

Week 9: Einstein's field equations

Riemann tensor and curvature

We are looking for an invariant characterisation of an manifold curved by gravity. As the discussion of normal coordinates showed, the first derivatives of the metric can be (at one point) always to be chosen to be zero. Hence this quantity will contain second derivatives of the metric, i.e. first ones of the Christoffel symbols.

The commutator of covariant derivatives will in general not vanish,

$$(\nabla_a \nabla_b - \nabla_b \nabla_a) T_{d...}^{c...} = [\nabla_a, \nabla_b] T_{d...}^{c...} \neq 0, \quad (101)$$

(think at the parallel transport from A first along $\mathbf{e}_a$, then along $\mathbf{e}_b$ to B and then back to A along $-\mathbf{e}_a$ and $-\mathbf{e}_b$ on a sphere), is obviously a tensor and contains second derivatives of the metric.

For the special case of a vector X^a we obtain with

$$\nabla_c X^a = \partial_c X^a + \Gamma_{bc}^a X^b \quad (102)$$

first

$$\nabla_d \nabla_c X^a = \partial_d (\partial_c X^a + \Gamma_{bc}^a X^b) + \Gamma_{ed}^a (\partial_c X^e + \Gamma_{bc}^e X^b) - \Gamma_{cd}^e (\partial_e X^a + \Gamma_{be}^a X^b) \quad (103)$$

and the same expression with $d \leftrightarrow c$,

$$\nabla_c \nabla_d X^a = \partial_c (\partial_d X^a + \Gamma_{bd}^a X^b) + \Gamma_{ec}^a (\partial_d X^e + \Gamma_{bd}^e X^b) - \Gamma_{dc}^e (\partial_e X^a + \Gamma_{be}^a X^b) \quad (104)$$

We subtract the two equations, using that $\partial_c \partial_d = \partial_d \partial_c$ and $\Gamma_{bc}^a = \Gamma_{cb}^a$,

$$[\nabla_c, \nabla_d] X^a = [\partial_c \Gamma_{bd}^a - \partial_d \Gamma_{bc}^a + \Gamma_{ec}^a \Gamma_{bd}^e - \Gamma_{ed}^a \Gamma_{bc}^e] X^b \equiv R_{bcd}^a X^b. \quad (105)$$

The tensor R_{bcd}^a is called *Riemann* or *curvature tensor*. Contracting the indices a and c , we obtain the *Ricci tensor*

$$\boxed{R_{ab} = R_{acb}^c = \partial_c \Gamma_{ab}^c - \partial_b \Gamma_{ac}^c + \Gamma_{ab}^c \Gamma_{cd}^d - \Gamma_{bc}^d \Gamma_{ad}^c.} \quad (106)$$

A further contraction gives the curvature scalar,

$$\boxed{R = R_{ab} g^{ab}.} \quad (107)$$

Symmetry properties Inserting the definition of the Christoffel symbols and using normal coordinates, the Riemann tensor becomes

$$R_{abcd} = \frac{1}{2} \{ \partial_d \partial_b g_{ac} + \partial_c \partial_a g_{bd} - \partial_d \partial_a g_{bc} - \partial_c \partial_b g_{ad} \} \quad (108)$$

The tensor is antisymmetric in the indices $c \leftrightarrow d$, antisymmetric in $a \leftrightarrow b$ and symmetric against an exchange of the index pairs $(ab) \leftrightarrow (cd)$. Moreover, there exists one algebraic identity,

$$R_{abcd} + R_{adbc} + R_{acdb} = 0. \quad (109)$$

Since each pair of indices (ab) and (cd) can take six values, we can combine the antisymmetrized components of $R_{[ab][cd]}$ in a symmetric six-dimensional matrix. The number of independent components of this matrix is thus for $n = 4$ space-time dimensions

$$\frac{6 \times 7}{2} - 1 = 20,$$

where we accounted also for the constraint (110).

The Bianchi identity is a differential constraint,

$$\nabla_e R_{abcd} + \nabla_c R_{abde} + \nabla_d R_{abec} = 0, \quad (110)$$

that is checked again simplest using normal coordinates.

Variation of the metric determinant g

We consider a variation of a matrix M with elements $m_{ij}(x)$ under an infinitesimal change of the coordinates, $\delta x^a = \varepsilon x^a$,

$$\begin{aligned} \delta \ln \det M &\equiv \ln \det(M + \delta M) - \ln \det(M) \\ &= \ln \frac{\det(M + \delta M)}{\det(M)} \\ &= \ln \det[M^{-1}(M + \delta M)] = \ln \det[1 + M^{-1}\delta M] = \\ &= \ln[1 + \text{Tr}(M^{-1}\delta M)] + \mathcal{O}(\varepsilon^2) = \text{Tr}(M^{-1}\delta M) + \mathcal{O}(\varepsilon^2). \end{aligned} \quad (111)$$

In the last step, we used $\ln(1 + x) = x + \mathcal{O}(x^2)$. Hence

$$\text{Tr}(M^{-1} \frac{\partial}{\partial x^\mu} M) = \frac{\partial}{\partial x^\mu} \ln \det(M). \quad (112)$$

Applied to the contraction of Christoffel symbols, we obtain

$$\Gamma^a_{ab} = \frac{1}{2} g^{ad} (\partial_a g_{db} + \partial_b g_{ad} - \partial_d g_{ab}) = \frac{1}{2} g^{ad} \partial_b g_{ad} = \frac{1}{2} \partial_b \ln g = \frac{1}{\sqrt{|g|}} \partial_b (\sqrt{|g|}) \quad (113)$$

and

$$\nabla_a X^a = \partial_a X^a + \Gamma^a_{ca} X^c = \partial_a X^a + \frac{1}{\sqrt{|g|}} (\partial_a \sqrt{|g|}) X^a = \frac{1}{\sqrt{|g|}} \partial_a (\sqrt{|g|} X^a). \quad (114)$$

An immediate consequence is a covariant form of Gauß theorem: If the current X^a vanishes at infinity, then

$$\int_{\Omega} d^4x \sqrt{|g|} \nabla_a X^a = \int_{\Omega} d^4x \partial_a (\sqrt{|g|} X^a) = 0. \quad (115)$$

Einstein-Hilbert action

Our main guide in choosing the appropriate action for the gravitational field is simplicity. Among the possible terms we can select are

$$\mathcal{L} = \sqrt{|g|} \left\{ \Lambda + bR + c\nabla_a \nabla_b R^{ab} + d(\nabla_a \nabla_b R^{ab})^2 + \dots + f(R) + \dots \right\} \quad (116)$$

Choosing only the first term, a constant, will not give dynamical equations. The next simplest possibility is to pick out only the second term, as it was done originally by Hilbert. We will see later that, if we do not include a constant term Λ in the gravitational action, it will pop up on the matter side. Thus we include Λ right from the start and define as the Einstein-Hilbert Lagrange density for the gravitational field

$$\boxed{\mathcal{L}_{\text{EH}} = \sqrt{|g|}(R - 2\Lambda)}. \quad (117)$$

We derive the resulting field equations for the metric tensor g_{ab} directly from the action principle

$$\delta S_{\text{EH}} = \delta \int_{\Omega} d^4x \sqrt{|g|} (R - 2\Lambda) = \delta \int_{\Omega} d^4x \sqrt{|g|} (g^{ab} R_{ab} - 2\Lambda) = 0. \quad (118)$$

We allow for variations of the metric g_{ab} restricted by the condition that the variation of g_{ab} and its first derivatives vanish on the boundary $\partial\Omega$.

$$\delta S_{\text{EH}} = \int_{\Omega} d^4x \left\{ \sqrt{|g|} g^{ab} \delta R_{ab} + \sqrt{|g|} R_{ab} \delta g^{ab} + (R - 2\Lambda) \delta \sqrt{|g|} \right\}. \quad (119)$$

Our task is to rewrite the first and third term as variations of δg^{ab} or to show that they are equivalent to boundary terms.

Let us start with the first term. Choosing normal coordinates, the Ricci tensor at the considered point P becomes

$$R_{ab} = \partial_c \Gamma_{ab}^c - \partial_b \Gamma_{ac}^c. \quad (120)$$

Hence

$$g^{ab} \delta R_{ab} = g^{ab} (\partial_c \delta \Gamma_{ab}^c - \partial_b \delta \Gamma_{ac}^c) = g^{ab} \partial_c \delta \Gamma_{ab}^c - g^{ac} \partial_c \delta \Gamma_{ab}^b, \quad (121)$$

where we exchanged the indices b and c in the last term. Since $\partial_c g_{ab} = 0$ at P , we can rewrite the expression as

$$g^{ab} \delta R_{ab} = \partial_c (g^{ab} \delta \Gamma_{ab}^c - g^{ac} \delta \Gamma_{ab}^b) = \partial_c X^c. \quad (122)$$

The quantity $\mathbf{X}$ is a vector, since the difference of two connection coefficients transforms as a tensor. Replacing in Eq. (122) the partial derivative by a covariant one promotes it therefore in a valid tensor equation,

$$g^{ab} \delta R_{ab} = \frac{1}{\sqrt{|g|}} \partial_a (\sqrt{|g|} X^a). \quad (123)$$

Thus this term corresponds to a surface term that vanishes.

Next we rewrite the third term using

$$\delta\sqrt{|g|} = \frac{1}{2\sqrt{|g|}}\delta|g| = \frac{1}{2}\sqrt{|g|}g^{ab}\delta g_{ab} = -\frac{1}{2}\sqrt{|g|}g_{ab}\delta g^{ab} \quad (124)$$

and obtain

$$\delta S_{\text{EH}} = \int_{\Omega} d^4x \sqrt{|g|} \left\{ R_{ab} - \frac{1}{2} g_{ab} R + \Lambda g_{ab} \right\} \delta g^{ab} = 0. \quad (125)$$

Hence the metric tensor fulfills in vacuum the equation

$$\boxed{\frac{1}{\sqrt{|g|}} \frac{\delta S_{\text{EH}}}{\delta g^{ab}} = R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} \equiv G_{ab} + \Lambda g_{ab} = 0,} \quad (126)$$

where we introduced the Einstein tensor G_{ab} . The constant Λ is called cosmological constant.

Einstein equations

We consider now the combined action of gravity and matter, as the sum of the Einstein-Hilbert Lagrange density $\mathcal{L}_{\text{EH}}/2\kappa$ and the Lagrange density $\mathcal{L}_{\text{m}}$ including all relevant matter fields,

$$\boxed{\mathcal{L} = \frac{1}{2\kappa}\mathcal{L}_{\text{EH}} + \mathcal{L}_{\text{m}} = \frac{1}{2\kappa}\sqrt{|g|}(R - 2\Lambda) + \mathcal{L}_{\text{m}}.} \quad (127)$$

In $\mathcal{L}_{\text{m}}$, the effects of gravity are accounted for by the replacements $\{\partial_a, \eta_{ab}\} \rightarrow \{\nabla_a, g_{ab}\}$, while we have to adjust later the constant κ such that we reproduce Newtonian dynamics in the weak-field limit. We expect that the source of the gravitational field is the energy-momentum tensor. More precisely, the Einstein tensor (“geometry”) should be determined by the matter, $\mathbf{G} = \kappa\mathbf{T}$. Since we know already the result of the variation of S_{EH} , we conclude that the variation of S_{m} should give

$$\boxed{\frac{2}{\sqrt{|g|}} \frac{\delta S_{\text{m}}}{\delta g^{ab}} = T_{ab}.} \quad (128)$$

The tensor T_{ab} defined by this equation is called *dynamical energy-momentum tensor*. In order to show that this rather bold definition makes sense, we have to prove that $\nabla^a(\delta S_{\text{m}}/\delta g^{ab}) = 0$ and we have to convince ourselves that this definition reproduces the standard results we know already. Einstein’s field equation follows then as

$$\boxed{G_{ab} + \Lambda g_{ab} = \kappa T_{ab}.} \quad (129)$$

Dynamical energy-momentum tensor

We start by proving that the dynamical energy-momentum tensor defined by Eq. (128) is conserved. We consider the change of the matter action under variations of the metric,

$$\delta S_m = \frac{1}{2} \int_{\Omega} d^4x \sqrt{|g|} T_{ab} \delta g^{ab} = -\frac{1}{2} \int_{\Omega} d^4x \sqrt{|g|} T^{ab} \delta g_{ab}. \quad (130)$$

We allow infinitesimal but otherwise arbitrary coordinate transformations,

$$\bar{x}^i = x^i + \xi(x^k). \quad (131)$$

For the resulting change in the metric δg_{ab} we can use our use Eqs. (6) and (7),

$$\delta g_{ij} = \nabla_i \xi_j + \nabla_j \xi_i. \quad (132)$$

We use that T^{ab} is symmetric and that $\delta S_m = 0$ for a coordinate transformation,

$$\delta S_m = - \int_{\Omega} d^4x \sqrt{|g|} T^{ab} \nabla_a \xi_b = 0. \quad (133)$$

Next we apply the product rule,

$$\delta S_m = - \int_{\Omega} d^4x \sqrt{|g|} (\nabla_a T^{ab}) \xi_b + \int_{\Omega} d^4x \sqrt{|g|} \nabla_a (T^{ab} \xi_b) = 0. \quad (134)$$

The second term is a four-divergence and thus a boundary term that we can neglect. The remaining first term vanishes for arbitrary ξ , if the energy-momentum tensor is conserved,

$$\boxed{\nabla_a T^{ab} = 0.} \quad (135)$$

Hence the local conservation of energy-momentum is a consequence of the general covariance of the gravitational field equations, in the same way as current conservation follows from gauge invariance in electromagnetism.

We now evaluate the dynamical energy-momentum tensor for the examples of the massless Klein-Gordon and the photon field. We rewrite Eq. (??) including a potential $V(\phi)$ as

$$\mathcal{L} = \frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - V(\phi) \quad (136)$$

Varying the action gives

$$\begin{aligned} \delta S_{\text{KG}} &= \frac{1}{2} \int_{\Omega} d^4x \left\{ \sqrt{|g|} \nabla_a \phi \nabla_b \phi \delta g^{ab} + [g^{ab} \nabla_a \phi \nabla_b \phi - 2V(\phi)] \delta \sqrt{|g|} \right\} \\ &= \int_{\Omega} d^4x \sqrt{|g|} \delta g^{ab} \left\{ \frac{1}{2} \nabla_a \phi \nabla_b \phi - \frac{1}{2} g_{ab} \mathcal{L} \right\}. \end{aligned} \quad (137)$$

and thus

$$T_{ab} = \frac{2}{\sqrt{|g|}} \frac{\delta S_m}{\delta g^{ab}} = \nabla_a \phi \nabla_b \phi - g_{ab} \mathcal{L}. \quad (138)$$

Next we consider the free electromagnetic action,

$$\delta S_{\text{em}} = -\frac{1}{4} \int_{\Omega} d^4x \sqrt{|g|} g^{ab} g^{cd} F_{ab} F_{cd} \quad (139)$$

remember that $\mathbf{F}$ does not depend on $\mathbf{g}$ and get

$$\begin{aligned} \delta S_{\text{em}} &= -\frac{1}{4} \int_{\Omega} d^4x \left\{ (\delta \sqrt{|g|}) F_{cd} F^{cd} + \sqrt{|g|} \delta (g^{ab} g^{cd}) F_{ab} F_{cd} \right\} \\ &= -\frac{1}{4} \int_{\Omega} d^4x \sqrt{|g|} \delta g^{ab} \left\{ -\frac{1}{2} g_{ab} F_{cd} F^{cd} + 2 g^{cd} F_{ac} F_{bd} \right\}. \end{aligned} \quad (140)$$

Hence the dynamical energy-momentum tensor is

$$T_{ab} = -F_{ac} F_b{}^c + \frac{1}{4} g_{ab} F_{cd} F^{cd}. \quad (141)$$

Equations of motions

We show now that Einstein's equation imply that particles move along geodesics. By analogy with a fluid, $T^{ab} = \rho u^a u^b$, we postulate

$$T^{ab}(\bar{x}) = \frac{m}{\sqrt{|g|}} \int d\tau \frac{dx^a}{d\tau} \frac{dx^b}{d\tau} \delta^{(4)}(\bar{x} - x(\tau)) \quad (142)$$

for a point-particle moving along $x(\tau)$ with proper time τ . Inserting this into

$$\nabla_a T^{ab} = \partial_a T^{ab} + \Gamma^a_{ca} T^{cb} + \Gamma^b_{ca} T^{ac} = \frac{1}{\sqrt{|g|}} \partial_a (\sqrt{|g|} T^{ab}) + \Gamma^b_{ca} T^{ac} = 0 \quad (143)$$

gives

$$\int d\tau \dot{x}^a \dot{x}^b \frac{\partial}{\partial x^a} \delta^{(4)}(\bar{x} - x(\tau)) + \Gamma^b_{ca} \int d\tau \dot{x}^a \dot{x}^c \delta^{(4)}(\bar{x} - x(\tau)) = 0. \quad (144)$$

We can replace $\partial/\partial x^a = -\partial/\partial \bar{x}^a$ acting $\delta^{(4)}(\bar{x} - x(\tau))$ and use moreover

$$\dot{x}^a \frac{\partial}{\partial x^a} \delta^{(4)}(\bar{x} - x(\tau)) = \frac{d}{d\tau} \delta^{(4)}(\bar{x} - x(\tau)) \quad (145)$$

to obtain

$$-\int d\tau \dot{x}^a \frac{d}{d\tau} \delta^{(4)}(\bar{x} - x(\tau)) + \Gamma^b_{ca} \int d\tau \dot{x}^a \dot{x}^c \delta^{(4)}(\bar{x} - x(\tau)) = 0. \quad (146)$$

Integrating the first term by parts we obtain

$$\int d\tau (\ddot{x}^a + \Gamma^b_{ca} \dot{x}^a \dot{x}^c) \delta^{(4)}(\bar{x} - x(\tau)) = 0. \quad (147)$$

The integral vanishes only, when the worldline fulfills the geodesic equation. Hence Einstein's equation implies already the equation of motion, in contrast to linear theories as the Maxwell equations, where the Lorentz force law has to be postulated separately.