

Week 7: Schwarzschild metric

Isometries and Killing vectors

An isometry is a distance conserving mapping between different spaces. In the case of a Riemannian manifold, the transformed metric $\bar{g}^{ij}(\bar{x})$ has to be the same function of its argument $\bar{x}$ as the original metric $g^{ij}(x)$ of its argument x ,

$$\bar{g}^{ij}(x) = g^{ij}(x). \quad (1)$$

Note the difference to the definition of a scalar, $\bar{\phi}(\bar{x}) = \phi(x)$. The metric transforms under an arbitrary transformation as

$$\bar{g}^{ij}(\bar{x}) = \frac{\partial \bar{x}^i}{\partial x^k} \frac{\partial \bar{x}^j}{\partial x^l} g^{kl}(x) \quad (2)$$

or

$$g^{ij}(x) = \frac{\partial x^i}{\partial \bar{x}^k} \frac{\partial x^j}{\partial \bar{x}^l} \bar{g}^{kl}(\bar{x}). \quad (3)$$

For an isometry,

$$\bar{g}^{ij}(\bar{x}) = \frac{\partial x^i}{\partial \bar{x}^k} \frac{\partial x^j}{\partial \bar{x}^l} \bar{g}^{kl}(\bar{x}). \quad (4)$$

We consider now the effect of an infinitesimal coordinate transformation,

$$\bar{x}^i = x^i + \varepsilon \xi(x^k), \quad \varepsilon \ll 1 \quad (5)$$

on the metric,

$$\begin{aligned} \bar{g}^{ij}(\bar{x}) &= \frac{\partial \bar{x}^i}{\partial x^k} \frac{\partial \bar{x}^j}{\partial x^l} g^{kl}(x) \\ &= (\delta_k^i + \varepsilon \xi_{,k}^i)(\delta_l^j + \varepsilon \xi_{,l}^j) g^{kl}(x) \\ &= g^{ij}(x) + \varepsilon (\xi^{i,j} + \xi^{j,i}) + \mathcal{O}(\varepsilon^2). \end{aligned} \quad (6)$$

In order to check the condition (4), we have to know $\bar{g}$ as function of x . A Taylor expansion gives

$$\bar{g}^{ij}(\bar{x}) = \bar{g}^{ij}(x + \varepsilon \xi) = \bar{g}^{ij}(x) + \varepsilon \xi^k \partial_k \bar{g}^{ij}(x) + \mathcal{O}(\varepsilon^2). \quad (7)$$

The metric is kept invariant, if

$$\xi^{i,j} + \xi^{j,i} - \xi^k \partial_k g^{ij} = 0 \quad (8)$$

or

$$g^{jk} \partial_k \xi^i + g^{ik} \partial_k \xi^j - \xi^k \partial_k g^{ij} = 0. \quad (9)$$

Expressing partial derivatives as covariant ones, the terms containing connection coefficients cancel and we obtain the Killing equations

$$\boxed{\nabla_i \xi_j + \nabla_j \xi_i = 0.} \quad (10)$$

The vectors ξ are called Killing vectors of the metric.

Example: Killing vectors of $\mathbb{R}^3$:

Choosing Cartesian coordinates, $dl^2 = dx^2 + dy^2 + dz^2$, makes it obvious that translations correspond to Killing vectors $\xi_1 = (1, 0, 0)$, $\xi_2 = (0, 1, 0)$, and $\xi_3 = (0, 0, 1)$. We find the Killing vectors describing rotational symmetry by writing for an infinitesimal rotation around, e.g., the z axis,

$$\begin{aligned} x' &= \cos \alpha x - \sin \alpha y \approx x - \alpha y, \\ y' &= \sin \alpha x + \cos \alpha y \approx y + \alpha x, \\ z' &= z. \end{aligned}$$

Hence $\xi_z = (-y, x, 0)$ and the other two follow by cyclic permutation. One of them, ξ_z , we could have also identified by rewriting the line-element in spherical coordinates and noting that dl does not contain ϕ dependent terms.

Conserved quantities along geodesics Assume that the metric is independent from one coordinate, e.g. x^1 . Then there exists a corresponding Killing vector, $\xi = (1, 0, 0, 0)$, and x^1 is a cyclic coordinate, $\partial L / \partial x^1 = 0$. With $L = d\tau/d\sigma$, the resulting conserved quantity $\partial L / \partial \dot{x}^1 = \text{const.}$ can be written as

$$\frac{\partial L}{\partial \dot{x}^1} = -g_{1\beta} \frac{dx^\beta}{L d\sigma} = -g_{1\beta} \frac{dx^\beta}{d\tau} = -\xi \cdot \mathbf{u}. \quad (11)$$

Hence the quantity $\xi \cdot \mathbf{u}$ is conserved along the solutions $x^a(\sigma)$ of the Lagrange equation, i.e. along geodesics.

Schwarzschild Metric

The metric outside of a radial-symmetric mass distribution is

$$\boxed{ds^2 = \frac{dr^2}{1 - \frac{2M}{r}} + r^2(d\vartheta^2 + \sin^2 \vartheta d\phi^2) - dt^2 \left(1 - \frac{2M}{r}\right)}. \quad (12)$$

Its main properties are

- symmetries: The metric is time-independent and spherically symmetric. Hence two Killing vectors are $\xi = (1, 0, 0, 0)$ and $\eta = (0, 0, 0, 1)$, where we order coordinates as $\{t, r, \phi, \vartheta\}$.
- asymptotically flat.
- potential singularities at $r = 2M$ and $r = 0$. The radius $2M$ is called Schwarzschild radius and has the value

$$R_S = \frac{2GM}{c^2} = 3 \text{ km} \frac{M}{M_\odot} \quad (13)$$

Gravitational redshift

We follow Sec. 9.2.

Radial equation and effective potential for massive particles

Spherically symmetry means that the movement if a test particle is contained in a plane. We choose $\vartheta = \pi/2$ and $u_\vartheta = 0$. We replace in the normalization condition $\mathbf{u} \cdot \mathbf{u} = -1$ written out for the Schwarzschild metric,

$$-1 = -\left(1 - \frac{2M}{r}\right) \left(\frac{dt}{d\tau}\right)^2 + \left(1 - \frac{2M}{r}\right)^{-1} \left(\frac{dr}{d\tau}\right)^2 + r^2 \left(\frac{d\phi}{d\tau}\right)^2 \quad (14)$$

the velocities u_t and u_r by the conserved quantities

$$e \equiv -\xi \cdot \mathbf{u} = \left(1 - \frac{2M}{r}\right) \frac{dt}{d\tau} \quad (15)$$

$$l \equiv \eta \cdot \mathbf{u} = r^2 \sin^2 \vartheta \frac{d\phi}{d\tau}. \quad (16)$$

Reordering gives

$$\mathcal{E} \equiv \frac{e^2 - 1}{2} = \frac{1}{2} \left(\frac{dr}{d\tau}\right)^2 + V_{\text{eff}} \quad (17)$$

with

$$V_{\text{eff}} = -\frac{M}{r} + \frac{l^2}{2r^2} - \frac{Ml^2}{r^3} = V_0 - \frac{Ml^2}{r^3}. \quad (18)$$

Hence the energy of a test particle in the Schwarzschild metric can be, as in the Newtonian case, divided into kinetic energy and potential energy. The latter contains the additional term Ml^2/r^3 , suppressed by $1/c^2$, that becomes important at small r .

The asymptotic behavior of V_{eff} for $r \rightarrow 0$ and $r \rightarrow \infty$ is

$$V_{\text{eff}}(r \rightarrow \infty) \rightarrow -\frac{M}{r} \quad \text{and} \quad V_{\text{eff}}(r \rightarrow 0) \rightarrow -\frac{Ml^2}{r^3}, \quad (19)$$

while $V(2M) = 1/2$ independent of M .

We determine the extrema of V_{eff} by solving $dV_{\text{eff}}/dr = 0$ and find

$$r_{1,2} = \frac{l^2}{2M} \left[1 \pm \sqrt{1 - 12M^2/l^2}\right] \quad (20)$$

Hence the potential has no extrema for $l/M > \sqrt{12}$ and is always negative: A particle can reach $r = 0$ for small enough but finite angular momentum, in contrast to the Newtonian case. By the same argument, there exists a last stable orbit at $r = 6M$, when the two extrema r_1 and r_2 coincide for $l/M = \sqrt{12}$.

For a discussion of the four different types of orbits see the book.

Radial infall

We consider the free fall of a particle that is at rest at infinity, $dt/d\tau = 1$, $\mathcal{E} = 0$ and $l = 0$. The radial equation (17) simplifies to

$$\frac{1}{2} \left(\frac{dr}{d\tau} \right)^2 = \frac{M}{r} \quad (21)$$

and can be integrated by separation of variables,

$$\int_r^0 dr r^{1/2} = \sqrt{2M} \int_{\tau_*}^{\tau} d\tau \quad (22)$$

with the result

$$\frac{2}{3} r^{3/2} = \sqrt{2M} (\tau_* - \tau). \quad (23)$$

Hence a freely falling particle needs only a finite proper time to fall from finite r to $r = 0$. In particular, it passes the Schwarzschild radius $2M$ in finite proper time.

We can answer the same question using the coordinate time t by combining Eqs. (15) and (21),

$$\frac{dt}{dr} = - \left(\frac{2M}{r} \right)^{-1/2} \left(1 - \frac{2M}{r} \right)^{-1}. \quad (24)$$

Integrating gives

$$\begin{aligned} t &= \int dr \left(\frac{2M}{r} \right)^{-1/2} \left(1 - \frac{2M}{r} \right)^{-1} \\ &= t' + 2M \left\{ -\frac{2}{3} \left(\frac{r}{2M} \right)^{3/2} - 2 \left(\frac{r}{2M} \right)^{1/2} + \ln \left| \frac{\sqrt{r/2M} + 1}{\sqrt{r/2M} - 1} \right| \right\} \\ &\rightarrow \infty \quad \text{for } r \rightarrow 2M. \end{aligned} \quad (25)$$

Since the coordinate time t is the proper time for an observer at infinity, a freely falling particle reaches the Schwarzschild radius $r = 2M$ only for $t \rightarrow \infty$ for such an observer.

The last result can be derived immediately for light-rays. Choosing a light-ray in radial direction with $d\phi = d\vartheta = 0$, the metric (12) simplifies with $ds^2 = 0$ to

$$\frac{dr}{dt} = 1 - \frac{2M}{r}. \quad (26)$$

Thus light travelling towards the star, as seen from the outside, will travel slower and slower as it comes closer to the Schwarzschild radius $r = 2\alpha$. The coordinate time is $\propto \ln |1 - 2M/r|$ and thus for an observer at infinity the signal will reach $r = 2M$ again only asymptotically for $t \rightarrow \infty$.

Perihelion precession

We recall first the derivation of the $r = r(\phi)$ in the Newtonian case. We solve the Lagrange equations for $L = (1/2)m(\dot{r}^2 + r^2\dot{\phi}^2) - GMm/r$, obtaining

$$r^2\dot{\phi} = l, \quad (27)$$

$$\ddot{r} = \frac{l}{r^2} - \frac{GM}{r}. \quad (28)$$

We eliminate t by

$$\frac{dr}{dt} = \frac{dr}{d\phi} \frac{d\phi}{dt} = \frac{dr}{d\phi} \frac{l}{r^2} \equiv r' \frac{l}{r^2} \quad (29)$$

and introduce $u = 1/r$,

$$u'' + u = \frac{GM}{l^2}. \quad (30)$$

The solution follows as

$$u = \frac{GM}{l^2}(1 + e \cos \phi) \quad (31)$$

We redo the same steps, starting from Eq. (17) for the Schwarzschild metric,

$$\dot{r}^2 + \frac{l^2}{r^2} = e^2 - 1 + \frac{2M}{r} - \frac{2Ml^2}{r^3}. \quad (32)$$

We eliminate first t and introduce $u = 1/r$,

$$(u')^2 + u^2 = \frac{e^2 - 1}{l^2} + \frac{2Mu}{l^2} + 2Mu^3. \quad (33)$$

Differentiating with respect to ϕ , it follows

$$u'' + u = \frac{M}{l^2} + 3Mu^2 = \frac{GM}{l^2} + \frac{3GM}{c^2}u^2. \quad (34)$$

In the last step we reintroduced c and G . Hence we see that the Newtonian limit corresponds to $c \rightarrow \infty$ (“instantaneous interactions”) or $v/c \rightarrow 0$ (“static limit”). The latter statement becomes clear, if one uses the virial theorem: $GMu = GM/r \sim v^2$.

In most situations, the relativistic correction is tiny. We use therefore perturbation theory to determine an approximate solution of (34), $u = u_0 + \delta u$, where u_0 is the Newtonian solution. Inserting u into (34), we obtain neglecting suppressed terms

$$(\delta u)'' + \delta u = \frac{3(GM)^3}{c^2 l^4}(1 + 2e \cos \phi + e^2 \cos^2 \phi) \quad (35)$$

Its solution is

$$\delta u = \frac{3(GM)^3}{c^2 l^4} \left[1 + e\phi \sin \phi + e^2 \left(\frac{1}{2} - \frac{1}{6} \cos(2\phi) \right) \right]. \quad (36)$$

While the first and third term lead only to extremely tiny changes in the orbital parameters, the second term is linear in ϕ and its effects accumulate therefore with time. Thus we include only $\delta u = Ae\phi \sin \phi$ in the approximate solution,

$$u = \frac{GM}{l^2}(1 + e(\cos \phi + \alpha \sin \phi)) = \frac{GM}{l^2}(1 + e \cos(\phi(1 - \alpha))) \quad (37)$$

Hence the period is $2\pi/(1 - \alpha)$, and the ellipse processes with

$$\Delta\phi = \frac{6\pi GM}{a(1 - e^2)c^2}. \quad (38)$$

Orbits of photons

We repeat the discussion of geodesics for massive particle for massless ones by changing $\mathbf{u} \cdot \mathbf{u} = -1$ into $\mathbf{u} \cdot \mathbf{u} = 0$ and by using an affine parameter λ instead of the proper-time. Reordering gives

$$\frac{1}{b^2} \equiv \frac{e^2}{l^2} = \frac{1}{l^2} \left(\frac{dr}{d\lambda} \right)^2 + W_{\text{eff}} \quad (39)$$

with the impact parameter $b = |l/e|$ and

$$W_{\text{eff}} = \frac{1}{r} \left(1 - \frac{2M}{r} \right). \quad (40)$$

The maximum of W_{eff} is at $3M$ with height $1/27M^2$. For impact parameters $b > 27M$, photon orbits have a turning point and photons escape to infinity. For $b < 27M$, they hit $r = 0$, while for $b = 27M$ a (unstable) circular orbit is possible.

Light deflection

We transform Eq. (39) as in the $m > 0$ case into a differential equation for $u(\phi)$. For small deflections, we use again perturbation theory. In zeroth order in v/c , we can set the RHS of

$$u'' + u = \frac{3GM}{c^2} u^2 \quad (41)$$

to zero. The solution u_0 is a straight line,

$$u_0 = \frac{\sin \phi}{b}. \quad (42)$$

Inserting $u = u_0 + \delta u$ gives

$$(\delta u)'' + \delta u = \frac{3GM}{c^2} \frac{\sin^2 \phi}{b^2}. \quad (43)$$

A particular solution is

$$\delta u = \frac{3GM}{2c^2 b^2} (1 + 1/3 \cos(2\phi)). \quad (44)$$

Thus the complete approximate solution is

$$u = u_0 + \delta u = \frac{\sin \phi}{b} + \frac{3GM}{2c^2 b^2} (1 + 1/3 \cos(2\phi)). \quad (45)$$

Considering the limit $r \rightarrow \infty$ or $u \rightarrow 0$ of this equation gives half of the deflection angle of a light-ray with impact parameter b to a point mass M ,

$$\Delta\phi = \frac{4GM}{c^2 b}. \quad (46)$$

For a numerical estimate, we use a light-ray grazing the solar surface, $b = R_\odot$,

$$\Delta\phi_\odot = \frac{4GM_\odot}{c^2 R_\odot} = \frac{2R_s}{R_\odot} \approx 10^{-5} \approx 2''. \quad (47)$$