

10 Cosmological models for an homogeneous, isotropic universe

10.1 Friedmann-Robertson-Walker metric for an homogeneous, isotropic universe

Einstein's cosmological principle Einstein postulated that the Universe is homogeneous and isotropic at each moment of its evolution. Note that a space isotropic around at least two points is also homogeneous, while a homogeneous space is not necessarily isotropic. The CMB provides excellent evidence that the universe is isotropic around us. Barring suggestions that we live at a special place, the universe is also homogeneous.

Weyl's postulate In 1923, Hermann Weyl postulated the existence of a privileged class of observers in the universe, namely those following the “average” motion of galaxies. He postulated that these observers follow time-like geodesics that never intersect. They may however diverge from a point in the (finite or infinite) past or converge towards such a point in the future.

Weyl's postulate implies that we can find coordinates such that galaxies are at rest. These coordinates are called *comoving coordinates* and can be constructed as follows: One chooses first a space-like hypersurface. Through each point in this hypersurface lies a unique worldline of a privileged observer. We choose the coordinate time such that it agrees with the proper-time of all observers, $g_{00} = 1$, and the spatial coordinate vectors such that they are constant and lie in the tangent space $\mathbf{T}$ at this point. Then $u^a = \delta_0^a$ and for $\mathbf{n} \in \mathbf{T}$ it follows $n^a = (0, \vec{n})$ and

$$0 = u_a n^a = g_{ab} u^a n^b = g_{0\beta} n^\beta. \quad (10.1)$$

Since $\mathbf{n}$ is arbitrary, it follows $g_{0\beta} = 0$. Hence as a consequence of Weyl's postulate we may choose the metric as

$$ds^2 = -dt^2 + dl^2 = -dt^2 + g_{\alpha\beta} dx^\alpha dx^\beta. \quad (10.2)$$

The cosmological principle constrains further the form of dl^2 : Homogeneity requires that the $g_{\alpha\beta}$ can depend on time only via common factor $S(t)$, while isotropy requires that only $\vec{x} \cdot \vec{x}$, $d\vec{x} \cdot \vec{x}$, and $d\vec{x} \cdot d\vec{x}$ enter dl^2 . Hence

$$dl^2 = C(r)(\vec{x} \cdot d\vec{x})^2 + D(r)(d\vec{x} \cdot d\vec{x})^2 = C(r)r^2 dr^2 + D(r)[r^2 + r^2 d\vartheta^2 + r^2 \sin^2 \vartheta d\phi^2] \quad (10.3)$$

We can eliminate the function $D(r)$ by the rescaling $r^2 \rightarrow Dr^2$. Thus the line-element becomes

$$dl^2 = S(t) [B(r)dr^2 + r^2 d\Omega] \quad (10.4)$$

with $d\Omega = d\vartheta^2 + \sin^2 \vartheta d\phi^2$, while $B(r)$ is a function that we have still to specify.

Maximally symmetric spaces are spaces with constant curvature. Hence the Riemann tensor of such spaces can depend only on the metric tensor and a constant K specifying the curvature. The only form that respects the (anti-)symmetries of the Riemann tensor is

$$R_{abcd} = K(g_{ac}g_{bd} - g_{ad}g_{bc}). \quad (10.5)$$

Contracting R_{abcd} with g^{ac} , we obtain in three dimensions for the Ricci tensor

$$R_{bd} = g^{ac}R_{abcd} = Kg^{ac}(g_{ac}g_{bd} - g_{ad}g_{bc}) = K(3g_{bd} - g_{bd}) = 2Kg_{bd}. \quad (10.6)$$

A final contraction gives as curvature R of a three-dimensional maximally symmetric space

$$R = g^{ab}R_{ab} = 2K\delta_a^a = 6K. \quad (10.7)$$

A comparison of Eq. (10.6) with the Ricci tensor for the metric (10.4) will fix the still unknown function $B(r)$. We proceed in the standard way: Calculation of the Christoffel symbols with the help of the geodesic equations, then use of the definition (??) for the Ricci tensor,

$$R_{rr} = \frac{1}{rB} \frac{dB}{dr} = 2Kg_{rr} = 2KB \quad (10.8)$$

$$R_{\vartheta\vartheta} = 1 + \frac{r}{2B^2} \frac{dB}{dr} - \frac{1}{B} = 2Kg_{\vartheta\vartheta} = 2Kr^2. \quad (10.9)$$

(The $\phi\phi$ equation contains no additional information.) Integration of (10.8) gives

$$B = \frac{1}{A - Kr^2} \quad (10.10)$$

with A as integration constant. Inserting the result into (10.9) determines A as $A = 1$. Thus we have determined the line-element of a maximally symmetric 3-space with curvature K as

$$dl^2 = \frac{dr^2}{1 - Kr^2} + r^2(\sin^2\vartheta d\phi^2 + d\vartheta^2). \quad (10.11)$$

Going over to the full four-dimensional line-element, we rescale for $K \neq 0$ the r coordinate by $r \rightarrow |K|^{1/2}r$. Then we absorb the factor $1/|K|$ in front of dl^2 by defining the scale factor $R(t)$ as

$$R(t) = \begin{cases} S(t)/|K|^{1/2}, & K \neq 0 \\ S(t), & K = 0 \end{cases} \quad (10.12)$$

As result we obtain the Friedmann-Robertson-Walker (FRW) metric for an homogeneous, isotropic universe

$$\boxed{ds^2 = -dt^2 + R^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2(\sin^2\vartheta d\phi^2 + d\vartheta^2) \right]} \quad (10.13)$$

with $k = \pm 1$ (positive/negative curvature) or $k = 0$ (flat three-dimensional space). Finally, we give two alternatives forms of the FRW metric that are also often used. The first one uses the conformal time $\eta = t/R$,

$$ds^2 = R^2(\eta) [-d\eta^2 + dl^2] \quad (10.14)$$

and gives for $k = 0$ a conformally flat metric. In the second one, one introduces $r = \sin \chi$ for $k = 1$. Then $dr = \cos \chi d\chi = (1 - r^2)^{1/2} d\chi$ and

$$ds^2 = -dt^2 + R^2(t) [d\chi^2 + S^2(\chi)(\sin^2 \vartheta d\phi^2 + d\vartheta^2)] \quad (10.15)$$

with $S(\chi) = \sin \chi = r$. Defining

$$S(\chi) = \begin{cases} \sin \chi & \text{for } k = 1, \\ \chi & \text{for } k = 0, \\ \sinh \chi & \text{for } k = -1. \end{cases} \quad (10.16)$$

the metric (10.15) is valid for all three values of k .

10.2 Geometry of the Friedmann-Robertson-Walker metric

Geometry of the FRW spaces

Let us consider a sphere of fixed radius at fixed time, $dr = dt = 0$. The line-element ds simplifies then to $R^2(t)r^2(\sin^2 \vartheta d\phi + d\vartheta^2)$, which is the usual line-element of a sphere S^2 with radius $rR(t)$. Thus the area of the sphere is $A = 4\pi(rR(t))^2 = 4\pi S^2(\chi)R(t)^2$ and the circumference of a circle is $L = 2\pi rR(t)$, while $rR(t)$ has the physical meaning of a length.

By contrast, the radial distance between two points (r, ϑ, ϕ) and $(r + dr, \vartheta, \phi)$ is $dl = R(t)dr/\sqrt{1 - kr^2}$. Thus the radius of a sphere centered at $r = 0$ is

$$l = R(t) \int_0^r \frac{dr'}{\sqrt{1 - kr'^2}} = R(t) \times \begin{cases} \arcsin(r) & \text{for } k = 1, \\ r & \text{for } k = 0, \\ \operatorname{arcsinh}(r) & \text{for } k = -1. \end{cases} \quad (10.17)$$

Using χ as coordinate, the same result follows immediately

$$l = R(t) \int_0^{\chi(r)} d\chi = R(t)\chi. \quad (10.18)$$

Hence for $k = 0$, i.e a flat space, one obtains the usual result $L/l = 2\pi$, while for $k = 1$ (spherical geometry) $L/l = 2\pi r/\arcsin(r) < 2\pi$ and for $k = -1$ (hyperbolic geometry) $L/l = 2\pi r/\operatorname{arcsinh}(r) > 2\pi$.

For $k = 0$ and $k = -1$, l is unbounded, while for $k = +1$ there exists a maximal distance $l_{\max}(t)$. Hence the first two case correspond to open spaces with an infinite volume, while the latter is a closed space with finite volume.

Hubble's law

Hubble found empirically that the spectral lines of “distant” galaxies are redshifted, $z = \Delta\lambda/\lambda_0 > 1$, with a rate proportional to their distance d ,

$$cz = H_0 d. \quad (10.19)$$

If this redshift is interpreted as Doppler effect, $z = \Delta\lambda/\lambda_0 = v_r/c$, then the recession velocity of galaxies follows as

$$v = H_0 d. \quad (10.20)$$

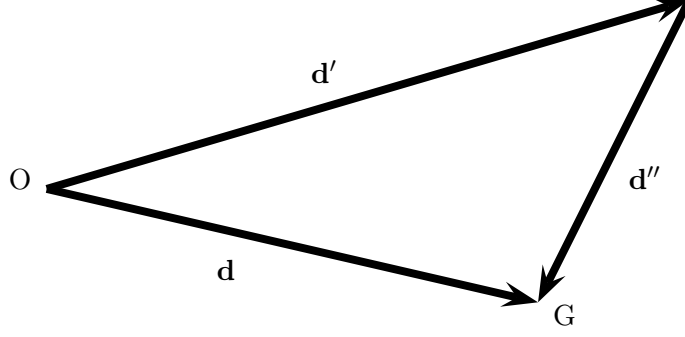


Figure 10.1: An observer at position $\mathbf{d}'$ sees the galaxy G receding with the speed $H(\mathbf{d} - \mathbf{d}') = H\mathbf{d}''$, if the Hubble relation is linear.

The restriction “distant galaxies” means more precisely that $H_0 d \gg v_{\text{pec}} \sim \text{few} \times 100 \text{ km/s}$. In other words, the peculiar motion of galaxies caused by the gravitational attraction of nearby galaxy clusters should be small compared to the Hubble flow $H_0 d$. Note that the interpretation of v as recession velocity is problematic. The validity of such an interpretation is certainly limited to $v \ll c$.

The parameter H_0 is called Hubble constant and has the value $H_0 \approx 71_{-3}^{+4} \text{ km/s/Mpc}$. We will see soon that the Hubble law Eq. (10.20) is an approximation valid for $z \ll 1$. In general, the Hubble constant is not constant but depends on time, $H = H(t)$, and we will call it therefore Hubble parameter for $t \neq t_0$.

We can derive Hubble’s law by a Taylor expansion of $R(t)$,

$$R(t) = R(t_0) + (t - t_0)\dot{R}(t_0) + \frac{1}{2}(t - t_0)^2\ddot{R}(t_0) + \dots \quad (10.21)$$

$$= R(t_0) \left[1 + (t - t_0)H_0 - \frac{1}{2}(t - t_0)^2 q_0 H_0^2 + \dots \right], \quad (10.22)$$

where

$$\boxed{H_0 \equiv \frac{\dot{R}(t_0)}{R(t_0)}} \quad \text{and} \quad \boxed{q_0 \equiv -\frac{\ddot{R}(t_0)R(t_0)}{\dot{R}^2(t_0)}} \quad (10.23)$$

is called deceleration parameter: If the expansion is slowing down, $\ddot{R} < 0$ and $q_0 > 0$.

Hubble’s law follows now as an approximation for small redshift: For not too large time-differences, we can use the expansion Eq. (10.21) and write

$$1 - z \approx \frac{1}{1 + z} = \frac{R(t)}{R_0} \approx 1 + (t - t_0)H_0. \quad (10.24)$$

Hence Hubble’s law, $z = (t_0 - t)H_0 = d/cH_0$, is valid as long as $z \approx H_0(t_0 - t) \ll 1$. Deviations from its linear form arises for $z \gtrsim 1$ and can be used to determine q_0 .

Hubble’s law as consequence of homogeneity Consider Hubble’s law as a vector equation with us at the center of the coordinate system,

$$\mathbf{v} = H\mathbf{d}. \quad (10.25)$$

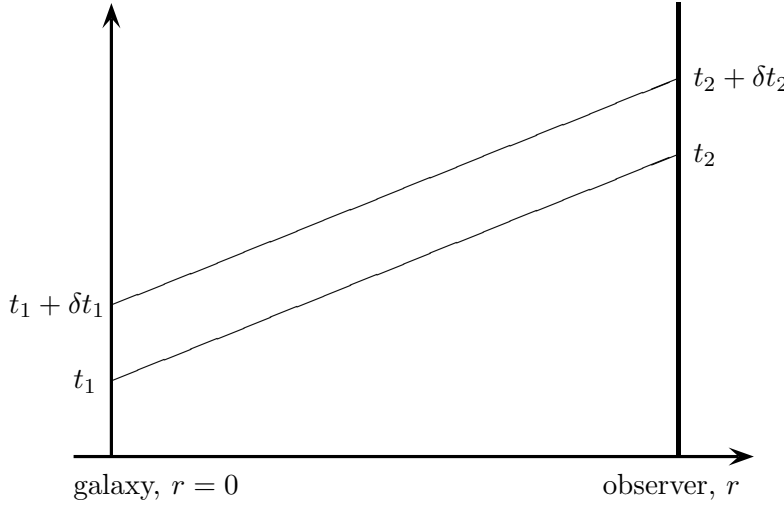


Figure 10.2: World lines of a galaxy emitting light and an observer at comoving coordinates $r = 0$ and r , respectively.

What sees a different observer at position $\mathbf{d}'$? He has the velocity $\mathbf{v}' = H\mathbf{d}'$ relative to us. We are assuming that velocities are small and thus

$$\mathbf{v}'' \equiv \mathbf{v} - \mathbf{v}' = H(\mathbf{d} - \mathbf{d}') = H\mathbf{d}'', \quad (10.26)$$

where $\mathbf{v}''$ and $\mathbf{d}''$ denote the position relative to the new observer. A linear relation between v and d as Hubble law is the only relation compatible with homogeneity and thus the “cosmological principle”.

Lemaître’s redshift formula

A light-ray propagates with $v = c$ or $ds^2 = 0$. Assuming a galaxy at $r = 0$ and an observer at r , i.e. light rays with $d\phi = d\vartheta = 0$, we rewrite the FRW metric as

$$\frac{dt}{R} = \frac{dr}{\sqrt{1 - kr^2}}. \quad (10.27)$$

We integrate this expression between the emission and absorption times t_1 and t_2 of the first light-ray,

$$\int_{t_1}^{t_2} \frac{dt}{R} = \int_0^r \frac{dr}{\sqrt{1 - kr^2}} \quad (10.28)$$

and between $t_1 + \delta t_1$ and $t_2 + \delta t_2$ for the second light-ray (see also Fig. 10.2),

$$\int_{t_1 + \delta t_1}^{t_2 + \delta t_2} \frac{dt}{R} = \int_0^r \frac{dr}{\sqrt{1 - kr^2}}. \quad (10.29)$$

The RHS’s are the same and thus we can equate the LHS’s,

$$\int_{t_1}^{t_2} \frac{dt}{R} = \int_{t_1 + \delta t_1}^{t_2 + \delta t_2} \frac{dt}{R}. \quad (10.30)$$

We change the integration limits, subtracting the common interval $[t_1 + \delta t_1 : t_2]$ and obtain

$$\int_{t_1}^{t_1+\delta t_1} \frac{dt}{R} = \int_{t_2}^{t_2+\delta t_2} \frac{dt}{R}. \quad (10.31)$$

Now we choose the time intervals δt_i as the time between two wave crests separated by the wave lengths λ_i of an electromagnetic wave. Since these time intervals are extremely short compared to cosmological times, $\delta t_i = \lambda_i/c \ll t_i$, we can assume $R(t)$ as constant performing the integrals and obtain

$$\frac{\delta t_1}{R_1} = \frac{\delta t_2}{R_2} \quad \text{or} \quad \frac{\lambda_1}{R_1} = \frac{\lambda_2}{R_2}. \quad (10.32)$$

The redshift z of an object is defined as the relative change in the wavelength between emission and detection,

$$z = \frac{\lambda_2 - \lambda_1}{\lambda_1} = \frac{\lambda_2}{\lambda_1} - 1 \quad (10.33)$$

or

$$\boxed{1 + z = \frac{\lambda_2}{\lambda_1} = \frac{R_2}{R_1}}. \quad (10.34)$$

This result is intuitively understandable, since the expansion of the universe stretches all lengths including the wave-length of a photon. For a massless particle like the photon, $\nu = c\lambda$ and $E = cp$, and thus its frequency (energy) and its wave-length (momentum) are affected in the same way. By contrast, the energy of a non-relativistic particle with $E \approx mc^2$ is nearly fixed.

A similar calculation as for the photon can be done for massive particles. Since the geodesic equation for massive particles leads to a more involved calculation, we use in this case however a different approach. We consider two comoving observer separated by the proper distance δl . A massive particle with velocity v needs the time $\delta t = \delta l/v$ to travel from observer one to observer two. The relative velocity of the two observer is

$$\delta u = \frac{\dot{R}}{R} \delta l = \frac{\dot{R}}{R} v \delta t = v \frac{\delta R}{R}. \quad (10.35)$$

Since we assume that the two observes are separated only infinitesimally, we can use the addition law for velocities from special relativity for the calculation of the velocity v' measured by the second observer,

$$v' = \frac{v - \delta u}{1 - v\delta u} = v - (1 - v^2)\delta u + \mathcal{O}(\delta u^2) = v - (1 - v^2)v \frac{\delta R}{R}. \quad (10.36)$$

Introducing $\delta v = v - v'$, we obtain

$$\frac{\delta v}{v(1 - v^2)} = \frac{\delta R}{R}. \quad (10.37)$$

and integrating this equation results in

$$p = \frac{mv}{\sqrt{1 - v^2}} = \frac{\text{const.}}{R}. \quad (10.38)$$

Thus not the energy but the momentum $p = \hbar/\lambda$ of massive particles is red-shifted: The kinetic energy of massive particles goes quadratically to zero, and hence peculiar velocities relative to the Hubble flow are damped.

Luminosity and angular diameter distance

In an expanding universe, the distance to an object depends on the expansion history, the behaviour of the scale factor $R(t)$, between the time of emission t of the observed light and its reception at t_0 . From the metric (10.15) we can define the coordinate distance

$$\chi = \int_t^{t_0} \frac{dt}{R(t)} \quad (10.39)$$

as well as the proper distance $d = g_{\chi\chi}\chi = R(t)\chi$. The proper distance is however only for a static metric a measurable quantity and cosmologists use therefore other, operationally defined measures for the distance. The two most important examples are the luminosity and the angular diameter distances.

Luminosity distance The luminosity distance d_L is defined such, that the inverse-square law between luminosity L of a source at distance d and the received energy flux $\mathcal{F}$ is valid,

$$d_L = \left(\frac{L}{4\pi\mathcal{F}} \right)^{1/2}. \quad (10.40)$$

Assume now that a (isotropically emitting) source with luminosity $L(t)$ and comoving coordinate χ is observed at t_0 by an observer at O . The cut at O through the forward light cone of the source emitted at t_e defines a sphere S^2 with proper area

$$A = 4\pi R^2(t_0) S^2(\chi) \quad (10.41)$$

Two additional effects are that a single photon is redshifted, $\nu_0 = \nu_e/(1+z)$, and that the arrival rate of photons is reduced by the same factor due to time-dilation. Hence the received flux is

$$\mathcal{F}(t_0) = \frac{1}{(1+z)^2} \frac{L(t_e)}{4\pi R_0^2 S^2(\chi)} \quad (10.42)$$

and the luminosity distance in a FRW universe

$$d_L = (1+z) R_0 S(\chi). \quad (10.43)$$

Note that d_L depends via χ on the expansion history of the universe between t_e and t_0 .

Observable are not the coordinates χ or r , but the redshift z of a galaxy. Differentiating $1+z \propto 1/R(t)$, we obtain

$$dz = -\frac{R_0}{R^2} dR = -\frac{R_0}{R^2} dR = -(1+z)H dt \quad (10.44)$$

or

$$t_0 - t = \int_t^{t_0} dt = \int_z^0 \frac{dz}{H(z)(1+z)}. \quad (10.45)$$

Inserting the relation (10.44) into Eq. (10.39), we find the coordinate χ of a galaxy at redshift z as

$$\chi = \int_t^{t_0} \frac{dt}{R(t)} = \frac{1}{R_0} \int_0^z \frac{dz}{H(z)} \quad (10.46)$$

For small redshift $z \ll 1$, we can use the expansion (10.22)

$$\chi = \int_t^{t_0} \frac{dt}{R_0} [1 - (t - t_0)H_0 + \dots]^{-1} \quad (10.47)$$

$$\approx \frac{1}{R_0 H_0} [(t - t_0) + \frac{1}{2}(t - t_0)^2 H_0 + \dots] = \frac{1}{R_0 H_0} [z - \frac{1}{2}(1 + q_0)z^2 + \dots] \quad (10.48)$$

Angular diameter distance Instead of basing a distance measurement on standard candles, one may use standard rods with known proper length l whose angular diameter $\Delta\vartheta$ can be observed. Then we define the angular diameter distance as

$$d_A = \frac{l}{\Delta\vartheta}. \quad (10.49)$$

$$d_A = \frac{R_0 S(\chi)}{1 + z}. \quad (10.50)$$

Thus at small distances, $z \ll 1$, the two definitions agree by construction, while for large redshifts the differences increase as $(1 + z)^2$.

10.3 Friedmann equations

Normal coordinates The equations governing the evolution of a homogeneous, isotropic universe can be derived in a straight-forward way from the FRW metric. We employ a different way, rewriting the FRW metric with normal coordinates centered at $\tilde{x} = 0$,

$$g_{ab} = \eta_{ab} + \frac{1}{2} \frac{\partial^2 g_{ab}(0)}{\partial \tilde{x}^c \partial \tilde{x}^d} \tilde{x}^c \partial \tilde{x}^d \quad (10.51)$$

In a small enough domain around 0, the metric is Minkowskian and we can use radar signals to construct $\tilde{x}$. Assume an observer at $x = (t_a, 0)$ emits a radar signal that is deflected back at (t, χ) and returns to $x = (t_b, 0)$. We label (t, χ) by (T, R) in normal coordinates (and use therefore a for the scale factor),

$$T = \frac{t_b + t_a}{2} - t_0 \quad (10.52)$$

$$R = \frac{t_b - t_a}{2} \quad (10.53)$$

The relation between the two sets of coordinates follows from Eq. (10.47): Evaluate it once from t_a to t , and once from t to t_b . Then take the sum and the difference,

$$T = t - t_0 + a\dot{a}r^2/2 + \mathcal{O}(r^3) \quad (10.54)$$

$$R = ar + \mathcal{O}(r^3). \quad (10.55)$$

We need the inverse transformations, $t(T, R)$ and $r(T, R)$. Inserting (10.71) into (10.70), it follows

$$t = t_0 + T - Hr^2/2 \quad (10.56)$$

$$r = R/a \quad (10.57)$$

As last step we have to eliminate the t dependence of a and $\dot{a}$, using their expansion around t_0 . Using also that $t - t_0 = T - R^2 H_0/2 + \mathcal{O}(r^3)$, we obtain finally

$$ds^2 = -(1 - \alpha R^2)dT^2 + (1 + \beta R^2)dR^2 + R^2 d\Omega \quad (10.58)$$

with the abbreviations

$$\alpha = -H_0^2 q_0 \quad \beta = H_0^2 + k/a_0^2. \quad (10.59)$$

Local energy conservation In comoving coordinates, the energy-momentum tensor of a fluid is diagonal,

$$T^{ab} = \text{diag}(\rho, P, P, P), \quad (10.60)$$

but we have to know the Christoffel symbol to evaluate $\nabla_a T^{ab} = 0$. By contrast, local energy conservation simplified in normal coordinates to $\nabla_a T^{ab} = \partial_a T^{ab} = 0$, but T^{ab} is no longer diagonal, since there is a momentum-flow relative to $\tilde{x}$. We can construct $T^{ab}(\tilde{x})$ for small $\tilde{x}$ by Lorentz transformations with $\beta = H_0 r \ll 1$,

$$\begin{pmatrix} \gamma\beta & \beta \\ \beta & \gamma\beta \end{pmatrix} \begin{pmatrix} \rho & 0 \\ 0 & P \end{pmatrix} \begin{pmatrix} \gamma\beta & \beta \\ \beta & \gamma\beta \end{pmatrix} = \begin{pmatrix} \rho & \beta(\rho + P) \\ \beta(\rho + P) & P \end{pmatrix} + \mathcal{O}(\beta^2). \quad (10.61)$$

Thus the components of the energy-momentum tensor are

$$T^{00}(\tilde{x}) = \rho \quad (10.62)$$

$$T^{0\alpha}(\tilde{x}) = x^\alpha H_0(\rho + P) \quad (10.63)$$

$$T^{\alpha\beta}(\tilde{x}) = P\delta^{\alpha\beta} \quad (10.64)$$

(Note that ρ and P are by construction the values measured by a comoving observer.) Thus $\partial_a T^{ab}(\tilde{x}) = 0$ becomes

$$\dot{\rho} + 3(\rho + P)H = 0. \quad (10.65)$$

The expansion decreases the energy density both by dilution and by the work required to expand a gravitating gas with pressure $P \geq 0$: Since the universe is homogenous, pressure does not result in forces, but acts as source of gravity.

Thermodynamics The first law of thermodynamics becomes for a perfect fluid with $dS = 0$ simply

$$dU = TdS - PdV = -PdV \quad (10.66)$$

or

$$d(\rho R^3) = -Pd(R^3). \quad (10.67)$$

Dividing by dt ,

$$R\dot{\rho} + 3(\rho + P)\dot{R} = 0, \quad (10.68)$$

we obtain our old result,

$$\dot{\rho} = -3(\rho + P)H. \quad (10.69)$$

Friedmann equations The Friedmann equations are linear functions of the coefficients α and β in the metric (10.58). Thus we make the ansatz

$$GT^{00} = G\rho = A\alpha + B\beta \quad (10.70)$$

$$GT^{\alpha\alpha} = GP = C\alpha + D\beta \quad (10.71)$$

We can determine the unknown coefficients A, B, C and D requiring $\nabla_a G^{ab} = 0$. Using (10.71) to eliminate α in (10.70), differentiating next, we find

$$\begin{aligned} \dot{\rho} &= (A/C)\dot{P} + \dot{\beta}(BC - AD)/(GC) \\ &= (A/C)\dot{P} + 2H(\alpha - \beta)(BC - AD)/(GC) \\ &= \frac{A}{C}\dot{P} - \frac{2H}{C}[(C + D)\rho + (A + B)P]. \end{aligned} \quad (10.72)$$

Local energy conservation, (10.69), requires that $A = 0$, $C = -2B/3$ and $D = -B/3$. Thus

$$G\rho = B \left[H^2 + \frac{k}{R^2} \right] \quad (10.73)$$

and

$$GP = -\frac{B}{3} \left[\frac{2\ddot{R}}{R} + H^2 + \frac{k}{R^2} \right]. \quad (10.74)$$

The still unknown coefficient B has to be determined by a comparison with the non-relativistic version of these equations, giving $B = 3/(8\pi)$. If we insert the value of B and show explicitly the contribution of a cosmological constant in (10.73), the usual Friedmann equation follows as

$$\boxed{H^2 \equiv \left(\frac{\dot{R}}{R} \right)^2 = \frac{8\pi}{3}G\rho - \frac{k}{R^2} + \frac{\Lambda}{3}.} \quad (10.75)$$

We define the *critical density* ρ_{cr} as the density for which the spatial geometry of the universe is flat. From $k = 0$, it follows

$$\rho_{\text{cr}} = \frac{3H_0^2}{8\pi G} \quad (10.76)$$

and thus ρ_{cr} is uniquely fixed by the value of H_0 . One “hides” this dependence by introducing h ,

$$H_0 = 100 h \text{ km}/(\text{s Mpc}).$$

Then one can express the critical density as function of h ,

$$\rho_{\text{cr}} = 2.77 \times 10^{11} h^2 M_{\odot}/\text{Mpc}^3 = 1.88 \times 10^{-29} h^2 \text{ g}/\text{cm}^3 = 1.05 \times 10^{-5} h^2 \text{ GeV}/\text{cm}^3.$$

Thus a flat universe with $H_0 = 100 h \text{ km}/\text{s}/\text{Mpc}$ requires an energy density of ~ 10 protons per cubic meter. We define the abundance Ω_i of the different players in cosmology as their energy density relative to ρ_{cr} , $\Omega_i = \rho/\rho_{\text{cr}}$.

In the following, we will often include Λ as other contributions to the energy density ρ via

$$\frac{8\pi}{3}G\rho_{\Lambda} = \frac{\Lambda}{3}. \quad (10.77)$$

Thereby one recognizes also that the cosmological constant acts as a constant energy density

$$\rho_\Lambda = \frac{\Lambda}{8\pi G} \quad \text{or} \quad \Omega_\Lambda = \frac{\Lambda}{3H_0^2}. \quad (10.78)$$

Adding the two equations, the “acceleration equation” follows as

$$\boxed{\frac{\ddot{R}}{R} = \frac{\Lambda}{3} - \frac{4\pi G}{3}(\rho + 3P)}. \quad (10.79)$$

This equation determines the (de-) acceleration of the Universe as function of its matter and energy content. “Normal” matter is characterized by $\rho > 0$ and $P \geq 0$. Thus a static solution is impossible for a universe with $\Lambda = 0$. Such a universe is decelerating and since today $\dot{R} > 0$, $\ddot{R}$ was always negative and there was a “big bang”.

We can understand better the physical properties of the cosmological constant by replacing Λ by $(8\pi G)\rho_\Lambda$. Now we can compare the effect of normal matter and of the Λ term on the acceleration,

$$\frac{\ddot{R}}{R} = \frac{8\pi G}{3}\rho_\Lambda - \frac{4\pi G}{3}(\rho + 3P) \quad (10.80)$$

Thus Λ is equivalent to matter with an E.o.S. $w_\Lambda = P/\rho = -1$. This property can be checked using only thermodynamics: With $P = -(\partial U/\partial V)_S$ and $U_\Lambda = \rho_\Lambda V$, it follows $P = -\rho$.

The borderline between an accelerating and decelerating universe is given by $\rho = -3P$ or $w = -1/3$. The condition $\rho < -3P$ violates the so-called strong energy condition for “normal” matter in equilibrium. An accelerating universe requires therefore a positive cosmological constant or a dominating form of matter that is not in equilibrium.

Note that the energy contribution of relativistic matter, photons and possibly neutrinos, is much smaller than the one of non-relativistic matter (stars and cold dark matter). Thus the pressure term in the acceleration equation can be neglected at the present epoch. Measuring $\ddot{R}/R$, $\dot{R}/R$ and ρ fixes therefore the geometry of the universe.

10.4 Scale-dependence of different energy forms

The dependence of different energy forms as function of the scale factor R can be derived from energy conservation, $dU = -PdV$, if an E.o.S. $P = P(\rho) = w\rho$ is specified. For $w = \text{const.}$, it follows

$$d(\rho R^3) = -3PR^2 dR \quad (10.81)$$

or eliminating P

$$\frac{d\rho}{dR} R^3 + 3\rho R^2 = -3w\rho R^2. \quad (10.82)$$

Separating the variables,

$$-3(1+w)\frac{dR}{R} = \frac{d\rho}{\rho}, \quad (10.83)$$

we can integrate and obtain

$$\boxed{\rho \propto R^{-3(1+w)}} = \begin{cases} R^{-3} & \text{for matter } (w = 0), \\ R^{-4} & \text{for radiation } (w = 1/3), \\ \text{const.} & \text{for } \Lambda \quad (w = -1). \end{cases} \quad (10.84)$$

This result can be understood also from heuristic arguments:

- (Non-relativistic) matter means that $kT \ll m$. Thus $\rho = nm \gg nT = P$ and non-relativistic matter is pressure-less, $w = 0$. The mass m is constant and $n \propto 1/R^3$, hence ρ is just diluted by the expansion of the universe, $\rho \propto 1/R^3$.
- Radiation is not only diluted but the energy of each single photon is additionally redshifted, $E \propto 1/R$. Thus the energy density of radiation scales as $\propto 1/R^4$. Alternatively, one can use that $\rho = aT^4$ and $T \propto \langle E \rangle \propto 1/R$.
- Cosmological constant Λ : From $\frac{8\pi}{3}G\rho_\Lambda = \frac{\Lambda}{3}$ one obtains that the cosmological constant acts as an energy density $\rho_\Lambda = \frac{\Lambda}{8\pi G}$ that is constant in time, independent from a possible expansion or contraction of the universe.
- Note that the scaling of the different energy forms is very different. It is therefore surprising that “just today”, the energy in matter and due to the cosmological constant is of the same order (“coincidence problem”).

Let us rewrite the Friedmann equation for the present epoch as

$$\frac{k}{R_0^2} = H_0^2 \left(\frac{8\pi G}{3H_0^2} \rho_0 + \frac{\Lambda}{3H_0^2} - 1 \right) = H_0^2 (\Omega_{\text{tot},0} - 1). \quad (10.85)$$

We express the curvature term for arbitrary times through $\Omega_{\text{tot},0}$ and the redshift z as

$$\frac{k}{R^2} = \frac{k}{R_0^2} (1+z)^2 = H_0^2 (\Omega_{\text{tot},0} - 1) (1+z)^2. \quad (10.86)$$

Dividing the Friedmann equation (10.75) by $H_0^2 = 8\pi G\rho_{\text{cr}}/3$, we obtain

$$\begin{aligned} \frac{H^2(z)}{H_0^2} &= \sum_i \Omega_i(z) - (\Omega_{\text{tot},0} - 1)(1+z)^2 \\ &= \Omega_{\text{rad},0}(1+z)^4 + \Omega_{\text{m},0}(1+z)^3 + \Omega_\Lambda - (\Omega_{\text{tot},0} - 1)(1+z)^2 \end{aligned} \quad (10.87)$$

This expression allows us to calculate the age of the universe (10.45), distances (10.43), etc. for a given cosmological model, i.e. specifying the energy content $\Omega_{i,0}$ and the Hubble parameter H_0 at the present epoch.

10.5 Cosmological models with one energy component

We consider a flat universe, $k = 0$, with one dominating energy component with E.o.S $w = P/\rho = \text{const.}$. With $\rho = \rho_{\text{cr}} (R/R_0)^{-3(1+w)}$, the Friedmann equation becomes

$$\dot{R}^2 = \frac{8\pi}{3} G \rho R^2 = H_0^2 R_0^{3+3w} R^{-(1+3w)}, \quad (10.88)$$

where we inserted the definition of $\rho_{\text{cr}} = 3H_0^2/(8\pi G)$. Separating variables we obtain

$$R_0^{-(3+3w)/2} \int_0^{R_0} dR R^{(1+3w)/2} = H_0 \int_0^{t_0} dt = t_0 H_0 \quad (10.89)$$

and hence the age of the Universe follows as

$$\boxed{t_0 H_0 = \frac{2}{3+3w}} = \begin{cases} 2/3 & \text{for matter } (w = 0), \\ 1/2 & \text{for radiation } (w = 1/3), \\ \rightarrow \infty & \text{for } \Lambda \quad (w = -1). \end{cases} \quad (10.90)$$

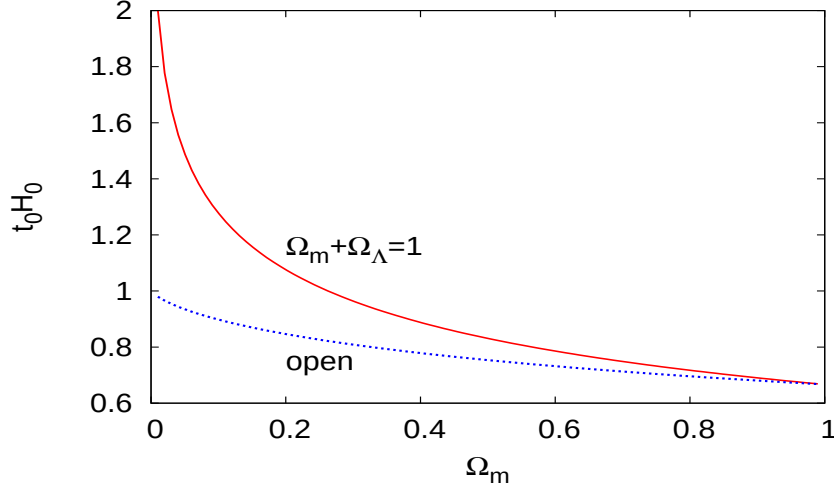


Figure 10.3: The product $t_0 H_0$ for an open universe containing only matter and for a flat cosmological model with $\Omega_\Lambda + \Omega_m = 1$.

Models with $w > -1$ needed a finite time to expand from the initial singularity $R(t=0) = 0$ to the current size R_0 , while a Universe with only a Λ has no “beginning”.

In models with a hot big-bang, $\rho, T \rightarrow \infty$ for $t \rightarrow 0$, and we should expect that classical gravity breaks down at some moment t_* . As long as $R \propto t^\alpha$ with $\alpha < 1$, most time elapsed during the last fractions of $t_0 H_0$. Hence our result for the age of the universe does not depend on unknown physics close to the big-bang as long as $w > -1/3$.

If we integrate (10.89) to the arbitrary time t , we obtain the time-dependence of the scale factor,

$$\boxed{R(t) \propto t^{2/(3+3w)}} = \begin{cases} t^{2/3} & \text{for matter } (w = 0), \\ t^{1/2} & \text{for radiation } (w = 1/3), \\ \exp(t) & \text{for } \Lambda \text{ } (w = -1). \end{cases} \quad (10.91)$$

Age problem of the universe. The age of a matter-dominated universe is (expanded around $\Omega_0 = 1$)

$$t_0 = \frac{2}{3H_0} \left[1 - \frac{1}{5}(\Omega_0 - 1) + \dots \right]. \quad (10.92)$$

Globular cluster ages require $t_0 \geq 13$ Gyr. Using $\Omega_0 = 1$ leads to $H_0 \leq 2/3 \times 13 \text{ Gyr} = 1/19.5 \text{ Gyr}$ or $h \leq 0.50$. Thus a flat universe with $t_0 = 13$ Gyr without cosmological constant requires a too small value of H_0 . Choosing $\Omega_m \approx 0.3$ increases the age by just 14%.

We derive the age t_0 of a flat Universe with $\Omega_m + \Omega_\Lambda = 1$ in the next section as

$$\frac{3t_0 H_0}{2} = \frac{1}{\sqrt{\Omega_\Lambda}} \ln \frac{1 + \sqrt{\Omega_\Lambda}}{\sqrt{1 - \Omega_\Lambda}}. \quad (10.93)$$

Requiring $H_0 \geq 65$ km/s/Mpc and $t_0 \geq 13$ Gyr means that the function on the RHS should be larger than $3 \times 13 \text{ Gyr} \times 0.65 / (2 \times 9.8 \text{ Gyr}) \approx 1.3$ or $\Omega_\Lambda \geq 0.55$.

10.6 The Λ CDM model

– just as supplement/remind for those interested –

We consider a flat Universe containing as its only two components pressure-less matter and a cosmological constant, $\Omega_m + \Omega_\Lambda = 1$. Then the curvature term in the Friedmann equation and the pressure term in the deceleration equation play no role and we can hope to solve these equations for $a(t)$. Multiplying the deceleration equation (10.79) by two and adding it to the Friedmann equation (10.75), we eliminate ρ_m ,

$$2 \frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a} \right)^2 = \Lambda. \quad (10.94)$$

Next we rewrite first the LHS and then the RHS as total time derivatives: With

$$\frac{d}{dt}(a\dot{a}^2) = \dot{a}^3 + 2a\dot{a}\ddot{a} = \dot{a}^2 \left[\left(\frac{\dot{a}}{a} \right)^2 + 2 \frac{\ddot{a}}{a} \right], \quad (10.95)$$

we obtain

$$\frac{d}{dt}(a\dot{a}^2) = \dot{a}^2 \Lambda = \frac{1}{3} \frac{d}{dt}(a^3) \Lambda. \quad (10.96)$$

Integrating is now trivial,

$$a\dot{a}^2 = \frac{\Lambda}{3} a^3 + C. \quad (10.97)$$

The constant C can be determined most easily by setting $a(t_0) = 1$ and comparing the Friedmann equation (10.75) with (10.97) for $t = t_0$ as $C = 8\pi G\rho_{m,0}/3$.

Next we introduce the new variable $x = a^{3/2}$. Then

$$\frac{da}{dt} = \frac{dx}{dt} \frac{da}{dx} = \frac{dx}{dt} \frac{2x^{-1/3}}{3}, \quad (10.98)$$

and we obtain as new differential equation

$$\dot{x}^2 - \Lambda x^2/4 + C/3 = 0. \quad (10.99)$$

Inserting the solution $x(t) = A \sinh(\sqrt{\Lambda}t/2)$ of the homogeneous equation fixes the constant A as $A = \sqrt{3C/\Lambda}$. We can express A also by the current values of Ω_i as $A = \Omega_m/\Omega_\Lambda = (1 - \Omega_\Lambda)/\Omega_\Lambda$. Hence the time-dependence of the scale factor is

$$a(t) = A^{1/3} \sinh^{2/3}(\sqrt{3\Lambda}t/2). \quad (10.100)$$

The time-scale of the expansion is set by $t_\Lambda = 2/\sqrt{3\Lambda}$.

The present age t_0 of the universe follows by setting $a(t_0) = 1$ as

$$t_0 = t_\Lambda \operatorname{arctanh}(\sqrt{\Omega_\Lambda}). \quad (10.101)$$

The deceleration parameter $q = -\ddot{a}/aH^2$ is an important quantity for observational tests of the Λ CDM model. We calculate first the Hubble parameter

$$H(t) = \frac{\dot{a}}{a} = \frac{2}{3t_\Lambda} \coth(t/t_\Lambda) \quad (10.102)$$

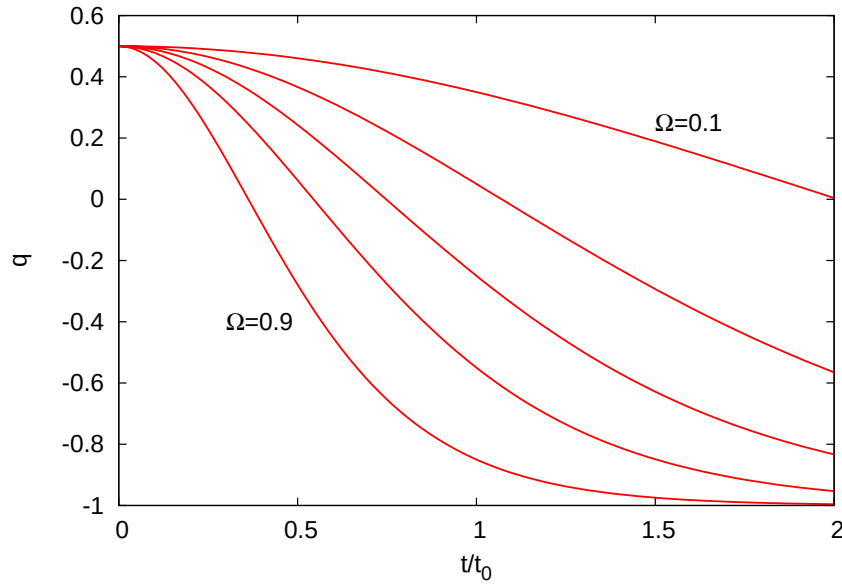


Figure 10.4: The deceleration parameter q as function of t/t_0 for the Λ CDM model and various values for Ω_Λ (0.1, 0.3, 0.5, 0.7 and 0.9 from the top to the bottom).

and then

$$q(t) = \frac{1}{2}[1 - 3 \tanh^2(t/t_\Lambda)]. \quad (10.103)$$

The limiting behavior of q corresponds with $q = 1/2$ for $t \rightarrow 0$ and $q = -1$ for $t \rightarrow \infty$ as expected to the one of a flat $\Omega_m = 1$ and a $\Omega_\Lambda = 1$ universe. More interesting is the transition region and, as shown in Fig. 10.4, the transition from a decelerating to an accelerating universe happens for $\Omega_\Lambda = 0.7$ at $t \approx 0.55t_0$. This can easily be converted to redshift, $z_* = a(t_0)/a(t_*) - 1 \approx 0.7$, that is directly measured in supernova observations.