



EXAMINATION TASK MAT-1004
(ENGLISH)

Exam in : **Mat-1004 Linear algebra.**
Date : **Thursday June 15.**
Time : **09 a.m. – 1 p.m.**
Place : **Åsgårdveien 9.**
Allowed materials : **Approved calculator,**
two A4 sheets of own notes and
Rottmann's tables.

The examination set is on 3 pages ex. front cover, and contains 10 subproblems: 1ab, 2abcd, 3abcd.

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Explain your answers, show the solution process clearly!

1 PROBLEM

Consider the following vectors in $\mathbb{R}^3$:

$$\begin{aligned}\mathbf{u} &= (-1, 2, 3); \\ \mathbf{v} &= (1, -2, 3); \\ \mathbf{w} &= (1, 1, 1).\end{aligned}$$

Let

$$\mathbf{x} = (x_1, x_2, x_3)$$

be an unknown vector.

- a) Find a condition for $\mathbf{x}$ guaranteeing that the vector equation

$$(a\mathbf{u} + b\mathbf{v}) \times \mathbf{w} = \mathbf{x}$$

has at least one solution (a, b) , $a, b \in \mathbb{R}$.

“ $\times$ ” means the cross product. The condition should be given in the form

$$\begin{aligned}k_1x_1 + k_2x_2 + k_3x_3 &= m, \\ k_1, k_2, k_3, m &\in \mathbb{R}.\end{aligned}$$

- b) Find all pairs (a, b) satisfying the equation

$$(a\mathbf{u} + b\mathbf{v}) \times \mathbf{w} = (9, -18, 9).$$

2 PROBLEM

a) Let f be the polynomial

$$f(x) = x^{10} - x^9 - 2x^8 + x + 2.$$

Consider an arbitrary $n \times n$ matrix B . Explain how the matrix $f(B)$ is defined.

Let P be an invertible $n \times n$ matrix. Prove, using the properties of matrix operations, that

$$f(PBP^{-1}) = Pf(B)P^{-1}.$$

b) Let

$$A = \begin{bmatrix} 2 & -3 & -3 \\ 3 & -4 & -3 \\ -3 & 3 & 2 \end{bmatrix}.$$

Show that $\lambda = -1$ is an eigenvalue of A . Find then all eigenvalues of A .

For each eigenvalue λ find the dimension and a basis for the eigenspace $E(\lambda)$.

c) Explain what the rank of a matrix is. Let $r(s)$ be the rank of the matrix $A - sI$:

$$r(s) = \text{rank}(A - sI),$$

where s is a real number (not necessarily an eigenvalue!) and I is the identity matrix. Investigate for which s

- (i) $r(s) = 3$;
- (ii) $r(s) = 2$;
- (iii) $r(s) = 1$;
- (iv) $r(s) = 0$.

d) Find an invertible matrix P and a diagonal matrix D such that

$$P^{-1}AP = D.$$

Compute then $f(A)$ where f is the polynomial from subproblem a).

3 PROBLEM

Let P_2 be the vector space of polynomials of degree ≤ 2 , and let β be the standard basis for P_2 :

$$\beta = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \{1, x, x^2\}.$$

a) Show that the set

$$\gamma = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\} = \{1, 1+x, 1+x+x^2\}$$

forms a basis for P_2 , and find the transition matrix Q from the basis γ to the basis β . Find then the transition matrix R from the basis β to the basis γ .

b) Let

$$\mathbf{u} = 3x + 4x^2 \in P_2.$$

Find the coordinate columns $[\mathbf{u}]_\beta$ and $[\mathbf{u}]_\gamma$ for $\mathbf{u}$ with respect to the bases β and γ .

Let

$$T: P_2 \longrightarrow P_2$$

be the following operator:

$$T(f(x)) = f(1-x) - f(x),$$

for instance

$$\begin{aligned} T(-1 + 3x + 4x^2) &= (-1 + 3(1-x) + 4(1-x)^2) - (-1 + 3x + 4x^2) = \\ &= (4x^2 - 11x + 6) - (-1 + 3x + 4x^2) = 7 - 14x. \end{aligned}$$

T is a linear operator (you do not need to prove that).

c) Find the matrix $[T]_\gamma$ for the operator T with respect to the basis γ (Notice: γ , not β !).

d) Find the dimension and a basis for the range $\text{Im}(T)$ of the operator T (in the textbook the range is denoted $R(T)$).

GOOD LUCK!