Universitetet i Tromsø



Det Matematisk Naturvitenskapelige Fakultet
Institut for Matematikk og Statistikk

Eksamen i:	MAT-3110, Differential Geometry-1
Eksamenstid:	Tirsdag, 6. Juni, 2006. Kl. 14.00-18.00
Eksamenssted:	Åsgårdveien 9
Hjelpemidler:	4 håndskrevene A4-ark med egne notater (kan skrives på begge sider)
Oppgavesettet:	Totalt på 2 sider
Kontaktperson:	Professor *Boris Kruglikov*, tlf. 93802163.

Problem I

Let G be a Lie group and $\mathcal{G}$ its Lie algebra.

1. Prove that $G = SO(2) \simeq \mathbb{S}^1$ and find the group structure. Calculate the corresponding Lie algebra so(2) (with the structure).
2. Calculate the Lie algebra $\mathcal{G}$ and the map $\exp : \mathcal{G} \to G$ for the following Lie groups:
$$(\mathbb{R}^1, +), \qquad (\mathbb{R}_+, \times), \qquad (SO(2), \cdot).$$
3. Show that for every $\xi \in \mathcal{G}$ the map $G \to G$, $g \mapsto \exp(\xi) \cdot g$, is a diffeomorphism. Is the map $\exp : \mathcal{G} \to G$ a diffeomorphism?

Problem II

Given a map $h : \mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2 \to \mathbb{R}^2$, $(\varphi, \psi) \mapsto (\sin(\frac{\varphi}{2} - \psi), \cos\psi)$, and a pair of points $\mathbf{a} = (\pi, \frac{\pi}{2}) \in \mathbb{T}^2$, $\mathbf{b} = h(\mathbf{a}) \in \mathbb{R}^2$, calculate:

1. $d_{\mathbf{a}}h$ and $S^2 d_{\mathbf{a}}h$,
2. $d^*_{\mathbf{a}}h$ and $\Lambda^2 d^*_{\mathbf{a}}h$.

For this choose bases $e_1 = \partial_\varphi, e_2 = \partial_\psi \in T_{\mathbf{a}}\mathbb{T}^2$, $f_1 = \partial_x, f_2 = \partial_y \in T_{\mathbf{b}}\mathbb{R}^2$, write the map $d_{\mathbf{a}}h : T_{\mathbf{a}}\mathbb{T}^2 \to T_{\mathbf{b}}\mathbb{R}^2$ as a matrix and express the other maps as matrices.

Problem III

Consider the manifold $M^4 = T^*\mathbb{R}^2 \simeq \mathbb{R}^4(x^1, x^2, p_1, p_2)$ equipped with the symplectic form $\omega = dx^1 \wedge dp_1 + dx^2 \wedge dp_2$. Let $\theta = dp_1 \wedge dx^2 - dp_2 \wedge dx^1 - c \cdot p_2\, dx^1 \wedge dx^2 \in \Omega^2(M)$ be another 2-form.

1. Since ω is non-degenerate, we can define the operator $A : TM \to TM$ by the formula $\theta(v, w) = \omega(Av, w)\ \forall v, w \in TM$. Calculate its matrix in the basis $\partial_{x^1}, \partial_{x^2}, \partial_{p_1}, \partial_{p_2}$. Show that the operator A satisfies $A^2 = -\mathbf{1}$.

2. Show that for any operator $A : TM \to TM$ satisfying $A^2 = -\mathbf{1}$ the map (Nijenhuis tensor)

   $$N_A(\xi, \eta) = [A\xi, A\eta] - A[A\xi, \eta] - A[\xi, A\eta] - [\xi, \eta],\ \xi, \eta \in D(M),$$

   is $C^\infty(M)$-linear, i.e. $N_A(f\xi, \eta) = fN_A(\xi, \eta)$ and $N_A(\xi, f\eta) = fN_A(\xi, \eta)$, $f \in C^\infty(M)$ and skew-symmetric, i.e. $N_A(\xi, \eta) = -N_A(\eta, \xi)$. Therefore it defines the map $N_A : \Lambda^2 TM \to TM$.

3. Is $N_A = 0$ for the operator A from (1)? Calculate $N_A(\partial_{x^1}, \partial_{p_1})$.

Problem IV

Consider the manifold $M = J^1(\mathbb{R}^2) \simeq \mathbb{R}^5(x^1, x^2, u, p_1, p_2)$ equipped with the Cartan 1-form $\alpha = du - p_1\, dx^1 - p_2\, dx^2$. Let $\mathcal{C} = \mathrm{Ker}(\alpha) \subset TM$ be the Cartan distribution.

1. Show that $\mathcal{C}$ is spanned by the vector fields $\mathcal{D}_{x^1} = \partial_{x^1} + p_1\partial_u, \mathcal{D}_{x^2} = \partial_{x^2} + p_2\partial_u, \partial_{p_1}, \partial_{p_2}$. Calculate $\omega = d\alpha$.

2. Show that $\omega \in \Omega^2(M)$ is degenerate (has kernel), but the restriction 2-form $\omega|_{\mathcal{C}}$ is non-degenerate, i.e. $\forall v \in \mathcal{C},\ v \neq 0$ we have: $i_v\omega|_{\mathcal{C}} \neq 0$, so that $\exists u \in \mathcal{C} : \omega(v, u) \neq 0$.

3. Let $X_f = f\partial_u - f_{p_1}\mathcal{D}_{x^1} - f_{p_2}\mathcal{D}_{x^2} + \mathcal{D}_{x^1}(f)\partial_{p_1} + \mathcal{D}_{x^2}(f)\partial_{p_2} \in D(M)$, $f \in C^\infty(M)$. Calculate $\alpha(X_f)$ and $i_{X_f}\omega$. Calculate then $L_{X_f}\alpha$. Deduce that the field X_f is a symmetry of the distribution $\mathcal{C}$.

4. Let $\mathcal{E}^4 \subset M^5$ be given by the equation $p_2 = F(x^1, x^2, u, p_1)$ (corresponding PDE is $u_y = F(x, y, u, u_x)$). Denote the inclusion by $j : \mathcal{E} \hookrightarrow M$. Find the condition on F such that the 2-form $\theta = j^*\omega \in \Omega^2(\mathcal{E})$ is decomposable.