

Week 4: Classical mechanics

Calculus of variations

We call a map $F[f(x)]$ from a certain space of functions $f(x)$ into $\mathbb{R}$ a functional. We will consider functionals from the space $C_2[a : b]$ of twice differentiable functions between fixed points a and b . Extrema of functionals are obtained by the calculus of variations. Let us consider as functional

$$S[L(q^i, \dot{q}^i)] = \int_a^b dt L(q^i, \dot{q}^i, t), \quad (59)$$

where L is a function of the $2n$ independent functions q^i and $\dot{q}^i = dq^i/dt$ as well as of the parameter t . Consider an infinitesimal variation of the path, $q^i(t) \rightarrow q^i(t) + \delta q^i(t)$ with $\delta q^i(t) = \varepsilon q(t)$ that keeps the endpoints fixed, but is otherwise arbitrary. The resulting variation of the functional is

$$\delta S = \int_a^b dt \delta L(q^i, \dot{q}^i, t) = \int_a^b dt \left(\frac{\partial L}{\partial q^i} \delta q^i + \frac{\partial L}{\partial \dot{q}^i} \delta \dot{q}^i \right). \quad (60)$$

We can eliminate the variation $\delta \dot{q}^i$ of the velocities, integrating the second term by parts using $\delta(\dot{q}^i) = d/dt(\delta q^i)$,

$$\delta S = \int_a^b dt \left[\frac{\partial L}{\partial q^i} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^i} \right) \right] \delta q^i + \left[\frac{\partial L}{\partial \dot{q}^i} \delta q^i \right]_a^b. \quad (61)$$

The boundary term vanishes, because we required that the variations δq^i are zero at a and b . Since the variations are otherwise arbitrary, the sum in the first bracket has to be zero for an extremal curve, $\delta S = 0$. The resulting equations are the Euler-Lagrange equations² of the functional S ,

$$\boxed{\frac{\delta S}{\delta q^i} = \frac{\partial L}{\partial q^i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} = 0.} \quad (62)$$

The Lagrangian L is not uniquely fixed: Adding a total time-derivative, $L' = L + df(q, t)/dt$ does not change the resulting Lagrange equations,

$$S' = S + \int_a^b dt \frac{df}{dt} = S + f(q(b), t_b) - f(q(a), t_a), \quad (63)$$

since the last two terms vanish varying the action with the restriction of fixed endpoints a and b .

²Physicists call these equations often only Lagrange equations (or, especially in classical mechanics, Lagrange equations of the second kind).

Hamilton's principle and the Lagrange function

The observation that the solutions of the equation of motions can be obtained as the extrema of an appropriate functional (‘‘the action S ’’) of the Lagrangian L is called *Hamilton's principle*.

We derive now the Lagrangian L of a free non-relativistic particle from the Galilean principle of inertia. More precisely, we use that the homogeneity of space and time forbids that L depends on $\vec{x}$ and t , while the isotropy of space implies that L depends only on the norm of the velocity vector, but not on its direction,

$$L = L(v^2).$$

Let us consider two inertial frames moving with the infinitesimal velocity $\vec{\varepsilon}$ relative to each other. Then a Galilean transformation connects the velocities measured in the two frames as $\vec{v}' = \vec{v} + \vec{\varepsilon}$. The Galilean principle of relativity requires that the laws of motion have the same form in both frames, and thus the Lagrangians can differ only by a total time-derivative. Expanding the difference δL in ε gives with $\delta v^2 = 2\vec{v}\vec{\varepsilon}$

$$\delta L = \frac{\partial L}{\partial v^2} \delta v^2 = 2\vec{v}\vec{\varepsilon} \frac{\partial L}{\partial v^2}. \quad (64)$$

The difference has to be a total time-derivative. Since $v = \dot{q}$, the derivative term $\partial L / \partial v^2$ has to be independent of $\vec{v}$. Hence, $L \propto v^2$ and we call the proportionality constant $m/2$, and the total expression kinetic energy T ,

$$L = T = \frac{1}{2}mv^2. \quad (65)$$

Example: Check the relativity principle for finite relative velocities:

For

$$L' = \frac{1}{2}mv'^2 = \frac{1}{2}m(\vec{v} + \vec{V})^2 = \frac{1}{2}mv^2 + m\vec{v} \cdot \vec{V} + \frac{1}{2}mV^2$$

or

$$L' = L + \frac{d}{dt} \left(m\vec{x} \cdot \vec{V} + \frac{1}{2}mV^2 t \right).$$

Thus the difference is indeed a total time derivative.

We can write the velocity with $dl^2 = dx^2 + dy^2 + dz^2$ as

$$v^2 = \frac{(dl)^2}{(dt)^2} = g_{ik} \frac{dx^i}{dt} \frac{dx^k}{dt}, \quad (66)$$

where g is again the metric tensor. For instance, in spherical coordinates $dl^2 = dr^2 + r^2 \sin^2 \vartheta d\phi^2 + r^2 d\vartheta^2$ and thus

$$T = \frac{1}{2}m \left(\dot{r}^2 + r^2 \sin^2 \vartheta \dot{\phi}^2 + r^2 \dot{\vartheta}^2 \right). \quad (67)$$

For a system of non-interacting particles, L is additive, $L = \sum_a \frac{1}{2} m_a v_a^2$. If there are interactions (assumed for the moment to depend only on the coordinates), then we add a function $-U(\vec{x}_1, \vec{x}_2, \dots)$ called potential energy.

We can now derive the equations of motions for a system of n interacting particles,

$$L = \sum_a \frac{1}{2} m_a v_a^2 - U(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_n). \quad (68)$$

using the Lagrange equations,

$$m_a \frac{d\vec{v}_a}{dt} = -\frac{\partial U}{\partial \vec{x}_a} = \vec{F}_a. \quad (69)$$

We can change from Cartesian coordinates to arbitrary (or “generalized”) coordinates for the n particles,

$$x^a = f^a(q^1, \dots, q^n), \quad \dot{x}^a = \frac{\partial f^a}{\partial q^k} \dot{q}^k. \quad (70)$$

Substituting gives

$$L = \frac{1}{2} a_{ik} \dot{q}^i \dot{q}^j - U(q_i), \quad (71)$$

where $a_{ik}(q)$ is a quadratic function of the velocity that is, apart from the factors m_a identical to the metric tensor on the configuration space q^n . Finally, we define the canonically conjugated momentum p_i as

$$p_i = \frac{\partial L}{\partial \dot{q}^i}. \quad (72)$$

Feynman's approach to quantum theory:

The whole information about a quantum mechanical system is contained in its time-evolution operator $U(t, t')$. Its matrix elements $K(x', t'; x, t)$ (propagator or Green's function) in the coordinate basis connect wavefunctions at different times as

$$\psi(x', t') = \int d^3x K(x', t'; x, t) \psi(x, t).$$

Feynman proposed the following connection between the propagator K and the classical action S ,

$$K(x', t'; x, t) = N \int_q^{q'} \mathcal{D}q \exp(-iS),$$

where $\mathcal{D}q$ denotes the “integration over all paths.” Hence the difference between the classical and quantum world is that in the former only paths extremizing the action S are allowed while in the latter all paths weighted by $\exp(-iS)$ contribute.

Symmetries and Noether's theorem

Quantities that remain constant during the evolution of a mechanical system are called integrals of motions. Seven of them that are connected to the fundamental symmetries of space and time are of special importance: These are the conserved quantities energy, momentum and angular momentum.

Energy The Lagrangian of a closed system depends, because of the homogeneity of time, not on time. Its total time derivative is

$$\frac{dL}{dt} = \frac{\partial L}{\partial q^i} \dot{q}^i + \frac{\partial L}{\partial \dot{q}^i} \ddot{q}^i. \quad (73)$$

Replacing $\partial L / \partial q^i$ by $(d/dt) \partial L / \partial \dot{q}^i$, it follows

$$\frac{dL}{dt} = \dot{q}^i \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} + \frac{\partial L}{\partial \dot{q}^i} \ddot{q}^i = \frac{d}{dt} \left(\dot{q}^i \frac{\partial L}{\partial \dot{q}^i} \right). \quad (74)$$

Hence the quantity

$$E \equiv \dot{q}^i \frac{\partial L}{\partial \dot{q}^i} - L \quad (75)$$

remains constant during the evolution of a closed system. This holds also more generally, e.g. in the presence of static external fields, as long as the Lagrangian is not time-dependent.

We have still to show that E coincides indeed with the usual definition of energy. Using as $L = T(q, \dot{q}) - U(q)$, where T is quadratic in the velocities, we have

$$\dot{q}^i \frac{\partial L}{\partial \dot{q}^i} = \dot{q}^i \frac{\partial T}{\partial \dot{q}^i} = 2T \quad (76)$$

and thus $E = 2T - L = T + U$.

Momentum Homogeneity of space implies that an translation by a constant vector of a closed system does not change its properties. Thus an infinitesimal translation from $\vec{r}$ to $\vec{r} + \vec{\varepsilon}$ should not change L . Since velocities are unchanged, we have (summation over a particles)

$$\delta L = \sum_a \frac{\partial L}{\partial \vec{r}_a} \cdot \delta \vec{r}_a = \vec{\varepsilon} \cdot \sum_a \frac{\partial L}{\partial \vec{r}_a}. \quad (77)$$

The condition $\delta L = 0$ is true for arbitrary $\vec{\varepsilon}$, if

$$\sum_a \frac{\partial L}{\partial \vec{r}_a} = 0. \quad (78)$$

Using again Lagrange's equations, we obtain

$$\sum_a \frac{d}{dt} \frac{\partial L}{\partial \vec{v}_a} = \frac{d}{dt} \sum_a \frac{\partial L}{\partial \vec{v}_a} = 0. \quad (79)$$

Hence, in a closed mechanical system the *momentum* vector of the system

$$\vec{p}_{\text{tot}} = \sum_a \frac{\partial L}{\partial \vec{v}_a} = \sum_a m_a \vec{v}_a = \text{const.} \quad (80)$$

is conserved.

The condition (78) signifies with $\partial L / \partial \vec{r}_a = -\partial U / \partial \vec{r}_a$ that the sum of forces on all particles is zero, $\sum_a \vec{F}_a = 0$. For the particular case of two-particle system, $\vec{F}_a = -\vec{F}_b$, we have thus derived Newton's third law, the equality of action and reaction.

Isotropy We consider now the consequences of the isotropy of space, i.e. search the conserved quantity that follows from a Lagrangian invariant under rotation. Under an infinitesimal rotation by $\delta\vec{\phi}$ both coordinates and velocities change,

$$\delta\vec{r} = \delta\vec{\phi} \times \vec{r}, \quad (81)$$

$$\delta\vec{v} = \delta\vec{\phi} \times \vec{v}. \quad (82)$$

Inserting the expression into

$$\delta L = \sum_a \left(\frac{\partial L}{\partial \vec{r}_a} \delta \vec{r}_a + \frac{\partial L}{\partial \vec{v}_a} \delta \vec{v}_a \right) = 0 \quad (83)$$

gives, using also the definition $\vec{p}_a = \partial L / \partial \vec{v}_a$ as well as the Lagrange equation $\vec{p}_a = \partial L / \partial \vec{r}_a$,

$$\delta L = \sum_a \left(\vec{p}_a \cdot \delta\vec{\phi} \times \vec{r}_a + \vec{p}_a \cdot \delta\vec{\phi} \times \vec{v}_a \right) = 0 \quad (84)$$

Permuting the factors and extracting $\delta\vec{\phi}$ gives

$$\delta\vec{\phi} \cdot \sum_a \left(\vec{r}_a \times \vec{p}_a + \vec{v}_a \times \vec{p}_a \right) = \delta\vec{\phi} \cdot \frac{d}{dt} \sum_a \vec{r}_a \times \vec{p}_a = 0. \quad (85)$$

Thus the angular momentum

$$\vec{M} = \sum_a \vec{r}_a \times \vec{p}_a = \text{const.} \quad (86)$$

is conserved.

Hamilton's equations

In the Lagrange formalism, we describe a system by specifying its generalized coordinates and velocities using the Lagrangian, $L = L(q^i, \dot{q}^i)$. An alternative is to use generalized coordinates and canonically conjugated momenta p_i defined as

$$p_i = \frac{\partial L}{\partial \dot{q}^i}. \quad (87)$$

The passage from $\{q^i, \dot{q}^i\}$ to $\{q^i, p_i\}$ is a special case of a Legendre transformation. We write first the differential of L as

$$dL = \frac{\partial L}{\partial q^i} dq^i + \frac{\partial L}{\partial \dot{q}^i} d\dot{q}^i. \quad (88)$$

or with the definition in Eq. (87) and $\partial L / \partial q^i = d/dt(\partial L / \partial \dot{q}^i) = \dot{p}_i$

$$dL = \dot{p}_i dq^i + p_i d\dot{q}^i. \quad (89)$$

We write next

$$p_i d\dot{q}^i = d(p_i \dot{q}^i) - dp_i \dot{q}^i,$$

bring the $d(pq)$ term on the LHS and reverse the sign,

$$d(p_i \dot{q}^i - L) = -\dot{p}_i dq^i + \dot{q}^i dp_i. \quad (90)$$

The function on the LHS to which d is applied is the energy, cf. Eq. (75), expressed by p_i and q^i . It is called Hamiltonian or *Hamilton function* H with

$$H(q^i, p_i, t) = p_i \dot{q}^i - L. \quad (91)$$

If we consider H as function of q^i and p_i , then its differential is

$$dH = \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial q^i} dq^i \quad (92)$$

A comparison with Eq. (90) gives *Hamilton's equations*,

$$\boxed{\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q^i}} \quad (93)$$

Consider an observable $O = O(x, p, t)$. Its time-dependence is given by

$$\frac{dO}{dt} = \frac{\partial O}{\partial p_i} \dot{p}_i + \frac{\partial O}{\partial q^i} \dot{q}^i + \frac{\partial O}{\partial t} = -\frac{\partial O}{\partial p_i} \frac{\partial H}{\partial q^i} + \frac{\partial O}{\partial q^i} \frac{\partial H}{\partial p_i} + \frac{\partial O}{\partial t} \quad (94)$$

where we used Hamilton's equations. If we define the Poisson brackets $\{A, B\}$ between two Observables A and B as

$$\{A, B\} = \frac{\partial A}{\partial q^i} \frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial B}{\partial q^i}, \quad (95)$$

then we can rewrite Eq. (94) as

$$\frac{dO}{dt} = \{O, H\} + \frac{\partial O}{\partial t}. \quad (96)$$

Note that we have obtained a formal correspondence between classical and quantum mechanics: The time-evolution of an observable O in the Heisenberg picture is given by the same equation as in classical mechanics, if the Poisson bracket is changed against a commutator.

Since the Poisson bracket is antisymmetric, we find

$$\frac{dH}{dt} = \frac{\partial H}{\partial t}. \quad (97)$$

Hence the Hamiltonian is a conserved quantity, if and only if he is time-independent.