


# FINAL EXAM IN STA-2002

| | | |
|---|---|---|
| EXAM IN | : | Sta-2002, Theoretical statistics |
| DATE | : | Monday 12 December 2005 |
| TIME | : | 0900–1300 |
| PLACE | : | Åsgårdsveien 9. |
| APPROVED REMEDIES | : | 4 handwritten pages.<br>Approved calculator. |

The exam contains 6 pages including this cover page.

Contact person : Fred Godtliebsen, tlf. 77 64 40 19.

FACULTY OF SCIENCE
**University of Tromsø, N-9037 Tromsø, Tel. 77 64 40 01, Fax 77 64 47 65**

## EXERCISE 1

In this exercise we will study a special type of hand dryers that, after switched on, blows warm air a time $t_0$ and thereafter is automatically switched off. If the hands are not dry within the time $t_0$, the dryer must be switched on again.

We suppose that the time an arbitrary person needs to get dry hands is a random variable $T$ (in seconds) with probability density function

$$f_T(t) = \frac{t}{\theta^2} e^{-\frac{t}{\theta}} \quad ; \; t > 0$$

where $\theta > 0$ is an unknown parameter.

**a)** Find the moment generating function (MGF) for $T$. Use this result to show that

$$\begin{aligned} \mathrm{E}(T) &= 2\theta \\ \mathrm{Var}(T) &= 2\theta^2. \end{aligned}$$

**b)** Show that

$$Z = \frac{2T}{\theta}$$

has a Chi square distribution with 4 degrees of freedom, i.e. that

$$f_Z(z) = \frac{1}{4} z e^{-\frac{z}{2}} \quad ; \; z > 0$$

Suppose we have $n$ observations from $n$ persons, i.e. we have $T_1, \ldots, T_n$ that are assumed independent and identically distributed (iid) with probability density function $f_T(t)$. Find a confidence interval for $\theta$ by using a pivotal quantity.

**c)** Show that the Maximum Likelihood Estimator (MLE) for $\theta$ based on $T_1, \ldots, T_n$ is

$$\theta^* = \frac{1}{2n} \sum_{j=1}^{n} T_j$$

Find the Cramer-Rao lower bond for this estimator. What can you conclude about $\theta^*$ based on the Cramer-Rao lower bond?

## EXERCISE 2

A new type of car tires, denoted A tires, will be compared to a traditional type of tires (denoted K tires). A bus company would like to find out whether the wearing of A tires is less than for K tires. The company does the following experiment:

On each of 9 buses one of the two rear tires is randomly given an A tire and the other rear tire is given a K tire. After a certain time (mileage), the wearing is measured on each of the

18 tires. The wearing on A tires and K tires for bus $j$ is denoted by $X_j$ and $Y_j$, respectively. We introduce

$$D_j = Y_j - X_j \quad ; \; j = 1, \ldots, 9$$

and assume that $D_1, \ldots, D_9$ are iid $\mathrm{N}(\delta, \sigma_0^2)$ where $\sigma_0^2 = 36$ is known. Based on the observed values of $D_1, \ldots, D_9$, the bus company would like to decide whether A tires have less wearing than K tires.

**a)** Why is it natural to treat this situation as a hypothesis testing problem where we would like to test

$$H_0 : \delta \leq 0 \quad \text{against} \quad H_1 : \delta > 0?$$

Propose a test for this hypothesis testing problem based on

$$\bar{D} = \frac{1}{9} \sum_{j=1}^{9} D_j.$$

What is the conclusion if the result of the experiment is as given at the end of this exercise. Use significance level $\alpha = 0.09$.

**b)** Alternatively, we might test whether A tires have less wearing than K tires by introducing

$$Y = \text{number of } D_j \text{ that are greater than 0.}$$

Since

$$Y \sim \mathrm{bin}(9, p)$$

where

$$p = P(D_j > 0)$$

the hypothesis testing problem can be solved by testing

$$H_0 : p \leq 0.5 \quad \text{against} \quad H_1 : p > 0.5.$$

Find the rejection criterion for this hypothesis testing problem when the test is based on $Y$. Use significance level $\alpha = 0.09$.
Compare this testing result with what you found in Exercise 2a). Explain what happens. Why might one in some cases use the testing procedure in Exercise 2b) instead of the one used in Exercise 2a)?

**c)** Find the power functions, $\beta_a(\delta)$ and $\beta_b(p)$ for the testing problems in 2a) and 2b) respectively. Which testing procedure would you prefer in this situation? Explain.

**Experimental results :**

| Bus $j$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| K tire ($Y_j$) | 35 | 45 | 61 | 89 | 58 | 74 | 120 | 34 | 55 |
| A tire ($X_j$) | 31 | 42 | 55 | 79 | 59 | 79 | 108 | 36 | 46 |
| Difference ($D_j$) | 4 | 3 | 6 | 10 | -1 | -5 | 12 | -2 | 9 |

## EXERCISE 3

Suppose we have $X_1, \ldots, X_n$ independently and identically distributed with probability mass function

$$P(X = x|p) = p^x(1-p)^{1-x} \quad ; \quad x = 0, 1,$$

where $p$ is a parameter in the interval $[0, 1]$.

**a)** Define the function $h(p)$ by

$$h(p) = P(\sum_{i=1}^{n} X_i > X_{n+1}|p),$$

the probability that the first $n$ observations exceed the $(n+1)$st. Show that

$$T(X_1, \ldots, X_{n+1}) = \begin{cases} 1 & \text{if } \sum_{i=1}^{n} X_i > X_{n+1} \\ 0 & \text{otherwise} \end{cases}$$

is an unbiased estimator of $h(p)$.
Find the best unbiased estimator of $h(p)$.

**b)** Let

$$Y = \sum_{i=1}^{n} X_i.$$

Then

$$Y \sim \text{bin}(n, p),$$

i.e.

$$p(y) = \frac{n!}{y!(n-y)!} p^y (1-p)^{n-y}, \quad y = 0, 1, \ldots, n.$$

Assume a prior distribution $\pi(p)$ for $p$ to be Beta$(\alpha, \beta)$, i.e.

$$\pi(p) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\alpha-1}(1-p)^{\beta-1} \quad ; \quad 0 \leq p \leq 1,$$

and $\alpha$ and $\beta$ are positive parameters. Find the Bayes estimator of $p$ if we assume a quadratic loss function. Give an interpretation of your estimator.

# N(0,1) DISTRIBUTION

$\Phi(x) = P(X \leq x)$

| | .00 | .01 | .02 | .03 | .04 | .05 | .06 | .07 | .08 | .09 |
|---|---|---|---|---|---|---|---|---|---|---|
| .0 | .500 | .504 | .508 | .512 | .516 | .520 | .524 | .528 | .532 | .536 |
| .1 | .540 | .544 | .548 | .552 | .556 | .560 | .564 | .567 | .571 | .575 |
| .2 | .579 | .583 | .587 | .591 | .595 | .599 | .603 | .606 | .610 | .614 |
| .3 | .618 | .622 | .626 | .629 | .633 | .637 | .641 | .644 | .648 | .652 |
| .4 | .655 | .659 | .663 | .666 | .670 | .674 | .677 | .681 | .684 | .688 |
| .5 | .691 | .695 | .698 | .702 | .705 | .709 | .712 | .716 | .719 | .722 |
| .6 | .726 | .729 | .732 | .736 | .739 | .742 | .745 | .749 | .752 | .755 |
| .7 | .758 | .761 | .764 | .767 | .770 | .773 | .776 | .779 | .782 | .785 |
| .8 | .788 | .791 | .794 | .797 | .800 | .802 | .805 | .808 | .811 | .813 |
| .9 | .816 | .819 | .821 | .824 | .826 | .829 | .831 | .834 | .836 | .839 |
| 1.0 | .841 | .844 | .846 | .849 | .851 | .853 | .855 | .858 | .860 | .862 |
| 1.1 | .864 | .867 | .869 | .871 | .873 | .875 | .877 | .879 | .881 | .883 |
| 1.2 | .885 | .887 | .889 | .891 | .893 | .894 | .896 | .898 | .900 | .901 |
| 1.3 | .903 | .905 | .907 | .908 | .910 | .911 | .913 | .915 | .916 | .918 |
| 1.4 | .919 | .921 | .922 | .924 | .925 | .926 | .928 | .929 | .931 | .932 |
| 1.5 | .933 | .934 | .936 | .937 | .938 | .939 | .941 | .942 | .943 | .944 |
| 1.6 | .945 | .946 | .947 | .948 | .949 | .951 | .952 | .953 | .954 | .954 |
| 1.7 | .955 | .956 | .957 | .958 | .959 | .960 | .961 | .962 | .962 | .963 |
| 1.8 | .964 | .965 | .966 | .966 | .967 | .968 | .969 | .969 | .970 | .971 |
| 1.9 | .971 | .972 | .973 | .973 | .974 | .974 | .975 | .976 | .976 | .977 |
| 2.0 | .977 | .978 | .978 | .979 | .979 | .980 | .980 | .981 | .981 | .982 |
| 2.1 | .982 | .983 | .983 | .983 | .984 | .984 | .985 | .985 | .985 | .986 |
| 2.2 | .986 | .986 | .987 | .987 | .987 | .988 | .988 | .988 | .989 | .989 |
| 2.3 | .989 | .990 | .990 | .990 | .990 | .991 | .991 | .991 | .991 | .992 |
| 2.4 | .992 | .992 | .992 | .992 | .993 | .993 | .993 | .993 | .993 | .994 |
| 2.5 | .9940 | .9940 | .9942 | .9943 | .9945 | .9946 | .9948 | .9949 | .9951 | .9952 |
| 2.6 | .9953 | .9955 | .9956 | .9957 | .9959 | .9960 | .9961 | .9962 | .9963 | .9964 |
| 2.7 | .9965 | .9966 | .9967 | .9968 | .9969 | .9970 | .9971 | .9972 | .9973 | .9974 |
| 2.8 | .9974 | .9975 | .9976 | .9977 | .9977 | .9979 | .9979 | .9980 | .9980 | .9981 |
| 2.9 | .9981 | .9982 | .9982 | .9983 | .9984 | .9984 | .9985 | .9985 | .9986 | .9986 |
| 3.0 | .9987 | .9987 | .9988 | .9988 | .9988 | .9989 | .9989 | .9989 | .9990 | .9990 |
| 3.1 | .9990 | .9991 | .9991 | .9991 | .9992 | .9992 | .9992 | .9992 | .9993 | .9993 |
| 3.2 | .9993 | .9993 | .9994 | .9994 | .9994 | .9994 | .9994 | .9995 | .9995 | .9995 |
| 3.3 | .9995 | .9995 | .9996 | .9996 | .9996 | .9996 | .9996 | .9996 | .9996 | .9997 |
| 3.4 | .9997 | .9997 | .9997 | .9997 | .9997 | .9997 | .9997 | .9997 | .9998 | .9998 |
| 3.5 | .9998 | .9998 | .9998 | .9998 | .9998 | .9998 | .9998 | .9998 | .9998 | .9998 |
| 3.6 | .9998 | .9998 | .9999 | .9999 | .9999 | .9999 | .9999 | .9999 | .9999 | .9999 |
| 3.7 | .9999 | .9999 | .9999 | .9999 | .9999 | .9999 | .9999 | .9999 | .9999 | .9999 |

# BINOMIAL DISTRIBUTION

$P(X \leq x)$

| n | x \ p | .05 | .10 | .15 | .20 | .25 | .30 | .40 | .50 |
|---|---|---|---|---|---|---|---|---|---|
| 9 | | | | | | | | | |
| | 0 | .630 | .387 | .232 | .134 | .075 | .040 | .010 | .002 |
| | 1 | .929 | .775 | .599 | .436 | .300 | .196 | .071 | .020 |
| | 2 | .992 | .947 | .859 | .738 | .601 | .463 | .232 | .090 |
| | 3 | .999 | .992 | .966 | .914 | .834 | .730 | .483 | .254 |
| | 4 | 1.000 | .999 | .994 | .980 | .951 | .901 | .733 | .500 |
| | 5 | 1.000 | 1.000 | .999 | .997 | .990 | .975 | .901 | .746 |
| | 6 | 1.000 | 1.000 | 1.000 | 1.000 | .999 | .996 | .975 | .910 |
| | 7 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | .996 | .980 |
| | 8 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | .998 |
| 10 | | | | | | | | | |
| | 0 | .599 | .349 | .197 | .107 | .056 | .028 | .006 | .001 |
| | 1 | .914 | .736 | .544 | .376 | .244 | .149 | .046 | .011 |
| | 2 | .988 | .930 | .820 | .678 | .526 | .383 | .167 | .055 |
| | 3 | .999 | .987 | .950 | .879 | .776 | .650 | .382 | .172 |
| | 4 | 1.000 | .998 | .990 | .967 | .922 | .850 | .633 | .377 |
| | 5 | 1.000 | 1.000 | .999 | .994 | .980 | .953 | .834 | .623 |
| | 6 | 1.000 | 1.000 | 1.000 | .999 | .996 | .989 | .945 | .828 |
| | 7 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | .998 | .988 | .945 |
| | 8 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | .998 | .989 |
| | 9 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | .999 |
| 11 | | | | | | | | | |
| | 0 | .569 | .314 | .167 | .086 | .042 | .020 | .004 | .000 |
| | 1 | .898 | .697 | .492 | .322 | .197 | .113 | .030 | .006 |
| | 2 | .985 | .910 | .779 | .617 | .455 | .313 | .119 | .033 |
| | 3 | .998 | .981 | .931 | .839 | .713 | .570 | .296 | .113 |
| | 4 | 1.000 | .997 | .984 | .950 | .885 | .790 | .533 | .274 |
| | 5 | 1.000 | 1.000 | .997 | .988 | .966 | .922 | .753 | .500 |
| | 6 | 1.000 | 1.000 | 1.000 | .998 | .992 | .978 | .901 | .726 |
| | 7 | 1.000 | 1.000 | 1.000 | 1.000 | .999 | .996 | .971 | .887 |
| | 8 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | .999 | .994 | .967 |
| | 9 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | .999 | .994 |
| | 10 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 12 | | | | | | | | | |
| | 0 | .540 | .282 | .142 | .069 | .032 | .014 | .002 | .000 |
| | 1 | .882 | .659 | .443 | .275 | .158 | .085 | .020 | .003 |
| | 2 | .980 | .889 | .736 | .558 | .391 | .253 | .083 | .019 |
| | 3 | .998 | .974 | .908 | .795 | .649 | .493 | .225 | .073 |
| | 4 | 1.000 | .996 | .976 | .927 | .842 | .724 | .438 | .194 |
| | 5 | 1.000 | .999 | .995 | .981 | .946 | .882 | .665 | .387 |
| | 6 | 1.000 | 1.000 | .999 | .996 | .986 | .961 | .842 | .613 |
| | 7 | 1.000 | 1.000 | 1.000 | .999 | .997 | .991 | .943 | .806 |
| | 8 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | .998 | .985 | .927 |
| | 9 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | .997 | .981 |
| | 10 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | .997 |
| | 11 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |