

The Tragedy of Logicism

The material conditional

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I The material conditional

The fact that Frege's truth functional logic is not an ideal or even an adequate tool for natural language, is commonly blamed on the material conditional interpretation of natural conditionals *If A then B*. The material conditional $A \supset B$ is defined as true whenever A is false or B is true, which means that the following conditionals will come out as true if interpreted as material:

1. If Ingrid Bergman is a brilliant actress, then rhinoceroses are majestic animals. (TT)
2. If rhinoceroses are majestic animals, then Ingrid Bergman is a brilliant actress. (TT)
3. If Gene Hackman looks like a rhinoceros, then Henry Kissinger has got the Nobel Peace Prize. (FT)
4. If Henry Kissinger is a philosopher, then Gene Hackman is a majestic animal. (FF)
5. If Liv Ullmann is a brilliant actress, then Henry Kissinger is a philosopher. (FF)

The discrepancy between natural and material conditionals is that while a natural conditional seems to express a hypothetical and intentional relation between the antecedent and the consequent, the material conditional is defined as purely truth functional and extensional.¹ That the material conditional is truth functional means that it is a function of the truth-values of the antecedent and the consequent. One is then not concerned with the internal relation between the antecedent and the consequent, but only with the combination of truth-values assigned to them.

The inadequacy of the material interpretation of natural conditionals is often explained away by reference to the fact that Frege designed his logical system as a tool for carrying out his logicist program of reducing arithmetic to logic, and not for interpreting natural language. So when applied to set theory, the extensional material conditional is thought to be both adequate and useful. The problems related to the material interpretation of natural conditionals will then allegedly not occur when logic is applied to mathematics. It seems however that these problems cannot be avoided in mathematics either. Indeed, natural

¹ We here use "extensional" in a non-technical sense, and not in the Fregean sense where the extension of a statement is its reference, namely The True or The False.

language is and has to be the basic language also of mathematics, since this is where we start in all our reasoning. Barwise argues in a similar fashion:

I think that the language of mathematics is continuous with ordinary language, since discourse about mathematical objects and mathematical activity takes place in English or some other natural language... To one who takes this line, mathematical conditionals simply are natural language conditionals, so an adequate account of mathematical conditionals must be part and parcel of an adequate account of natural language conditionals.²

It seems undeniable that we use natural language to express mathematical statements, since no mathematical formulas or statements will make any sense unless we basically have a natural language reading of them. For instance, when we state the mathematical truth ' $2+2=4$ ', this does not mean anything at all unless we know that these symbols represent 'two plus two equals four', and could we not even say 'If we add two and two, (then) we get four'?

One might think that mathematical statements are of a totally different kind from for instance empirical statements, arguing that mathematical statements are analytic and a priori, which means that mathematics is independent of empirical or contextual elements. Understood in this way, mathematical statements assert no more than what is already thought in the terms. The only language needed to handle mathematical statements is therefore logic.

But are mathematical statements completely independent of contextual elements in the manner apparently demanded by a truth functional logic? How about the context that is added by the encircled passage in which it occurs, for instance in a mathematical problem? According to Barwise, it is not obvious how a mathematical statement should be understood if taken out of context. For instance, he points out that while one in natural language often has some sort of grammatical indicator of whether a conditional statement is used to express a general or specific situation, mathematical conditionals lack this sort of indicator:

² Barwise (1986), p. 21.

Sometimes only context can determine whether a conditional statement is specific or general, especially in mathematics, where the simple present tense is ubiquitous. There is simply nothing about the sentence:

$$\text{If } x = a \cdot b, \text{ then } \sqrt{x} = \sqrt{a} \cdot \sqrt{b}$$

that determines whether some statement made with it is making a general claim about all numbers under discussion, say, or a claim about three specific numbers.³

In the following we will argue that mathematical statements are no less context-dependent than natural language statements; hence no more suited for truth functional logical treatment. This is confirmed by the fact that if we give ordinary mathematical conditionals a material interpretation, we will get the same problems as we did with natural language conditionals. For instance, the following mathematical conditionals will all come out as true with a material interpretation:

1. If 3 is a prime number, then 113 is a prime number. (TT)
2. If 113 is a prime number, then 3 is a prime number. (TT)
3. If 12 is a prime number, then 2 is a prime number. (FT)
4. If 50 is a prime number, then 115 is a prime number. (FF)
5. If 115 is a prime number, then 50 is a prime number. (FF)

This indicates that mathematical conditionals are no closer to the material conditional than natural language conditionals. For instance, the intentional content of the example ‘If n is an even number, it is divisible on 2’ is not normally understood as a general material conditional $(\forall x)(Fx \supset Gx)$, which basically means that for all the single instantiations $\{Fa \supset Ga, Fb \supset Gb, Fc \supset Gc, \dots Fn \supset Gn\}$ one will not get that the antecedent is true and the consequent is false. It should consequently be impossible to assign a truth-value to this conditional unless we could insert different numbers for n one by one and ascertain whether we ever get the antecedent true and the consequent false at the same time. As we know, we would never get the combination (TF), but neither would we be able to complete the test. Accordingly, the truth functional first order logic cannot help us prove this simple mathematical theorem if interpreted as a general material conditional, when the intentional relation between antecedent and consequent is then disregarded. Accordingly, the inadequacy of the material interpretation of mathematical conditionals is critical for the logical reduction of mathematics, since numerous mathematical axioms and theorems are formulated as conditionals in this way.

³ Barwise (1986), p. 23.

Still, there is something about mathematics that makes us want to treat mathematical conditionals differently from natural language conditionals. We are inclined to consider the necessary and a priori nature of mathematics as an indication of mathematical statements being in essence logical statements. This leads us to believe that we can treat mathematical statements as epistemologically different from empirical statements, and to some extent, this is correct. Mathematical theorems are necessary in a sense that empirical necessities, as for instance the laws of physics, are not. The problem arises when we assume that the logical character of mathematics coincides with the truth functional logic of Frege. If this logic is not a well-suited tool for interpreting natural language, it seems bound to fail as a tool for mathematics as well. So although the a priori character of mathematical theorems makes mathematical laws essentially different from empirical laws, the truth functional logic is no more suited as a tool for mathematics than for the empirical sciences. We will try to strengthen this thesis by arguing that, if we accept the logicist program on its own premises, then the fact that the material conditional plays a major role in this program is a crucial problem for the accomplishment of a derivation of mathematics from logic.

II The logicist program

In his *Introduction to Mathematical Philosophy*, Russell presents the relationship between logic and mathematics as the relationship between boy and man: Logic is the youth of mathematics and mathematics is the manhood of logic.⁴ The view that mathematics is closely related to logic, in the sense that mathematics is reducible to logic, forms the main thesis of the logicist program. Mathematics is then considered as a branch of logic. The aim of this logicist program is to give mathematics a logical foundation in the way that one can define all mathematical concepts from logical concepts. More precisely, the logicist thesis can be divided into three parts:⁵

- I All so-called primitive notions of mathematics can be defined using only logical notions.
- II All so-called primitive propositions of mathematics can be logically deduced from logical axioms.
- III All theorems can be logically deduced from its so-called primitive propositions (and hence, by virtue of (II), from logical axioms).

⁴ Russell (1961), p.175.

⁵ Musgrave (1977), p. 101.

This is the logicist program in its boldest form, with no restrictions concerning which mathematical disciplines can be given a logical foundation. Musgrave calls this “old-style logicism”.

Frege’s logicist program was an old-style logicism, except that he wanted to derive only the laws of arithmetic, not the whole of mathematics, from the laws of logic. To Frege, no sharp boundary can be drawn between logic and arithmetic, since the propositions on which arithmetic is based are extremely general and basic in such a respect that they must extend to everything that can be thought at all. The limits of logic are the limits of arithmetic, so to speak. Frege’s logicist thesis is accordingly that concepts of arithmetic are derivable from logical concepts through explicit definitions, and that the theorems of arithmetic are derivable from logical axioms through purely logical deduction. This would prove that the truths of arithmetic are logical truths, being analytic a priori.⁶ Frege’s purpose with his logicist program was to be able to explain our knowledge of the basic laws of arithmetic by answering the question “How are numbers given to us?” without making any appeal to intuition (*Anschauung*).

One of the main reasons for Frege’s restricted logicist program, including arithmetic only, is his strict notion of axioms and definitions. To him, axioms are basic, unprovable propositions within a system that express the most basic, self-evident facts of the subject in question. Axioms can therefore not stipulate the reference of any new terms, but must rather presuppose that all the terms occurring in them are already known. In an axiom, then, all terms are known, either because they are primitive, or because they have been introduced by explicit definitions (using Frege’s truth functional logic) prior to the introduction of the axiom. According to Frege, the axioms constitute the ultimate ground of all inferences within a given system. And as opposed to an explicit definition, an axiom is informative and extends our knowledge.⁷ Expressing self-evident, basic facts, axioms are

⁶ According to Resnik, to Frege, a truth is analytic if and only if its proof depends only on general logical laws and definitions, including any proposition on which the correctness of the definitions may depend. Further, a truth is a priori if and only if its proof depends on general laws that neither need nor admit of proof and does not appeal to empirical facts, i.e. truths that cannot be proved and are about empirical objects.

⁷ But at the same time, an axiom of arithmetic is supposed to be analytic, either because it is self-evident on the basis of the senses of the terms used to state the axiom, or because it is derived from general logical laws and definitions. General logical laws are supposed to be self-evident on the basis of the senses of the terms used in stating them. Most of them can be shown to be self-evident by the principle of contradiction, but Basic Law V, though supposed self-evident on the basis of the senses of the terms used to state it, was never considered by Frege to be true by the principle of contradiction. Furthermore, ordinary arithmetical statements like ‘ $2+2=4$ ’, though considered to be analytic, since they are supposed to be derivable from

necessarily true, and therefore they cannot contradict each other. Explicit definitions, on the other hand, assign sense and reference to the signs that so far have no sense and reference within the system; thus allowing us to prove new sentences but not new thoughts. In an explicit definition, then, a term must be explained separately in a proposition in which all other words are known beforehand, and this must be fulfilled prior to a term's occurrence in an axiom.

The problem with giving geometry a logical foundation, then, is that the primitive notions of geometry, such as “point”, “straight line”, “plane figure” and “between” *cannot* be explicitly defined using only logical notions, but must rather be presupposed by the system as already intuitively known. Frege thus agreed with Kant that the truths of geometry are synthetic a priori, since, while the axioms of arithmetic express basic, self-evident and general facts of everything that can be thought, the axioms of geometry express our intuitions (*Anschauungen*) only of what is spatial.

In the correspondence between Frege and Hilbert, leading up to Frege's ‘Foundation of Geometry’, Frege argues against Hilbert's attempt to construct an axiomatic system of geometry that was supposed to enable one to prove the independence and consistency of the axioms of Euclidean geometry by using purely logical notions. Frege claims that Hilbert lets his axioms introduce new terms such as ‘line’, ‘between’, ‘plane figure’ and ‘point’ through indirect definitions though implicitly presupposing them as already known, and thereby blurring Frege's distinction between axioms and definitions.

We will in the following argue that Frege himself eventually acknowledged that he could not fulfill his own strict requirements for the introduction of primitive notions even in his limited logicist program for the reduction of arithmetic to logic, hence forcing him to eventually throw in the logicist towel.

arithmetical axioms and definitions, are according to Frege not true on the basis of senses of the terms used to state them, since the two sides of the equality have different senses. Thus we have three subclasses of analytical statements in arithmetic, according to Frege. First we have self-evident axioms whose negations can be shown to imply contradictions. Then we have Basic Law V, supposed to be true on the basis of senses. And finally we have ordinary arithmetical statements, like ‘ $7+5=12$ ’, which has to be proved, and that are not true on the basis of the senses of the terms used to state them. Cf. Resnik (1980), pp. 225ff. and Alnes (1999), p. 13, p. 20.

III Basic Law V, Hume's Principle and the Julius Caesar Problem

In order to be able to carry out his logicist program for arithmetic, Frege developed two formal systems, the propositional calculus and the first-order calculus. In *Begriffsschrift*, he formulated a second-order predicate calculus, which he used to define mathematical concepts and to state and prove mathematical propositions. In *Grundgesetze der Arithmetik*, he added an axiom known as Basic Law V:⁸

Basic Law V

$$\{x|Fx\} = \{y|Gy\} \equiv \forall z(Fz \equiv Gz)$$

With the introduction of Basic Law V, Frege was supposed to be able to introduce the numbers through an explicit definition, which allegedly allowed him to carry out the logicist program.

There are however some problematic features in Frege's formulation of Basic Law V clearly related to the use of material equivalence. The main material equivalence is here placed between the identity of two sets and the general material equivalence of certain related statements. This means that in order for Basic Law V to be true, that is, in order for the main material equivalence to be true, the identity statement on the left hand side and the general material equivalence on the right hand side must have identical truth-values (true or false). That is, both these statements must have reference in the Fregean sense. If the material equivalence $(\forall x)(Fx \equiv Gx)$ is true or false, the only thing asserted is an identity between truth-values for Fx and Gx , regardless of whether there are any x s that fall under F or G . However, if the identity statement $\{x/Fx\} = \{y/Gy\}$ is true or false, then both $\{x/Fx\}$ and $\{y/Gy\}$ must have both sense and reference, which means that these sets must exist.

So while we have no claims of existence of objects on the right hand side of Basic Law V, we have to presuppose existence of particular objects on the left hand side. But how can we have an equivalence between something that does not include claims of existence and something that does include claims of existence? (Or, at best, between two different types of claims of existence, namely of predicates on one side and sets on the other.) This

⁸ The original formulation was $\{f(\epsilon)\} = \{g(\alpha)\} \equiv \forall x[f(x) = g(x)]$. Cf. Zalta (2003), pp. 8-9.

discrepancy in logical properties involves a problem with Basic Law V.⁹ First of all, an analytic axiom is not supposed to yield new information, and secondly, the unaccounted for existence claims indicates that Basic Law V is not apt to prevent uninvited and unwanted elements from slipping through. The reason for this is the material equivalence. Being weaker than the identity sign, in the sense that $A \equiv B$ asserts only that we have identity between truth-values for A and B , while $A = B$ asserts identity between the two objects A and B ,¹⁰ the main material equivalence allows in what it should have excluded and abstracts from what it should have expressed.

In fact, the use of the material equivalence in Basic Law V seems to be what actually allows for Russell's Paradox to occur in the first place. Russell's paradox is a set-theoretical paradox where we consider the predicate 'sets that are not members of themselves'. If one then accepts Basic Law V, it follows from this that there might exist such things as the set of all sets that are not members of themselves. And if we assume that such a set does exist, Russell's Paradox follows, since such a set appears to be a member of itself if and only if it is not a member of itself. This means that the language and axioms of *Begriffsschrift* had not enabled Frege to exclude all terms without reference, allowing such problematic notions as 'the set of all sets that are not members of themselves'. In Basic Law V, the use of the material equivalence, however, forces truth-value assignments on both sides of the main equivalence, regardless of whether all the terms used in fact have sense or reference, and thus assigns existence to the alleged references of the problematic notions. Accordingly, Frege had not managed to introduce in a valid manner the notion of numbers through explicit definitions with the help of Basic Law V and thereby demonstrate that the basic laws of arithmetic are derivable from purely analytic laws of logic.

It is argued that the best strategy for Frege to avoid Russell's Paradox would be to replace Basic Law V with Hume's Principle.¹¹ This principle is an indirect definition of the cardinal numbers, eventually relying on explicit definitions of equinumerosity and numbers:

⁹ That Basic Law V involves claims of existence and extensionality is a problem that is thoroughly treated in Alnes (1999) and Zalta (2003), pp. 25-31.

¹⁰ Treating truth-values as objects, Frege contributes to camouflaging the significance of this distinction.

¹¹ In *Grundgesetze*, Frege attempted to derive Hume's Principle and the basic laws of arithmetic from Basic Law V in second-order logic. However, already in *Grundlagen*, he sketched the derivations of the basic laws of numbers from Hume's Principle. This is called Frege's Theorem.

Hume's Principle

The number of *F*s is identical to the number of *G*s if and only if *F* and *G* are equinumerous (formalized as: $\#F = \#G \equiv F \approx G$).¹²

It is however to be expected that Hume's Principle, having a similar structure to Basic Law V, involves many of the same problems. Frege probably realized that Russell's Paradox was just the top of the iceberg. Instead of replacing Basic Law V with Hume's Principle, he renounced his logicist program. First of all, Frege did not regard Hume's Principle as strong enough, epistemologically, to give an answer to the question "How are numbers given to us?" without appealing to intuition (*Anschauung*). Secondly he realized that he had not been able to give an explicit definition of 'the number of *F*s', since the concept of equinumerosity between *F* and *G* was defined so that *F* and *G* are in a one-to-one correspondence, which one may argue presupposes an intuitive idea of numbers and counting. This is evident in what is called the Julius Caesar Problem: that Hume's Principle does not give an explicit definition of numbers, which means that it does not describe completely the conditions under which an arbitrary object, say Julius Caesar, is to be identified as a cardinal number or not.

The Julius Caesar Problem occurs because, in order for arithmetic to be logical, it must quantify over all objects: truth-values, numbers, chairs and persons, including Julius Caesar. The explicit definition of numbers must also make possible decisions concerning the identity of a number and any other object. And what tools are best suited for such an all-embracing account? Apparently not solely the identity sign applied directly to the level of objects, since this would seem to force identity between for instance five pears and five apes. The material equivalence, on the other hand, opens for an account of numbers in terms of sets and members of sets, regardless of what kinds of objects are counted, added or multiplied. The problem is of course that since the material equivalence explicitly asserts identity merely between truth-values, and not between objects or numbers, it can never be a precise enough tool for explicitly defining numbers. The material equivalence eliminates intentional content, and since both Hume's Principle and Basic Law V have a material equivalence as their main connective, Frege gets a purely extensional definition of numbers.

¹² Zalta (2003), pp. 12-14.

Since Hume's Principle only offers identity conditions when x is an object that we already know is a cardinal number, one will have to circumscribe the domain for the definition in order to avoid the Julius Caesar problem. But this means that we must assume what we wanted to achieve, namely that x is a number. This is not a satisfactory foundation for arithmetic, and it does not satisfy Frege's criteria for axioms and definitions in logic and mathematics.

It seems evident, then, that Frege could not adopt an *ad hoc* solution to Russell's Paradox, for instance by means of replacing Basic Law V with Hume's Principle, without at the same time giving up on his fundamental principles about logic and mathematics. The dispute with Hilbert about the foundations of Geometry in the same period of time strongly suggests that he was not willing to compromise on his views on axioms and definitions.¹³

IV If-thenism, or Deductivism

As a result of the unveiled inconsistency in Frege's axiomatic system, any mathematical theorem derived from it could now be questioned, since anything can be derived from an inconsistency. This problem is however thought to be easily solved by adopting a less ambitious logicist program, suggested by Hilbert, according to which the ontological question of truth for a mathematical theorem T is replaced with the question of the deducibility of T from a set of axioms A . So instead of forcing a decision between mutually incompatible mathematical theorems (as for instance Euclidean and non-Euclidean geometry) or assertion of the axioms (as Basic Law V), "the deductivist can adopt a stance of agnosticism concerning the existence of mathematical objects and truths."¹⁴ This was Hilbert's suggestion for giving geometry a logical foundation: If A represents the conjunction of the axioms, then a theorem T should be expressed fully as $A \supset T$ in Hilbert's system. Mathematical inferences are thus thought to become less problematic when translated into conditional statements, so that for instance set theory, viewed as a body of conditional statements, could nevertheless be derived from logical axioms. Resnik calls this logicist program "Deductivism", while Musgrave calls it "If-thenism":

- I A mathematical statement is a conditional statement with a conjunction of 'mathematical axioms' as antecedent and a 'mathematical theorem' as consequent.

¹³ See the discussion with Hilbert in Frege (1971), pp. 6-112.

¹⁴ Resnik (1980), p. 105.

II All true mathematical statements can be deduced from logical axioms.¹⁵

So the point with the If-thenist maneuver is to be able to assert only the purely hypothetical conditional relation between the axioms and the theorems, without being committed to having asserted anything about the axioms and theorems as such. Leibniz pointed very clearly to this hypothetical understanding of mathematical truths: “As to *eternal truths*, it is to be noted that at bottom they are all conditional, and say in effect; Granted such a thing, such another thing is. For instance, when I say ‘Every figure which has three sides will also have three angles’, I say nothing but this, that supposing there is a figure with three sides, this same figure will have three angles.”¹⁶

The If-thenist maneuver allegedly precludes Russell’s Paradox, since one instead of asserting a possibly inconsistent set of axioms only asserts the conditionals linking the axioms to the theorems. However, it was impossible for Frege to resign himself to this weaker version of logicism, since it, among other things, involved a total abandonment of his conception of axioms. According to Frege, it is in the nature of the axioms that they are true, and that they therefore cannot contradict one another. To derive mathematical theorems from a possibly inconsistent set of axioms would then violate the essence of both logic and mathematics. Moreover, to use the logical axioms as antecedents and the mathematical theorems as consequents with the purpose not to assert anything about the truth of the axioms or theorems, would, according to Frege, not solve the problem of giving mathematics a logical foundation.

Another problem with this If-thenist maneuver is Frege’s concept of conditionals. To him, a conditional only has sense and reference as a whole. The antecedent and the consequent *per se* are both pseudo-propositions, having neither sense nor reference:

What I call a proposition *tout court* or a real proposition is a group of signs that expresses a thought; however, whatever has only the grammatical form of a proposition I call a pseudo-proposition. Examples of the latter are often to be found as antecedent and consequent propositions of conditional propositional complexes... Let us consider the proposition “If something is greater than 1, then it is a positive number.” “Something” and “it” refer to one another. If we break this connection by separating the propositions, each of them becomes senseless.¹⁷

¹⁵ Musgrave (1977), p. 117-118.

¹⁶ Leibniz (1981), book IV, chapter ii, section 14.

¹⁷ Frege (1906), part II, p. 69.

So only together in a conditional, the antecedent and the consequent form a real proposition.¹⁸ If the conjunction of axioms are taken as the antecedent and the conjunction of theorems is taken as the consequent (as the label “If-thenism” indicates), it would seem that the axioms are not even real propositions any more, having neither sense nor reference. On the other hand, if the axioms are taken as premises, and the theorems as conclusions derived from the premises (as the label “Deductivism” indicates), then the axioms must be true, since Frege emphasizes that “only true thoughts can be premises of inferences.”¹⁹ And this is particularly important when the premises are supposed to be axioms in the Fregean sense. This is because, to Frege, in a logical inference from premises to conclusion, “each premise is a definite thought which has been recognized as true, and the conclusion is also a definite thought which is recognized as true”²⁰, and every mathematical proof has axioms as its ultimate premises.

A final objection against Hilbert and If-thenism follows from Frege’s claim that the assertions of mathematics are not hypothetical, but categorical, truths.²¹ And logical and mathematical inferences are not hypothetical in the sense that they infer a conclusion from the assumption of a set of premises. In a conditional proof, then, where a premise occurs as a condition, this condition must also appear in the conclusion. But then it is not a premise, but an assumption. This means that all the so-called mathematical proofs of If-thenism must have a conditional form $A \supset B$ in the conclusion, while to Frege, a mathematical proof should have the form $P, \text{ therefore } C$, where P is the true axioms and C is the true theorem.

It seems that the If-thenist maneuver is the ultimate *ad hoc* maneuver in the attempt to give mathematics a logical foundation. Instead of going for Frege’s uncompromising line, admitting defeat when the basic concepts of mathematics are at stake, the If-thenists keep deleting principles as pragmatically required. Where Frege initially limited his logicist program, leaving only arithmetic to be reduced to logic, because he saw that the basic

¹⁸ This account of conditionals seems to be in conflict with Frege’s material conditional account, since a truth functional material conditional cannot be assigned a truth-value unless the antecedent and the consequent have been assigned a truth-value, expressing nothing but a certain combination of truth-values for the antecedent and the consequent.

¹⁹ Frege (1906), p. 105.

²⁰ Resnik (1980), p. 174.

²¹ It may seem that Frege here forsakes his formal interpretation of categorical statements *All Fs are Gs* as general material conditionals ($\forall x)(Fx \supset Gx)$, and thus, his thesis that the distinction between hypothetical and categorical statements is only grammatical.

terms of geometry cannot be given explicit logical definitions, the If-thenists include the whole of mathematics, ignoring all the problems pointed out by Frege and by Russell's Paradox.

Now, regardless of Frege's objections to If-thenism, we have some arguments against this logicist position related to their (mis)use of conditionals. In fact, we want to call this If-thenist maneuver a "chickening out"²² maneuver, since one here uses the conditional form in a strategy to assert *less*. In mathematics, conditionals are usually used to express a fact or a theorem, based on the network of axioms. So while mathematical conditionals are necessary, a priori, assertive and informative, in such a way that given the axioms, we have numerous kinds of conditional relations holding between numbers, figures, and so on, the If-thenists want to exploit the hypothetical aspect of the conditional form to place the axioms and the derived theorems in doubt. This means that they use conditionals to say as little as possible. What they say may be precise enough, but still what they say is much less than they make as if to say. This is to use conditionals in an unproductive manner.

An analogy to this chickening out use of conditionals could be the following situation: In being asked my opinion on death penalty, instead of giving an answer to the question, I say:

1. If one thinks that the human worth is inviolable, and one considers that death penalty involves a risk of executing innocent people, then death penalty can be considered as highly problematic.

But,

2. If one considers that it is important to protect the public against murders and rapists, and at the same time give the criminals an adequate punishment according to their crime, then death penalty can be considered as a constructive solution.

What have I actually said? Well, I am asked whether *A* or *not-A*, and my answer is the following conditionals:

1. *If B, then not-A*
2. *If C, then A*

²² This term is borrowed from Diamond (1991).

It is important to recognize that the conditional statements above are used to express arguments, which means that they are kind of deductions in which some of the premises are explicit and some are implicit. To use conditionals in such a deductive manner, where one says that this conclusion will follow from this set of premises, is different from asserting a conditional relation holding between for instance death penalty and reduction of crime, namely *If A then B*. In asserting a conditional relation between two logically independent matters, as for instance smoking and cancer, one has not asserted only this particular relation, but a whole cluster of facts and conditional relations, for instance about the human body. Mathematical conditionals are of the latter kind *If A then B*, while the If-thenist deductions of theorems from axioms are of the first kind, where the relation expressed in *If B, then not-A* and *If C, then A* is a logical necessity and therefore has no assertive power. It seems, then, that it is not indifferent whether we call this logicist position “If-thenism” or “Deductivism”, since it now should be clear that they use a deductivist maneuver. We will shortly return to the discussion of the divergence between conditionals and inferences.

But if we now assume, for the sake of argument, that the If-thenist maneuver does in fact solve the problem with the inconsistent Basic Law V, and that the use of conditionals to assert less about the axioms and the derived theorems is actually satisfactory: could we then say that they have managed to give mathematics a logical foundation? It seems not. In aiming to assert only the conditional relation between the axioms and the theorems, the Deductivists are not willing to abandon the material conditional. And in order for a material conditional to be true, its antecedent and consequent must be assigned a certain truth value, namely (TT), (FT) or (FF). But once we assign a truth-value to the antecedent in a material conditional $A \supset T$, we have already asserted or denied the axioms, and accordingly, the problem with truth of the axioms presupposed and the existence of the mathematical and logical objects referred to will hit us like a boomerang.²³

This means that if the Deductivists insist on a material conditional interpretation, not only of mathematical conditionals, but also of the conditional statements linking the axioms to

²³ Sandmel comments that: “But, the Deductivist might object, this assignment of truth-values to the antecedent and the consequent is, of course, only conditional. I am not really assigning the value T or F to A or B: In asserting $A \supset B$ all I am saying is that if I assign the value T to A, then I am bound to assign the same value T to B as well. This maneuver on the part of the Deductivist, however, leads to an infinite regress if this new conditional is also to be interpreted materially, $(A \text{ has value } T) \supset (B \text{ has value } T)$, for then what we really assert is that if “A has value T” has value T, then “B has value T” also has value T, etc.”

the theorems, they end up having to assert or deny the axioms as well as the theorems. However, what the logicians wanted to achieve with the If-thenist maneuver, was to escape Russell's Paradox by asserting only the purely hypothetical relation between axioms and theorems, instead of asserting the inconsistent set of axioms. But in making this conditional material, the Deductivist position sacrifices hypotheticality, hence shoots itself in the foot. This means that no problems get solved with the If-thenist maneuver, and accordingly, the problems unearthed by Russell's paradox are still all over us. The material conditional, together with the truth functional logic in general, seems to have blocked even the most out-chickening of the chickening out maneuvers in the name of logicism.

V Logical implication versus the material conditional

Now, even if we were to suppose that the use of the material conditional in the If-thenist maneuver does not eliminate hypotheticality, and that this maneuver therefore is a productive solution to Russell's Paradox, we could still not agree that the Deductivists have given mathematics a logical foundation. The reason for this is that the logical implication, which they need for the logical derivation of the theorems, is defined merely as a tautological material conditional:

To every valid truth-functional argument there corresponds a tautology; the tautology is a conditional whose antecedent is the conjunction of the premises of the argument and whose consequent is the conclusion of the argument. For example, the tautology " $((p \supset q) \ \& \ p) \supset q$ " corresponds to modus ponens. To say that a form of argument is valid is to say that, if its premises are true, then its conclusion must be true. Under such circumstances the corresponding conditional cannot have a true antecedent and a false consequent and so will necessarily be true, a tautology. Thus we can say that a truth-functional argument is valid if and only if its corresponding conditional is a tautology.²⁴

So while the material conditional $A \supset B$ is a statement, in which what is stated is supposed to be a conditional relation between the antecedent and the consequent, the logical implication $A \Rightarrow B$ is supposed to be a consequence relation between premises and a conclusion:

²⁴ Barker (1965), p. 120.

$((A \supset B) \& A) \supset B$	$A \supset B$ (P1)
T	A (P2)
T	$\Downarrow$
T	B (Concl.)
T	

This means that the logical implication $\Rightarrow$ is logically stronger than the material conditional $\supset$. In a logical implication $A \Rightarrow B$, we claim that $A \supset B$ is a tautology. We can also say that A is the premise(s) and B is the conclusion in a truth functionally valid inference. In a material conditional, $A \supset B$, we have a truth functional conditional statement that is true unless A is true and B is false. In a logical implication, on the other hand, A and/or B must be composites and of such a nature that we can never get the truth-value T for A and the truth-value F for B .

We thus have three different levels: 1) Natural language conditionals that are used to express relations between facts or events in the world. 2) The material conditional $A \supset B$, being merely a truth function, expresses only a certain combination of truth-values between linguistic entities. It is therefore on a meta-level in relation to the natural conditional. 3) The logical implication $A \Rightarrow B$ expresses a consequence relation between premises and conclusion, consisting of truth functional statements, and belongs therefore to a meta-level with respect to the material conditional. Hence the logical implication is on a meta-meta-level in relation to natural conditionals. The difference between natural conditionals, the material conditional and the logical implication can be illustrated through the following statements:

1. If n is an even number, then it is divisible on 2. (*If A then B*)
2. A and B have one of the following combinations of truth-values: (TT), (FT) or (FF). ($A \supset B$)
3. $A \supset B$ is a truth functionally defined tautology. ($A \Rightarrow B$)

Here, 1 and 2 would be true, while 3 actually would be false, if we do not find a way to express A and B as composites.

We see that since Frege's truth functional logic is concerned with the relations between truth-values of linguistic entities, and not explicitly with the relations between numbers, figures, statements, and so on, the movement between meta- and meta-meta-levels will not get us down to the level where natural language operates.

Because of the correspondence between the material conditional and the logical implication, we lose the intuitive consequence relation between premises and conclusion, since within a truth functional account the consequence relation is determined as equivalent to the corresponding material conditional being a tautology. In fact, a formally valid argument, using material conditionals and logical implication, can have true premises and false conclusion. If we insert statements for A , B and C in the following form of argument $((A \ \& \ B) \supset C) \Rightarrow ((A \supset C) \vee (B \supset C))$, we will for instance get that:

If I have both matches and cigarettes, I can smoke.²⁵
Therefore, if I have matches, I can smoke, or if I have cigarettes, I can smoke.

In this example, many will regard the conclusion as false, considered as based on the truth of the premise. However, these statements are actually defined as logically equivalent, which means that the inference goes both ways. The problem here is that the truth conditions of the logical connectives are already truth functionally defined; the rules of the game are given. So if these definitions do not correspond to our natural language understanding, there will be a divergence between what we regard intuitively as a valid argument and what the truth functional logic determines as a valid inference from the formal rules of inference together with the definitions of the logical connectives.

The fact that the material conditional and the logical implication are artificially defined within a formal logical approach makes it crucial for us to distinguish between them. For instance, to misread a material conditional as a logical implication will result in the assumption of a tautological relation between the antecedent and the consequent. However, if one on the contrary misreads a tautological logical implication as a mere material conditional, one will believe that one has asserted a stronger, in the sense more informative, relation. According to this, the Deductivists thought that they were asserting informative mathematical conditionals of the form 'If n is an even number, then it is

²⁵ We got this example from a dedicated smoker.

divisible on 2' rather than tautological deductions as for instance modus ponens. But in mathematics, as in natural language, we use neither the material conditional nor the logical implication, formally defined. Rather, we use the natural conditional understanding of both the relation between antecedent and consequent and the relation between premises and conclusion.

This natural conditional understanding of logical validity is what the logicians need in order to be able to derive the mathematical theorems from logical axioms. With a natural conditional interpretation, for instance, one would not accept the inference from *A and not-A* to *A or B* and further to *B*, and hence Russell's Paradox would not have such powerful consequences for the attempt to build mathematics on logic. By using the truth functional and tautologically defined logical implication, on the other hand, the logicians have to accept that an argument is valid whenever its premises contain a contradiction (as in Basic Law V) or whenever its conclusion contains a tautology. The motivation for the If-thenist maneuver was, however, to avoid assertion of mutually inconsistent logical axioms, and instead to assert only the deduction of the theorems from the axioms.

VI Russell's confusion

Since the logical implication $A \Rightarrow B$ is defined by means of the material conditional $A \supset B$, the Deductivists must either accept the problems connected with the material conditional or try to explain them away. Russell, who was a devoted Deductivist, tried to take the latter way out and explain away the divergences between natural and material conditionals by distinguishing between 'material implication' and 'formal implication'. According to him, a material implication is a singular instance of a general formal implication. This means that 'For all x , if x is a man, then x is mortal' is a formal implication, while 'If Socrates is a man, then Socrates is mortal' is a material implication. In the following passage, Russell argues that the alleged paradoxes of the material conditional occur when we interpret the singular material implication as a general formal implication:

It would certainly not be commonly maintained that " $2+2=4$ " can be deduced from "Socrates is a man", or that both are implied by "Socrates is a triangle". But the reluctance to admit such implications is chiefly due, I think, to preoccupation with formal implication, which is a much more familiar notion, and is really before

the mind, as a rule, even where material implication is what is explicitly mentioned.²⁶

However, when we in natural language assert the singular conditional ‘If Socrates is a man, he is mortal’ (corresponding to a material implication), we do this because we do know that all men are mortal (corresponding to the formal implication). Hence, the truth of the singular conditional is naturally understood to depend on the truth of the corresponding general conditional. Russell continues his argument in this very direction:

In inferences from “Socrates is a man”, it is customary not to consider the philosopher who vexed the Athenians, but to regard Socrates merely as a symbol; capable of being replaced by any other man; and only a vulgar prejudice in favour of true propositions stands in the way of replacing Socrates by a number, a table, or a plum-pudding. Nevertheless, wherever, as in Euclid, one particular proposition is deduced from another, material implication is involved, though as a rule the material implication may be regarded as a particular instance of some formal implication, obtained by giving some constant value to the variable or variables involved in the said formal implication.²⁷

It seems, then, that Russell distinguishes between formal implications of the form $(\forall x)(Fx \supset Gx)$ and material implications of the form $A \supset B$, where the latter is interpreted as a particular instantiation of the former, which often means that the material implication actually has the structure $Fa \supset Ga$. This is a correct analysis if the implications are interpreted as natural conditionals. When we state a singular natural conditional, we often understand this as expressing the same conditional relation as the corresponding general conditional. This is so since in natural language, it is the conditional relation expressed that is in focus. In the truth functional logic, on the other hand, where we do not have a conditional relation at all; it is rather the combinations of truth-values for the antecedent and the consequent that are in focus. Disregarding this fact, Russell is led to believe that it is a logically stronger conditional relation in formal implications than in material implications, and thus that when it comes to mathematical conditionals, they can be understood as such material implications derived from corresponding formal implications.

However, Russell’s distinction between material and formal implication does not contribute significantly to the clarification of the problem of the material conditional

²⁶ Russell (1937), chapter III, section 37, p. 34.

²⁷ Russell (1937), p. 34.

interpretation of mathematical conditionals. On the contrary, it throws a veil of confusion over the relation between the material conditional and the logical implication.

[Russell fails] to distinguish a *rule of inference*, a *conditional statement* of the form ‘If A then B ’ and a *universal statement* of the form ‘ $(x) (Fx \supset Gx)$ ’. Some mathematical axioms fall into the third category. Applying the If-thenist manoeuvre results in statements of the second category. If we identify the two, we can still suppose that, even after adopting the If-thenist manoeuvre, we are deriving mathematical *axioms* from logic. And if we confuse both of these with rules of inference, we can suppose that we are deriving them from rules of inference. Because of these confusions, one always finds If-thenism rubbing shoulders with logicism proper in Russell’s writings.²⁸

According to Musgrave then, this correspondence between the material conditional and the logical implication is the reason for Russell’s confusion. By confusing these two concepts, Russell actually moves unnoticeably between the different levels of language, a misdeed that Frege strongly warned us against, though, to his own misfortune and consternation, Frege discovered that he himself had failed to take enough heed of this problem.

VII The tragedy of logicism

Even if Frege’s truth functional logic were in fact a suitable tool for natural language, and hence for mathematics, we could still not admit that the logicist program at least in principle could be fulfilled. This is because mathematics, and even arithmetic, is not reducible to logic at all, no matter how adequate the logic. In mathematics, there is always more contained in the conclusion than in the premises or definitions. Even the simplest calculation cannot be reduced to a logical statement. To know that, and why, $3+4 = 7$ is to possess a multiplicity of knowledge about numbers and the relation between them. For instance, we realize, that $7-4 = 3$, that $7-3 = 4$, that the calculation rule is universal, that it holds for all kinds of objects: apples, trees, money, and so on, and that the objects (numbers) in front of ‘ $=$ ’ must be the same as those behind: 5 apples and 4 trees can never equal 9 dollars or 9 trees. This supports Barwise’s claim that mathematical statements never are independent of context, strictly opposite to what is assumed in the logicist definition of numbers that quantifies over all objects, including Julius Caesar.

²⁸ Musgrave (1977), p. 112.

Mathematical knowledge gives us a certain kind of knowledge: a general, dispositional, conditional, necessary and a priori type of knowledge. The generality of for instance the singular conditional ‘If 4 is an even number, it is divisible on 2’ lies in the fact that it is an instantiation of the general conditional ‘If n is an even number, then it is divisible on 2’. Both conditionals express the same rule and the same conditional relation. This is because we consider the conditional to express the dispositional character of even numbers as divisible on 2. That all even numbers are divisible on 2 is a necessary feature of even numbers, and it is given to us a priori.

Since mathematics gives us substantial (not only analytical) knowledge, and involves a large amount of creativity and unpredictability, Kant calls mathematical knowledge *synthetic a priori*. This synthetic aspect of mathematics would completely detach it from logic, once and for all. In fact, in his last thoughts on arithmetic, Frege opened for the view that arithmetic is synthetic a priori. He then wanted to base arithmetic upon the “geometrical” source of knowledge and claimed that numbers are comparable to geometrical objects:

The more I have thought the matter over, the more convinced I have become that arithmetic and geometry have developed on the same basis – geometrical one in fact – so that mathematics in its entirety is really geometry. Only on this view does mathematics present itself as completely homogeneous in nature.²⁹

And further:

I distinguish the following sources of knowledge for mathematics and physics:

1. Sense perception
2. The Geometrical Source of Knowledge
3. The Logical Source of Knowledge.

The last of these is involved when inferences are drawn, and thus is almost always involved. Yet it seems that this on its own cannot yield us any objects. From the geometrical source of knowledge flows pure geometry. In the case of arithmetic, just as in the case of geometry, I exclude only sense perception as a source of knowledge. This is significant because it means that arithmetic and geometry, and hence the whole of mathematics flows from one and the same source of knowledge – that is the geometrical one.³⁰

²⁹ Frege (1924/1925a), p. 277.

³⁰ Frege (1924/1925b), p. 280.

So one problem is that no aspect of mathematics can be reduced to something analytic, truth functional and tautological; another is that no mathematical system is purely axiomatic in the sense that we have a complete set of axioms from which we can derive all the theorems. And even if one could: what about the axioms themselves? Since they have to be true, they have to be generated from something. So even though the Deductivist position is that one can deduce the mathematical theorems from a set of axioms, this deduction would never have a productive character. A mathematical theory can be formally axiomatized only after the creative mathematical work is done. Logic is not productive or creative in this way; it cannot help us attain or produce new knowledge, mathematical or otherwise.

We have chosen to call this paper “the Tragedy of Logicism”. This is because there are certain features of logicism that are parallel to the classical tragedy. A tragedy is a drama whose main theme is a tragic course: it is a conflict to which there is no solution. The main character cannot escape his or her destiny, because the conflict is caused by forces within the character. The conflict has to be of a certain dimension in order to qualify as instrumental to a tragedy. As for logicism, its greatness lies in its ambitious project to explain mathematics by means of logic and in its vast troop of advocates. Secondly, the features of logicism that led to its fall, are i) the material conditional that eliminates the conditional aspect, and ii) the reduction of mathematics to logic that undermines the productive aspect of mathematics. In this way, logicism contains the source of its fall within itself.

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