals theorem.

Further work

Finite trees with labels

- Wellordered set of labels
- Each node has a label
- $\langle A \rangle_i$ – sequence of i-subtrees
- Defines $<_i$ and $<_\infty$
- $A <_i B \Leftrightarrow A \leq_i \langle B \rangle_i \vee (\langle A \rangle_i < B \wedge A <_{i+} B)$
- $A <_\infty B$ — lexicographical ordering
- Linear wellfounded preorderings
- Takeutis ordinal diagrams