

Week 5: Classical field theory³

Lagrange formalism for fields

A field is a function of space-time x^μ with values in “field space.” The space of values the field can take may be characterized by its transformation properties under Lorentz transformations (scalar ϕ , vector A^μ , tensor $g^{\mu\nu}$, ... or spinor $\psi_a, \psi_{ab}, \dots$ fields) and (typically) Lie groups like $\mathbb{R}$, $\mathbb{R}^+$, $U(1)$, $SU(n)$, ... Thus we have to generalize Hamilton’s principle to a collection of fields $\phi_a(x^\mu)$, $a = 1, \dots, k$. To ensure Lorentz invariance, we consider a scalar Lagrange density $\mathcal{L}$ that may, analogously to $L(q, \dot{q})$, depend on the fields and its first derivatives $\partial_\mu \phi_a$. There is no explicit time-dependence, since “everything” should be explained by the fields and their interactions. The Lagrangian $L(\phi_a, \partial_\mu \phi_a)$ is obtained by integrating $\mathcal{L}$ over a given space volume V .

The action S is thus the four-dimensional integral

$$S[\mathcal{L}(\phi_a, \partial_\mu \phi_a)] = \int_a^b dt L(\phi_a, \partial_\mu \phi_a) = \int_\Omega d^4x \mathcal{L}(\phi_a, \partial_\mu \phi_a), \quad (113)$$

where $\Omega = V \times [t_a : t_b]$. A variation $\varepsilon \phi_a \equiv \delta \phi_a$ of the fields leads to a variation of the action,

$$\delta S = \int_\Omega d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi^a} \delta \phi^a + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi^a)} \delta (\partial_\mu \phi^a) \right), \quad (114)$$

where we have to sum over fields ($a = 1, \dots, k$) and the Lorentz index $\mu = 0, \dots, 3$. We eliminate again the variation of the field gradients $\partial_\mu \phi^a$ by a partial integration using Gauß’ theorem,

$$\delta S = \int_\Omega d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi^a} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi^a)} \right) \right] \delta \phi^a = 0. \quad (115)$$

The boundary term vanishes, since we require that the variation is zero on the boundary $\partial\Omega$. Thus the Lagrange equations for the fields ϕ^a are

$$\boxed{\frac{\partial \mathcal{L}}{\partial \phi^a} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi^a)} \right) = 0.} \quad (116)$$

If the Lagrange density $\mathcal{L}$ is changed by a four-dimensional divergence, the same equations of motions result.

Conservation laws

Let j^μ be a conserved vector field in Minkowski space,

$$\partial_\mu j^\mu = 0. \quad (117)$$

³We use in this week/section the metrik $\eta = \text{diag}(1, -1, -1, -1)$.

Then

$$\frac{d}{dt} \int_V d^3x j^0 = - \int_{\partial V} d\vec{S} \cdot \vec{j} \quad (118)$$

and

$$Q = \int_V d^3x j^0 \quad (119)$$

is a globally conserved quantity, if there is no outgoing flux $\vec{j}$ through the boundary ∂V . To show that Q is a Lorentz invariant quantity, we have to rewrite Eq. (119) as a tensor equation.

Consider

$$Q(t=0) = \int d^4x j^\mu(x) \partial_\mu \vartheta(\mathbf{n} \cdot \mathbf{x}) \quad (120)$$

with ϑ the step function and $\mathbf{n}$ a unit vector in time direction, $\mathbf{n} \cdot \mathbf{x} = x^0 = t$. Then

$$Q(t=0) = \int d^4x j^0(x) \partial_0 \vartheta(x^0) = \int d^4x j^0(x) \delta(x^0) = \int d^3x j^0(x) \quad (121)$$

and hence Eqs. (119) and (120) are equivalent. Since one of them is a tensor equation, Q is Lorentz invariant.

To show this more explicitly, we ask how a different observer O' in the frame x' calculates the charge. He finds

$$Q' = \int d^4x j'^\mu(x') \frac{\partial}{\partial x'^\mu} \vartheta(\mathbf{n} \cdot \mathbf{x}') = \int d^4x j^\mu(x) \partial_\mu \vartheta(\mathbf{n} \cdot \mathbf{x}). \quad (122)$$

Since

$$n_\alpha x'^\alpha = n_\alpha \frac{\partial x'^\alpha}{\partial x^\beta} x^\beta = n'_\alpha x^\alpha \quad (123)$$

it follows

$$Q - Q' = \int d^4x j^\mu(x) \partial_\mu [\vartheta(\mathbf{n} \cdot \mathbf{x}) - \vartheta(\mathbf{n}' \cdot \mathbf{x})]. \quad (124)$$

The current is conserved, $\partial_\mu j^\mu = 0$, and we can thus move ∂_μ to the left,

$$Q - Q' = \int d^4x \partial_\mu [j^\mu(x) (\vartheta(\mathbf{n} \cdot \mathbf{x}) - \vartheta(\mathbf{n}' \cdot \mathbf{x}))]. \quad (125)$$

Now we can integrate by parts and use that all surface terms vanish, because for large $|\vec{x}|$ the current vanishes, while for large $|t|$ the theta functions are identical and cancel. Thus a globally, Lorentz invariant conserved charge Q exists.

In the same way, we can construct globally conserved quantities Q for conserved tensors: If for instance $\partial_\mu T^{\mu\nu} = 0$, then

$$P^\nu = \int d^3x T^{0\nu} \quad (126)$$

is a globally conserved vector, and similarly for higher-rank tensors.

Canonical energy-momentum stress tensor

In forming the gradient $\partial_\mu \mathcal{L}$ of the Lagrange density, one has to remember that $\mathcal{L}$ depends via the fields and their gradients on x ,

$$\frac{\partial \mathcal{L}}{\partial x^\mu} = \frac{\partial \mathcal{L}}{\partial \phi^a} \frac{\partial \phi^a}{\partial x^\mu} + \frac{\partial \mathcal{L}}{\partial (\partial^\nu \phi^a)} \frac{\partial (\partial^\nu \phi^a)}{\partial x^\mu} = \frac{\partial \mathcal{L}}{\partial \phi^a} \frac{\phi^a}{\partial x^\mu} + \frac{\partial \mathcal{L}}{\partial \phi^a_{,\nu}} \frac{\partial \phi^a_{,\nu}}{\partial x^\mu}. \quad (127)$$

In the second step we introduced (partially) the short-hand notation $\partial_\nu \phi^a = \phi^a_{,\nu}$. Next we replace $\partial \mathcal{L} / \partial \phi^a$ with the help of the Lagrange equation, obtaining

$$\frac{\partial \mathcal{L}}{\partial x^\mu} = \frac{1}{\partial x^\nu} \left(\frac{\partial \mathcal{L}}{\partial \phi^a_{,\nu}} \right) \phi_{,\mu}^a + \frac{\partial \mathcal{L}}{\partial \phi^a_{,\nu}} \frac{\partial \phi^a_{,\nu}}{\partial x^\mu}. \quad (128)$$

Next we use the symmetry of second derivatives, $\phi^a_{,\nu,\mu} = \phi^a_{,\mu,\nu}$, and the product rule,

$$\frac{\partial \mathcal{L}}{\partial x^\mu} = \frac{1}{\partial x^\nu} \left(\phi_{,\mu}^a \frac{\partial \mathcal{L}}{\partial \phi^a_{,\nu}} \right). \quad (129)$$

We use now that

$$\frac{\partial \mathcal{L}}{\partial x^\mu} = \delta_\mu^\nu \frac{\partial \mathcal{L}}{\partial x^\nu} \quad (130)$$

and bring both terms on one side. Thus

$$0 = \partial_\nu T_\mu^\nu = \partial_\mu \left(\phi_{,\mu}^a \frac{\partial \mathcal{L}}{\partial \phi^a_{,\nu}} - \delta_\mu^\nu \mathcal{L} \right). \quad (131)$$

We will discuss the meaning of this conserved tensor soon.

Angular momentum If the tensor $T^{\mu\nu}$ is symmetric, we can construct six more conserved quantities. Define

$$M^{\mu\nu\lambda} = x^\nu T^{\mu\lambda} - x^\lambda T^{\mu\nu}, \quad (132)$$

then M is conserved with respect to the index μ ,

$$\partial_\mu M^{\mu\nu\lambda} = \delta_\mu^\nu T^{\mu\lambda} - \delta_\mu^\lambda T^{\mu\nu} = T^{\nu\lambda} - T^{\lambda\nu} = 0, \quad (133)$$

provided that $T^{\nu\lambda} = T^{\lambda\nu}$. In this case,

$$M^{\nu\mu} = \int d^3x M^{0\nu\mu} = x^\nu T^{0\mu} - x^\mu T^{0\nu}, \quad (134)$$

is a globally conserved tensor.

The conserved *energy-momentum stress tensor* that is defined by Eq. (131) is called *canonical*. Another possible derivation requires that $\mathcal{L}$ is translation invariant and gives the same result. In general, both methods do not guranty that $T^{\mu\nu}$ is symmetric. Since the Lagrange density is only determined up to a four-divergence, we can symmetrize always $T^{\mu\nu}$. We will learn later a third method that produces automatically a symmetric $T^{\mu\nu}$.

Klein-Gordon field

Real field The Klein-Gordon equation as a relativistic wave equation was derived by analogy with the (free) Schrödinger equation,

$$i\partial_t\psi = H_0\psi = \frac{p^2}{2m}, \quad (135)$$

and its replacements

$$E \rightarrow i\partial_t \quad \vec{p} \rightarrow -i\vec{\nabla}_x. \quad (136)$$

With the same replacements, the relativistic $E^2 = m^2 + p^2$ becomes

$$(\square + m^2)\phi = 0 \quad \text{with} \quad \square = \eta_{\mu\nu}\partial^\mu\partial^\nu. \quad (137)$$

How do we guess the correct $\mathcal{L}$? The correspondance $\dot{q} \leftrightarrow \partial_\mu\phi$ means that the “kinetic” field energy is quadratic in the field derivatives, while mass terms (missing in a nonrelativistic theory) are quadratic in the fields. Thus we try

$$\mathcal{L} = \frac{1}{2}\eta_{\mu\nu}\partial^\mu\phi\partial^\nu\phi - \frac{1}{2}m^2\phi^2 \quad (138)$$

With

$$\frac{\partial}{\partial\phi_{,\alpha}} (\eta^{\mu\nu}\phi_{,\mu}\phi_{,\nu}) = \eta^{\mu\nu} (\delta_\mu^\alpha\phi_{,\nu} + \delta_\nu^\alpha\phi_{,\mu}) = \eta^{\alpha\nu}\phi_{,\nu} + \eta^{\mu\alpha}\phi_{,\mu} = 2\phi^{,\alpha} \quad (139)$$

the Lagrange equation becomes

$$\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\alpha \left(\frac{\partial\mathcal{L}}{\partial\phi_{,\alpha}} \right) = -m^2\phi - \partial^\alpha\partial_\alpha\phi = 0. \quad (140)$$

Thus the Lagrange density (138) leads to the Klein-Gordon equation. Next we calculate the energy-momentum tensor,

$$T^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial\phi_{,\mu}} \phi^{,\nu} - \eta^{\mu\nu}\mathcal{L} = \phi^{,\mu}\phi^{,\nu} - \eta^{\mu\nu}\mathcal{L}. \quad (141)$$

that is already symmetric. The corresponding 00 component is

$$T^{00} = \phi_{,0}\phi_{,0} - \mathcal{L} = \frac{1}{2} [(\partial_t\phi)^2 + (\nabla\phi)^2 + m^2\phi^2] > 0. \quad (142)$$

Complex field If two field exist with the same mass m , one might wish to combine them in a complex field,

$$\phi = \frac{1}{\sqrt{2}} (\phi_1 + i\phi_2). \quad (143)$$

The resulting Lagrangian density is just the sum,

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 = \eta_{\mu\nu}\partial^\mu\phi^*\partial^\nu\phi - m^2|\phi|^2 \quad (144)$$

and

$$T^{00} = 2\phi_{,0}\phi_{,0}^* - \mathcal{L} = |\partial_t\phi|^2 + |\nabla\phi|^2 + m^2|\phi|^2 > 0. \quad (145)$$

We consider now plane-wave solutions to the Klein-Gordon equation,

$$\phi = N \exp(i\mathbf{k}\mathbf{x}) \quad (146)$$

If we insert $\partial_\mu\phi = ik_\mu\phi$ into $\mathcal{L}$, we find $\mathcal{L} = 0$ and thus

$$T^{00} = 2|N|^2 k^0 k^0. \quad (147)$$

The other components are necessarily

$$T^{\mu\nu} = 2|N|^2 k^\mu k^\nu. \quad (148)$$

Since $T^{\mu\nu}$ is symmetric, we can find a frame in which $T^{\mu\nu}$ is diagonal with $T \propto \text{diag}(E^2, p_x^2, p_y^2, p_z^2)$.

Maxwell field

Should we try to form the Lagrangian density out of the potentials $(\phi, \vec{A})$ or the field-strengths $(\vec{E}, \vec{B})$ for a (massive) spin-1 particle? In the first case, e.g.

$$\mathcal{L} = \partial^\mu A_\nu \partial_\mu A^\nu + m^2 A_\mu A^\mu. \quad (149)$$

This Lagrange density has four degrees of freedom, while a massive/massless spin-1 field has only 3/2 dynamical degrees of freedom. Moreover, either the A^0 or the spatial components $\vec{A}$ will give a negativ contribution to T^{00} .

Using the field strengths requires first to form out of $\vec{E}$ and $\vec{B}$ a Lorentz invariant tensor. We derive the connection of this tensor to A by postulating the coupling $j^\mu A_\mu$ between photons and matter.

Field tensor We consider a charged particle interacting with an external electromagnetic field A . As Lagrangian for the free particle we use $L = -mds$ or

$$S_0 = - \int_a^b ds m \quad (150)$$

Next we want to add an interaction term with an electromagnetic field. The simplest action is

$$S_{\text{em}} = -q \int dx^\mu A_\mu(x) = -q \int d\sigma \frac{dx^\mu}{d\sigma} A_\mu(x). \quad (151)$$

Any candidate for S_{em} should be invariant under a change of gauge,

$$A_\mu(x) \rightarrow A_\mu(x) + \partial_\mu \Lambda(x). \quad (152)$$

This the case, since the induced change in the action,

$$\delta_\Lambda S_{\text{em}} = -q \int_1^2 d\sigma \frac{dx^\mu}{d\sigma} \frac{\partial \Lambda(x)}{\partial x^\mu} = -q \int_1^2 d\Lambda = q[\Lambda(2) - \Lambda(1)] \quad (153)$$

drops out from δS for fixed endpoints.

With $ds = \sqrt{dx^\mu dx_\mu}$, the variation of the action is

$$\delta S = -\delta \int_a^b (m ds + e A_\mu dx^\mu) = -\int_a^b \left(m \frac{dx^\mu \delta dx_\mu}{ds} + e A_\mu d(\delta x^\mu) + e \delta A_\mu dx^\mu \right). \quad (154)$$

We integrate the first two terms partially,

$$\delta S = \int_a^b \left(m d \left(\frac{dx_\mu}{ds} \right) \delta x^\mu + e \delta x^\mu dA_\mu - e \delta A_\mu dx^\mu \right) \quad (155)$$

and use as “always” that the boundary terms vanish. Next we introduce $u_\mu = dx_\mu/ds$ and use

$$\delta A_\mu = \frac{\partial A_\mu}{\partial x^\nu} \delta x^\nu, \quad dA_\mu = \frac{\partial A_\mu}{\partial x^\nu} dx^\nu. \quad (156)$$

Then

$$\delta S = \int_a^b \left(m du_\mu \delta x^\mu + e \frac{\partial A_\mu}{\partial x^\nu} \delta x^\mu dx^\nu - e \frac{\partial A_\mu}{\partial x^\nu} \delta x^\nu dx^\mu \right). \quad (157)$$

Finally, we rewrite in the first term $du_\alpha = du_\alpha/ds ds$, in the second and third $dx^\alpha = u^\alpha ds$ and exchange the summation indices μ and ν in the third term. Then

$$\delta S = \int_a^b \left[m \frac{du_\mu}{ds} - e \left(\frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu} \right) u^\nu \right] \delta x^\mu ds = 0. \quad (158)$$

For arbitrary variations, the brackets has to be zero and we obtain as equation of motion

$$m \frac{du_\mu}{ds} = f_\mu = e \left(\frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu} \right) u^\nu \equiv e F_{\mu\nu} u^\nu. \quad (159)$$

Comparing this equation with its 3-dimensional correspondance, the equation of motion with the Lorentz force

$$m \vec{a} = \vec{F}_L = q(\vec{E} + \vec{v} \times \vec{B}) \quad (160)$$

or using the definition $\mathbf{F} = d\mathbf{A}$, we can read-off the compenents of $\mathbf{F}$ (the first index denotes rows, the second columns) as

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \quad (161)$$

and

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}. \quad (162)$$

Connection between 3- and 4-dim. formulation of electrodynamics

The first row of $F_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu}$ reads with $A_\mu = (\phi, -A_k)$ and $\partial_\nu = \partial/\partial x^\mu = (\partial/\partial t, \nabla_k)$ as

$$F_{0k} = E_k = \partial_0 A_k - \partial_k A_0 = -\frac{\partial A_k}{\partial t} - \nabla_k \phi$$

or $\vec{E} = -\partial_t \vec{A} - \nabla \phi$. We go in the opposite direction for $\vec{B} = \nabla \times \vec{A}$. In components, we have e.g.

$$B^1 = \partial_2 A^3 - \partial_3 A^2 = \partial_3 A_2 - \partial_2 A_3 = F_{32}$$

and similarly for the other components.

The force law $f_\mu = e F_{\mu\nu} u^\nu$ becomes simplest in a frame with $u = (1, 0, 0, 0)$. Then $f_\mu = e F_{\mu 0}$ or $-\vec{F} = -e \vec{E}$.

Now we can rewrite the Maxwell equations as

$$\partial_\alpha F^{\alpha\beta} = j^\beta \quad (163)$$

and

$$\partial_\alpha F^{\beta\gamma} + \partial_\beta F^{\gamma\alpha} + \partial_\gamma F^{\alpha\beta} = 0. \quad (164)$$

The last equation is completely antisymmetric in all three indices, and contains therefore only four independent equations. It is equivalent to

$$\partial_\alpha \tilde{F}^{\alpha\beta} = 0, \quad (165)$$

where

$$\tilde{F}^{\alpha\beta} = \frac{1}{2} \varepsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta} \quad (166)$$

and $\varepsilon^{\alpha\beta\gamma\delta}$ is the complete antisymmetric tensor in four dimensions,

$$\varepsilon^{0123} = +1, \quad (167)$$

and all even permutations, -1 for odd permutations and zero otherwise.

Current conservation and gauge invariance We take the divergence of Maxwells equation (163),

$$\partial_\nu \partial_\mu F^{\mu\nu} = \partial_\nu j^\nu. \quad (168)$$

Since $\partial_\nu \partial_\mu$ is symmetric and $F^{\mu\nu}$ antisymmetric, the summation of the two factors has to be zero,

$$\partial_\nu \partial_\mu F^{\mu\nu} = -\partial_\nu \partial_\mu F^{\nu\mu} = -\partial_\mu \partial_\nu F^{\nu\mu} = -\partial_\nu \partial_\mu F^{\mu\nu}. \quad (169)$$

Thus current conservation,

$$\partial_\nu j^\nu = 0, \quad (170)$$

follows from the anti-symmetry of $\mathbf{F}$. The latter followed in turn from the assumed gauge-invariant action $S_{\text{em}} = -q \int dx^\mu A_\mu$.

Consider next the transformation of $\mathbf{F}$ under a gauge transformation,

$$A_\mu \rightarrow A'_\mu = A_\mu + \partial_\mu \chi. \quad (171)$$

$$F'_{\mu\nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu = F_{\mu\nu} + \partial_\mu \partial_\nu \chi - \partial_\nu \partial_\mu \chi = F_{\mu\nu}. \quad (172)$$

Thus the gauge invariance of $\mathbf{F}$ is again closely connected to the fact that it is an antisymmetric tensor, formed by derivatives of $\mathbf{A}$.

Differential forms:

A surface in $\mathbb{R}^3$ can be described at any point either by its two tangent vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ or by the normal $\mathbf{n}$. They are connected by a cross product, $\mathbf{n} = \mathbf{e}_1 \times \mathbf{e}_2$, or in index notation,

$$n^i = \varepsilon^{ijk} e_{1,j} e_{2,k}. \quad (173)$$

In four dimensions, the ε tensor defines a map between 1-3 and 2-2 tensors. Since ε is antisymmetric, the symmetric part of tensors would be lost; Hence the map is suited for antisymmetric tensors.

Antisymmetric tensors of rank n can be seen also as differential forms: Functions are forms of order $n = 0$; differential of functions are an example of order $n = 1$,

$$df = \frac{\partial f}{\partial x^i} dx^i \quad (174)$$

Thus the dx^i form a basis, and one can write in general

$$\mathbf{A} = A_i dx^i. \quad (175)$$

For $n > 1$, the basis has to be antisymmetrized,

$$\mathbf{F} = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu \quad (176)$$

with $dx^\mu \wedge dx^\nu = -dx^\nu \wedge dx^\mu$. Looking at df , we can define a differentiation of a form ω with coefficients w and degree n as an operation that increases its degree by one to $n + 1$,

$$d\omega = dw_{\alpha,\dots,\beta} dx^{n+1} \wedge dx^\alpha \wedge \dots \wedge dx^\beta \quad (177)$$

Thus we have $\mathbf{F} = d\mathbf{A}$. Moreover, it follows $d^2\omega = 0$ for all forms. Hence a gauge transformation $\mathbf{F}' = d(\mathbf{A} + d\chi) = \mathbf{F}$.

Wave equation The Maxwells equation (163) consists of four equations for the six components of $\mathbf{F}$. Thus we need either a second equation, i.e. Eq. (164), or we should

transform Eq. (163) into an equation for the four components of the four-potential $\mathbf{A}$. Let's do the latter and insert the definition of $\mathbf{A}$,

$$\partial_\mu F^{\mu\nu} = \partial_\mu (\partial^\mu A^\nu - \partial^\nu A^\mu) = \square A^\nu - \partial_\mu \partial^\nu A^\mu = j^\nu. \quad (178)$$

If we choose the Lorentz gauge, $\partial_\mu A^\mu = 0$, we obtain the familiar wave equation for photons,

$$\square A^\nu = j^\nu. \quad (179)$$

Lagrange density The free field equation is

$$\partial_\nu F^{\mu\nu} = 0. \quad (180)$$

In order to find $\mathcal{L}$, we multiply by a variation δA that vanishes at the endpoints t_a and t_b . Then we integrate over $\Omega = V \times [t_a : t_b]$, and perform a partial integration,

$$\int_\Omega d^4x \partial_\nu F^{\mu\nu} \delta A_\mu = - \int_\Omega d^4x F^{\mu\nu} \delta(\partial_\nu A_\mu) = 0. \quad (181)$$

Next we note that

$$(A_{\alpha,\beta} - A_{\beta,\alpha})(A^{\alpha,\beta} - A^{\beta,\alpha}) = 2(A_{\alpha,\beta} - A_{\beta,\alpha})A^{\alpha,\beta} \quad (182)$$

and thus

$$F^{\mu\nu} \delta(\partial_\mu A_\nu) = \frac{1}{2} F^{\mu\nu} \delta F_{\mu\nu}. \quad (183)$$

Applying the product rule, we obtain as final result

$$-\frac{1}{4} \delta \int_\Omega d^4x F_{\mu\nu} F^{\mu\nu} = 0 \quad (184)$$

and

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}. \quad (185)$$

Note that we expressed $\mathcal{L}$ through F , but $\mathcal{L}$ should be viewed nevertheless as function of A .

Energy-momentum tensor I According to Eq. (131) we have

$$T_\mu^\nu = \frac{\partial A_\sigma}{\partial x^\mu} \frac{\partial \mathcal{L}}{\partial(\partial A_\sigma / \partial x^\nu)} - \delta_\mu^\nu \mathcal{L}. \quad (186)$$

Since $\mathcal{L}$ depends only on the derivatives $A_{,\nu}^\mu$, we can use the following short-cut: We know already that

$$\delta \mathcal{L} = -\frac{1}{4} \delta(F_{\mu\nu} F^{\mu\nu}) = F^{\mu\nu} \delta(\partial_\nu A_\mu). \quad (187)$$

Thus

$$\frac{\partial \mathcal{L}}{\partial(\partial A_\sigma / \partial x^\nu)} = F^{\sigma\nu} = -F^{\nu\sigma} \quad (188)$$

and

$$T_{\mu}^{\nu} = -\frac{\partial A_{\sigma}}{\partial x^{\mu}} F^{\nu\sigma} + \frac{1}{4} \delta_{\mu}^{\nu} F_{\sigma\tau} F^{\sigma\tau}. \quad (189)$$

Raising the index μ and rearranging σ , we have

$$T^{\mu\nu} = -\frac{\partial A^{\sigma}}{\partial x_{\mu}} F^{\nu}_{\sigma} + \frac{1}{4} \eta^{\mu\nu} F_{\sigma\tau} F^{\sigma\tau}. \quad (190)$$

This result is neither gauge invariant (contains A) nor symmetric. To symmetrize it, we should add

$$\frac{\partial A^{\mu}}{\partial x_{\sigma}} F^{\nu}_{\sigma} = \frac{\partial}{\partial x_{\sigma}} (A^{\mu} F^{\nu}_{\sigma}). \quad (191)$$

The last step is possible for a free electromagnetic field, $\partial_{\sigma} F^{\nu\sigma} = 0$, and shows that we are allowed to add the LHS. Then the two terms combine to F , and we get

$$T^{\mu\nu} = -F^{\mu\sigma} F^{\nu}_{\sigma} + \frac{1}{4} \eta^{\mu\nu} F_{\sigma\tau} F^{\sigma\tau}. \quad (192)$$

It is gauge invariant and symmetric. Its trace is zero, $T^{\mu}_{\mu} = 0$.

Energy-momentum tensor II We use now an alternative derivation of $\mathbf{T}$, using Maxwell's equations and Lorentz's law, Eq. (159). Rewriting the latter first as an equation between force density and current density, and inserting then Maxwell's law gives

$$f_{\mu} = F_{\mu\nu} j^{\nu} = F_{\mu\nu} \frac{\partial F^{\nu\lambda}}{\partial x^{\lambda}}. \quad (193)$$

We use now the product rule to rewrite this as

$$f_{\mu} = \frac{\partial}{\partial x^{\lambda}} (F_{\mu\nu} F^{\nu\lambda}) - F^{\nu\lambda} \frac{\partial F_{\mu\nu}}{\partial x^{\lambda}}. \quad (194)$$

We now try to rewrite the second term as a symmetric divergence. Starting from

$$F^{\nu\lambda} \frac{\partial F_{\mu\nu}}{\partial x^{\lambda}} = \frac{1}{2} \left(F_{\nu\lambda} \frac{\partial F_{\mu\nu}}{\partial x^{\lambda}} + F_{\lambda\nu} \frac{\partial F_{\mu\lambda}}{\partial x^{\nu}} \right) \quad (195)$$

we have exchanged the indices λ and ν in the second term. Then we use first in both factors of the second term the antisymmetry of F ,

$$= \frac{1}{2} F_{\nu\lambda} \left(\frac{\partial F_{\mu\nu}}{\partial x^{\lambda}} + \frac{\partial F_{\lambda\mu}}{\partial x^{\nu}} \right) \quad (196)$$

then Maxwell (164),

$$= -\frac{1}{2} F_{\nu\lambda} \frac{\partial F_{\nu\lambda}}{\partial x^{\mu}} \quad (197)$$

$$= -\frac{1}{4} \frac{\partial}{\partial x^{\mu}} (F_{\nu\lambda} F^{\nu\lambda}) = -\frac{1}{4} \delta_{\mu}^{\lambda} \frac{\partial}{\partial x^{\lambda}} (F_{\sigma\tau} F^{\sigma\tau}) \quad (198)$$

Combining, we get

$$f_\mu = \frac{\partial}{\partial x^\lambda} \left(F_{\mu\nu} F^{\nu\lambda} + \frac{1}{4} \delta_\mu^\lambda F_{\sigma\tau} F^{\sigma\tau} \right) = \frac{\partial T^{\lambda\sigma}}{\partial x^\lambda} . \quad (199)$$

Considering the electromagnetic field as an external (i.e. fixed) field, the divergence of its energy-momentum tensor corresponds to the four-force density on charges.