

Week 8: Gravitational lensing & black holes

Gravitational lensing

One distinguishes three different cases of gravitational lensing, depending on the strength of the lensing effect:

1. Strong lensing occurs when the lens is very massive and the source is close to it: In this case light can take different paths to the observer and more than one image of the source will appear, either as multiple images or deformed arcs of a source. In the extreme case that a point-like source, lens and observer are aligned the image forms an “Einstein ring”.
2. Weak Lensing: In many cases the lens is not strong enough to form multiple images or arcs. However, the source can still be distorted and its image may be both stretched (shear) and magnified (convergence). If all sources were well known in size and shape, one could just use the shear and convergence to deduce the properties of the lens.
3. Microlensing: One observes only the usual point-like image of the source. However, the additional light bent towards the observer leads to brightening of the source. Thus microlensing is only observable as a transient phenomenon, when the lens crosses approximately the axis observer-source.

The lens equation in the “thin lens” ($b \ll D_i$) and weak deflection ($\alpha \ll 1$) limit is

$$\vartheta D_{\text{os}} = \alpha D_{\text{ls}} + \beta D_{\text{os}}. \quad (48)$$

Inserting $\alpha = 4GM/(c^2 b^2)$ and $b = \beta D_{\text{ol}}$ gives a quadratic equation for ϑ ,

$$\vartheta^2 - \beta \vartheta - \vartheta_{\text{E}} = 0, \quad (49)$$

with the Einstein angle

$$\vartheta_{\text{E}} = \left(2R_S \frac{D_{\text{ls}}}{D_{\text{os}} D_{\text{ol}}} \right)^{1/2}. \quad (50)$$

For a numerical estimate, we set $D_{\text{ls}}/D_{\text{os}} \approx 1/2$ and obtain

$$\vartheta_{\text{E}} = 0.64'' \times 10^{-4} \left(\frac{M}{M_{\odot}} \frac{D_{\text{ol}}}{10 \text{ kpc}} \right)^{1/2}. \quad (51)$$

The two images of the source are deflected by the angles

$$\vartheta_{\pm} = \frac{1}{2} [\beta \pm (\beta^2 + 4\vartheta_{\text{E}}^2)^{1/2}] \quad (52)$$

from the line-of-sight to the lens. Since we do not know the lens locations, measuring the separation of two lenses images, $\vartheta_+ + \vartheta_-$, provides only an upper bound on the lens mass.

The lensed images are stretched: Their magnification is

$$a_{\pm} = \frac{\vartheta_{\pm} d\vartheta_{\pm}}{\beta d\beta} \quad (53)$$

and differentiating $\vartheta_{\pm}$ gives

$$a_{\text{tot}} = a_{+} + a_{-} = \frac{x^2 + 2}{x(x^2 + 4)^{1/2}} \quad (54)$$

with $x = \beta/\vartheta_E$. For large separation x , the magnification a_{tot} goes to one, while the magnification diverges for $x \rightarrow 0$ as $a_{\text{tot}} \sim 1/x$: We receive light from an infinite number of images on the Einstein circle. Physically, the approximation of a point source breaks down when x reaches the extension of the source.

In microlensing experiments that have tried to detect dark matter in the form of MACHOs (black holes, brown dwarfs, ...) one observed stars of the LMC. If a MACHO with speed $v \approx 220$ km/s moves through the line-of-sight of a monitored star, its light-curve is magnified temporally. If v is the perpendicular velocity of the source,

$$\beta(t) = \left[\beta_0^2 + \frac{v^2}{D_{\text{ol}}^2} (t - t_0)^2 \right]^{1/2} \quad (55)$$

The magnification $a(t)$ is symmetric around t_0 and its shape can be determined inserting typical values for D_{ol} , the MACHO mass.

Schwarzschild black holes

Eddington-Finkelstein coordinates The elements g_{rr} and g_{tt} of the metric tensor change sign at $r = 2M$. Thus t is only for $r > 2M$ time-like, becoming space-like for $r < 2M$ and vice versa for r . Geodesics cover only the region $r > 2M$ within finite coordinate time t : Integrating Eq. (26), we find

$$t = +r + 2M \ln \left| \frac{r}{2M} - 1 \right| + q \quad \text{outgoing rays,} \quad (56)$$

$$t = -r - 2M \ln \left| \frac{r}{2M} - 1 \right| + p \quad \text{ingoing rays,} \quad (57)$$

for radial light-rays, where q and p are integration constants. We extend now the Schwarzschild metric by using the integration constant of an infalling light-ray as a new variable, the “advanced time parameter p ”,

$$p = t + r + 2M \ln |r/2M - 1|. \quad (58)$$

Forming the differential,

$$dp = dt + \left(1 - \frac{2M}{r} \right)^{-1} dr \quad (59)$$

we can eliminate dt from the Schwarzschild metric and find

$$ds^2 = - \left(1 - \frac{2M}{r} \right) dp^2 + 2dpdr + r^2 d\Omega \quad (60)$$

This metric is regular at $2M$ and valid for all $r > 0$. For a discussion, see the book.

Kruskal coordinates A manifold is called *maximal*, if all geodesics can be extended to infinite values of their affine parameters (in both directions) or end at a singularity. A manifold is called *geodesically complete*, if all geodesics can be extended and do not end at a singularity. A trivial example for a geodesically complete manifold is Minkowski space.

The steps to find the maximal extension of the Schwarzschild metric, the Kruskal solution, can be motivated as follows: Use first not only the advanced time parameter p but also the retarded time parameter q and eliminate both t and r as independent variables. Find new coordinates for which the metric is *conformally flat*, i.e. a metric which is Minkowskian apart from an overall factor Ω ,

$$ds^2 = g_{ab}dx^a dx^b = \Omega^2(x)\eta_{ab}dx^a dx^b. \quad (61)$$

Moreover, eliminate singular points at $r > 0$.

The first step gives

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dp dq + r^2 d\Omega, \quad (62)$$

where r is considered to be a function of p and q ,

$$\frac{1}{2}(p - q) = r + 2M \ln \left| \frac{r}{2M} - 1 \right|. \quad (63)$$

We consider now only the metric $d\bar{s}^2$ of the two-dimensional p, q space and try to find coordinates that exhibit explicitly conformal flatness of this space. Defining

$$t = \frac{1}{2}(p + q), \quad \bar{r} = \frac{1}{2}(p - q), \quad (64)$$

gives

$$d\bar{s}^2 = -\left(1 - \frac{2M}{r}\right) (dt^2 - d\bar{r}^2), \quad (65)$$

We succeeded to find an explicitly conformal metric, but the conformal factor $1 - 2M/r$ is singular. Hence we should try different coordinates $\bar{p}$ and $\bar{q}$ that eliminate the singularity at $r = 2M$, but keep the conformal structure. A transformation of the type $\bar{p} = \bar{p}(p)$ and $\bar{q} = \bar{q}(q)$ will do the job, because

$$d\bar{s}^2 = -\left(1 - \frac{2M}{r}\right) \frac{dp}{d\bar{p}} \frac{dq}{d\bar{q}} d\bar{p} d\bar{q} \quad (66)$$

has the same structure as Eq. (62). Kruskal suggested

$$\bar{p} = \exp(p/(4M)), \quad \bar{q} = -\exp(-q/(4M)), \quad (67)$$

which gives

$$d\bar{s}^2 = -\frac{32M^3}{r} \exp\left(-\frac{r}{2M}\right) d\bar{p} d\bar{q} \quad (68)$$

The standard form follows by defining a time-like variable V and a space-like variable U by

$$V = \frac{1}{2}(\bar{p} + \bar{q}), \quad \bar{r} = \frac{1}{2}(\bar{p} - \bar{q}), \quad (69)$$

as

$$d\bar{s}^2 = -\frac{32M^3}{r} \exp\left(-\frac{r}{2M}\right) (dV^2 - dU^2) + r^2 d\Omega. \quad (70)$$

Kerr black holes

The metric outside of a rotating black hole is using Boyer-Lindquist coordinates given by

$$\begin{aligned} ds^2 = & -\left(1 - \frac{2Mr}{\rho^2}\right) dt^2 - \frac{4Mar \sin^2 \vartheta}{\rho^2} d\phi dt + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\vartheta^2 \\ & + \left(r^2 + a^2 + \frac{2Mra^2 \sin^2 \vartheta}{\rho^2}\right) \sin^2 \vartheta d\phi^2, \end{aligned} \quad (71)$$

with

$$a = \frac{J}{M}, \quad \rho = r^2 + a^2 \cos^2 \vartheta, \quad \Delta = r^2 - 2Mr + a^2. \quad (72)$$

Its main properties are

- symmetries: The metric is time-independent and axially symmetric. Hence two Killing vectors are $\xi = (1, 0, 0, 0)$ and $\eta = (0, 0, 0, 1)$, where we order coordinates as $\{t, r, \phi, \vartheta\}$.
- The metric is asymptotically flat.
- Potential singularities at $\rho = 0$ and $\Delta = 0$.
- The weak-field limit shows that J is the angular momentum of the rotating black hole.
- The presence of the mixed term $g_{t\phi}$ means that infalling particles (and thus space-time) is dragged around the rotating black hole.

Orbits in the equatorial plane $\vartheta = \pi/2$ could be derived in the same way as for the Schwarzschild case, for $\vartheta \neq \pi/2$ the discussion becomes much more involved.

Singularity The singularity at $\rho = 0 = r^2 + a^2 \cos^2 \vartheta$ corresponds to

$$r = 0 \quad \text{and} \quad \vartheta = \pi/2. \quad (73)$$

We would expect that $r = 0$ is compatible with all ϑ values. To understand this point, we consider the $M \rightarrow 0$ limit¹ of (71),

$$ds^2 = -dt^2 + \frac{\rho}{r^2 + a^2} dr^2 + r^2 d\vartheta^2 + (r^2 + a^2) \sin^2 \vartheta d\phi^2 = -dt^2 + dx^2 + dy^2 + dz^2. \quad (74)$$

¹Looks because of $a < M$ fishy, but gives fast the correct result.

The comparison with the Minkowski metric shows that

$$\begin{aligned} x &= \sqrt{r^2 + a^2} \sin \vartheta \cos \phi, \\ y &= \sqrt{r^2 + a^2} \sin \vartheta \sin \phi, \\ z &= r \cos \vartheta. \end{aligned} \quad (75)$$

Hence the singularity at $r = 0$ and $\vartheta = \pi/2$ corresponds to a ring of radius a in the equatorial plane $z = 0$ of the Kerr black hole.

Horizons We define an *event horizon* as a three-dimensional hypersurface, $f(x^a) = 0$, that is null. Hence we require that at each point of such a surface exists a null tangent vector $\mathbf{n}$ that is orthogonal to two space-like tangent vectors. From

$$0 = n_a n^a = g^{ab} n_a n_b \quad (76)$$

we see that $ds = 0$ along the horizon. Hence the (future) light cones at each point of an event horizon is tangents at this point, and the worldline of all particles lies entirely on one side of the surface.

In a stationary, axisymmetric space-time the general equation of a surface, $f(x^a) = 0$, becomes $f(r, \vartheta) = 0$. The condition for a null surface becomes

$$0 = g^{ab}(\partial_a f)(\partial_b f) = g^{rr}(\partial_r f)^2 + g^{\vartheta\vartheta}(\partial_\vartheta f)^2. \quad (77)$$

For a special set of coordinates, the condition for a surface may be even simpler, $f(r) = 0$ or

$$0 = g^{rr}(\partial_r f)^2. \quad (78)$$

Since we guess that the coordinate singularity at $\Delta = r^2 - 2Mr + a^2 = 0$ or

$$r_\pm = M \pm \sqrt{M^2 - a^2} \quad (79)$$

corresponds to horizons, we can use the latter condition. Then we require $g^{rr} = 0$ or $g_{rr} = 1/g^{rr} = \infty$. Hence, r_- and r_+ define indeed an inner and outer horizon around a Kerr black hole. (For an interpretation see the space-time diagram 1 that uses coordinates of the advanced Eddington-Finkelstein type.)

We can determine the null tangent vector checking the condition $\mathbf{n} \cdot \mathbf{n} = 0$ inserting r_+ and $dr = 0$ into the metric,

$$0 = g_{tt}(n^t)^2 + 2g_{t\phi}n^t n^\phi + g_{\phi\phi}(n^\phi)^2 + g_{\vartheta\vartheta}(n^\vartheta)^2 \quad (80)$$

or

$$\left(\frac{2Mr_+ \sin \vartheta}{\rho_+^2}\right)^2 \left(n^\phi - \frac{a}{2Mr_+} n^t\right)^2 + \rho_+^2 (n^\vartheta)^2 = 0. \quad (81)$$

Hence $n^\vartheta = 0$ and

$$n^a = C(1, 0, 0, a/(2Mr_+)) \quad (82)$$

with C as an arbitrary constant. The rotation velocity of the black hole is defined as the rotation velocity of the light-rays forming the horizon. With

$$\omega = \frac{d\phi}{dt} = \frac{d\phi}{d\lambda} \frac{d\lambda}{dt} = \frac{n^\phi}{n^t} \quad (83)$$

we obtain

$$\boxed{\omega_H = \frac{a}{2Mr_+}}. \quad (84)$$

The surface A of the horizon follows from inserting r_+ together with $dr = dt = 0$ into the metric,

$$ds^2 = \rho_+^2 d\vartheta^2 + \left(r_+^2 + a^2 + \frac{2Mr_+a^2 \sin^2 \vartheta}{\rho_+^2} \right) \sin^2 \vartheta d\phi^2, \quad (85)$$

Using $r_\pm^2 + a^2 = 2Mr_\pm$, it follows

$$ds^2 = \rho_+^2 d\vartheta^2 + \left(\frac{2Mr_+}{\rho_+} \right)^2 \sin^2 \vartheta d\phi^2. \quad (86)$$

Hence $\sqrt{g_{\vartheta\vartheta}} = 2Mr_+ \sin \vartheta$ and integration gives the area of the horizon as

$$A = 8\pi Mr_+ = 8\pi M(M + \sqrt{M^2 - a^2}). \quad (87)$$

Note that the area depends on the angular momentum of the black hole that can in turn be manipulated by dropping material into the hole.

Ergosphere We consider a light-ray with $d\vartheta = dr = 0$. Then the line-element becomes

$$g_{tt}dt^2 + 2g_{t\phi}dtd\phi + g_{\phi\phi}d\phi^2 = 0. \quad (88)$$

Dividing by dt^2 we find

$$\dot{\phi}_\pm = -\frac{g_{t\phi}}{g_{\phi\phi}} \pm \sqrt{\left(\frac{g_{t\phi}}{g_{\phi\phi}} \right)^2 - \frac{g_{tt}}{g_{\phi\phi}}} \quad (89)$$

For $g_{tt} = 0$, we find for the two possible light-rays

$$\dot{\phi}_+ = -2 \frac{g_{t\phi}}{g_{\phi\phi}}, \quad \dot{\phi}_- = 0. \quad (90)$$

Hence, the rotating black hole drags space-time at $g_{tt} = 0$ so strongly that even a photon can only co-rotate. Similarly, this condition specifies a surface inside no stationary observers are possible: The normalization condition $\mathbf{u} \cdot \mathbf{u} = -1$ is inconsistent with $u^a = (1, 0, 0, 0)$ and $g_{tt} > 0$.

Solving

$$g_{tt} = 1 - \frac{2Mr}{\rho^2} = 0 \quad (91)$$

we find the position of the two stationary limit surfaces at

$$r_{1/2} = M \pm \sqrt{M^2 - a \cos \vartheta}. \quad (92)$$

The ergosphere is the space bounded by these two surfaces.

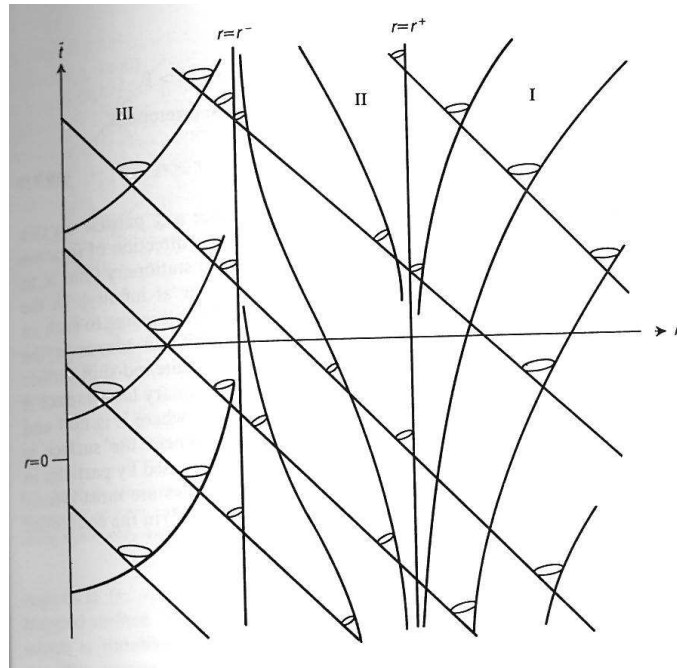


Figure 1: Space-time diagram in advanced Eddington-Finkelstein coordinates for a Kerr black hole with $a < M$. Between the two horizons $r_- < r < r_+$, light cones are oriented towards $r = 0$, particles will reach the singularity. Inside the inner horizon, geodesics are possible that do not reach $r = 0$ in finite time.

Black holes thermodynamics Classically, the area of a black hole can only increase with time. The only other quantity in physics with the same property is the entropy, $dS \geq 0$. This suggests a connection between the black hole area and its entropy. To derive this relation, we apply the first law of thermodynamics $dU = TdS + pdV + \dots$ to a black hole. Its internal energy U is given by $U = Mc^2$ and thus

$$dU = dMc^2 \stackrel{!}{=} TdS + \vec{\omega}d\vec{L}, \quad (93)$$

where $\vec{\omega}d\vec{L}$ denotes the mechanical work done on a rotating macroscopic body.

Our experience with the thermodynamics of non-gravitating systems suggests that the entropy is an extensive quantity and thus proportional to the volume, $S \propto V$. We sketch now one argument, that shows that $S \propto A$ in the case of a black hole. We introduce the “rationalized area” $\alpha = A/4\pi = 2Mr_+$. The parameters describing a Kerr black hole are its mass M and its angular momentum L and thus $\alpha = \alpha(M, L)$. We form the differential and find

$$\frac{\sqrt{M^2 - a^2}}{2\alpha} d\alpha = dM + \frac{\vec{a}}{\alpha} d\vec{L} = dM + \vec{\omega}d\vec{L}. \quad (94)$$

The factor in front of $d\alpha$ is positive, as its interpretation as temperature requires. Our confidence in the correspondence between the first law of black hole thermodynamics, (94), is strengthened by the fact that for a charged black hole the term Φdq appears on the RHS, i.e. the normal te representing the work done by adding the charge dq to the black hole with surface potential Φ .

We identify

$$TdS = \frac{\sqrt{M^2 - a^2}}{2\alpha} d\alpha \quad (95)$$

and thus $S = f(A)$. The validity of the area theorem requires that f is a linear function, the proportionality coefficient between S and A can be only determined by calculating the temperature of black hole. Hawking could show 1974 that a black hole in vacuum emits black-body radiation (“Hawking radiation”) with temperature

$$T = \frac{2\sqrt{M^2 - a^2}}{A} \quad (96)$$

and thus

$$S = \frac{kc^3}{4\hbar G} A = \frac{A}{4L_{\text{Pl}}^2}. \quad (97)$$

Thus a black hole surrounded by a cooler medium emits radiation and heats up the environment. The entropy of a black hole is large, because its basic unit of entropy, $4L_{\text{Pl}}^2$, is so tiny.