Høgskolen i Sør-Trøndelag
Avdeling for teknologi
Program for Elektro- og datateknikk

HØGSKOLEN I SØR-TRØNDELAG

GO326A Matematikk 2F

Øving 8 , Løsningsforslag

Emne: Digitale filtre og Z-transformasjonen

Oppgave 1 *Diskret 1.ordens LP filter med knekkfrekvens 100 Hz.*

Et slikt filter vil ha transferfunksjonen:

$$h_{LP}(s) = \frac{Y(s)}{U(s)} = \frac{K}{1+T_1 s} \quad \text{hvor } K = 1 \quad \text{og} \quad \omega_0 = \frac{1}{T_1} = 2\pi f_0 \Rightarrow T_1 = \frac{1}{2\pi f_0} \approx 1{,}5915 \text{ ms}$$

I denne likningen er *U(s)* og *Y(s)* henholdsvis inngang og utgang på filteret. Vi kryssmultipliserer likningen ovenfor og får:

$(1+T_1 s)Y(s) = K \cdot U(s) = U(s)$ Vi inverstransformerer denne og får :

$$y + T_1 \dot{y} = u \qquad (1)$$

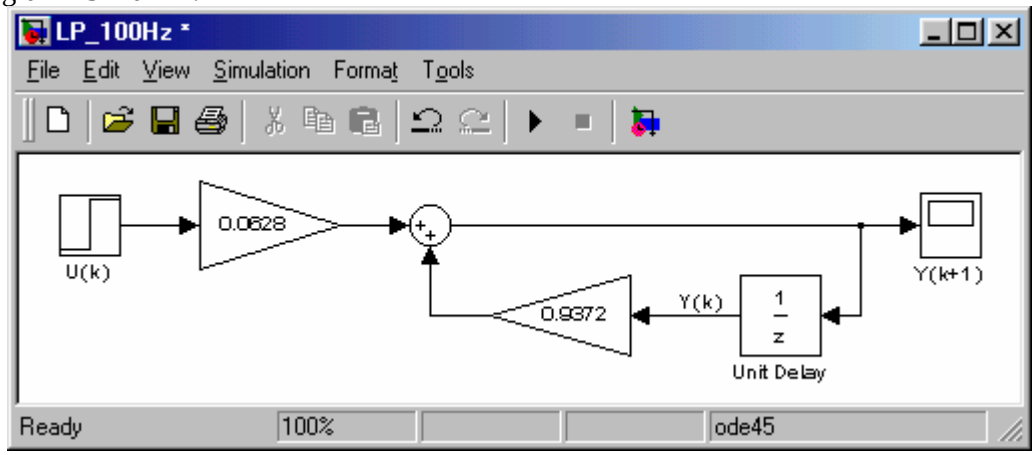
Vi innfører tilnærmimngen : $\dot{y} \approx \frac{y_{k+1} - y_k}{T}$ hvor T er samplingstida.

Som tomelfingerregel bruker vi at samplingstida bør være mindre enn 10 % av minste tidskonstant.

Vi velger : $T = 0{,}1$ ms Vi setter dette inn i likning (1) og får :

$$y_k + T_1 \cdot \frac{y_{k+1} - y_k}{T} = u_k \Rightarrow y_{k+1} = \left(1 - \frac{T}{T_1}\right) y_k + \frac{T}{T_1} u_k \Rightarrow y_{k+1} = 0{,}9372 \cdot y_k + 0{,}0628 u_k$$

Blokkdiagram i Simulink:



Oppgave 2

Avsnitt 22.6 (s.667) oppg.1:

1)

$x[n+2]+x[n+1]-x[n]=2 \qquad x[0]=3 \quad x[1]=5$

Finn $x[2], x[3], x[4], x[5]$

Løsning :

$n=0: \quad x[2]+x[1]-x[0]=2 \Rightarrow x[2]+5-3=2 \Rightarrow \underline{x[2]=0}$

$n=1: \quad x[3]+x[2]-x[1]=2 \Rightarrow x[3]+0-5=2 \Rightarrow \underline{x[3]=7}$

$n=2: \quad x[4]+x[3]-x[2]=2 \Rightarrow x[4]+7-0=2 \Rightarrow \underline{x[4]=-5}$

$n=3: \quad x[5]+x[4]-x[3]=2 \Rightarrow x[5]-5-7=2 \Rightarrow \underline{x[5]=14}$

2)

$z[n]\cdot z[n-1]=n^2 \qquad z[1]=7$

Finn $z[2], z[3], z[4]$

Løsning :

$n=2: \quad z[2]\cdot z[1]=2^2 \Rightarrow z[2]\cdot 7=4 \Rightarrow \underline{z[2]=\frac{4}{7}}$

$n=3: \quad z[3]\cdot z[2]=3^2 \Rightarrow z[3]\cdot \frac{4}{7}=9 \Rightarrow \underline{z[3]=\frac{63}{4}}$

$n=4: \quad z[4]\cdot z[3]=4^2 \Rightarrow z[4]\cdot \frac{63}{4}=16 \Rightarrow \underline{z[2]=\frac{64}{63}}$

3)

Finn $x[2]$ og $x[3]$ i følgende tilfeller:

(a) $\quad 2x[n+2]-5x[n+1]=4n \qquad x[1]=2$

$n=0: \quad 2x[2]-5x[1]=0 \Rightarrow 2x[2]-5\cdot 2=0 \Rightarrow \underline{x[2]=5}$

$n=1: \quad 2x[3]-5x[2]=4 \Rightarrow 2x[3]-5\cdot 5=4 \Rightarrow \underline{x[2]=\frac{29}{2}}$

(b) $\quad 6x[n]-x[n-1]+2x[n-2]=n^2-n \qquad x[0]=1 \quad x[1]=2$

$n=2: \quad 6x[2]-x[1]+2x[0]=2^2-2$

$\Rightarrow 6x[2]-2+2=2 \Rightarrow \underline{x[2]=\frac{1}{3}}$

$n=3: \quad 6x[3]-x[2]+2x[1]=3^2-3 \Rightarrow 6x[3]-\frac{1}{3}+4=6 \Rightarrow \underline{x[3]=\frac{7}{18}}$

(c) $3x[n-1]+x[n-2]-9x[n-3]=(n-1)^2$

$x[0]=3 \quad x[1]=2$

$n=3: \quad 3x[2]+x[1]-9x[0]=2^2 \Rightarrow 3x[2]+2-27=4 \Rightarrow \underline{x[2]=\frac{29}{3}}$

$n=4: \quad 3x[3]+x[2]-9x[1]=3^2 \Rightarrow 3x[3]+\frac{29}{3}-18=9 \Rightarrow \underline{x[3]=\frac{52}{9}}$

4) Beregn de fem første ledd i følgende differenslikninger dvs. fom. $x[0]$ tom. $x[4]$

a) $x[n+1]-x[n]=2 \qquad x[0]=3$

$n=0: \quad x[1]-x[0]=2 \Rightarrow x[1]-3=2 \Rightarrow \underline{x[1]=5}$

$n=1: \quad x[2]-x[1]=2 \Rightarrow x[2]-5=2 \Rightarrow \underline{x[2]=7}$

$n=2: \quad x[3]-x[2]=2 \Rightarrow x[3]-7=2 \Rightarrow \underline{x[3]=9}$

$n=3: \quad x[4]-x[3]=2 \Rightarrow x[4]-9=2 \Rightarrow \underline{x[4]=11}$

b) $x[n+2]+x[n+1]-x[n]=4 \Rightarrow \underline{x[n+2]=4+x[n]-x[n+1]}$

$\underline{x[0]=5} \qquad \underline{x[1]=7}$

$n=0: \quad \underline{x[2]}=4+x[0]-x[1]=4+5-7=\underline{2}$

$n=1: \quad \underline{x[3]}=4+x[1]-x[2]=4+7-2=\underline{9}$

$n=2: \quad \underline{x[4]}=4+x[2]-x[3]=4+2-9=\underline{-3}$

c) $x[p+2]-x[p+1]+2x[p]=2 \Rightarrow \underline{x[p+2]=x[p+1]-2x[p]+2}$

$\underline{x[0]=1} \qquad \underline{x[1]=1}$

$p=0: \quad \underline{x[2]}=x[1]-2x[0]+2=1-2+2=\underline{1}$

osv. for alle etterfølgende ledd. $\underline{x[p]=1 \text{ for alle } p}$

Avsnitt 22.7 (s.671) oppg.1(a,b,c):

Finn z - transformasjonen for de nedenstående sekvensene ut fra definisjonen :

$$F(z) = Z\{f[k]\} = \sum_{k=0}^{\infty} f[k] z^{-k}$$

a) $f[0] = 0,\ f[1] = 0,\ f[k] = 1$ for $k \geq 2$

$$F(z) = \sum_{k=2}^{\infty} 1 \cdot z^{-k} = z^{-2} + z^{-3} + z^{-4} + \cdots$$

$$zF(z) = z^{-1} + z^{-2} + z^{-3} + z^{-4} + \cdots$$

$$zF(z) - F(z) = z^{-1} \Rightarrow \underline{F(z) = \frac{z^{-1}}{z-1} = \frac{1}{z(z-1)}}$$

b) $f[k] = \begin{cases} 0 & k = 0,1,...,5 \\ 4 & k > 5 \end{cases}$

$$F(z) = \sum_{k=6}^{\infty} 4z^{-k} = 4\sum_{k=6}^{\infty} z^{-k} = 4\left(z^{-6} + z^{-7} + z^{-8} + \cdots\right)$$

$$zF(z) = 4\left(z^{-5} + z^{-6} + z^{-7} + z^{-8} + \cdots\right)$$

$$zF(z) - F(z) = 4z^{-5} \Rightarrow \underline{F(z) = \frac{4z^{-5}}{z-1} = \frac{4}{z^5(z-1)}}$$

c) $f[k] = 3k\ ,\ k \geq 0$

$$F(z) = \sum_{k=0}^{\infty} 3k \cdot z^{-k} = 3\sum_{k=0}^{\infty} k \cdot z^{-k} = 3\left(z^{-1} + 2z^{-2} + 3z^{-3} + 4z^{-4} + \cdots\right)$$

$$zF(z) = 3\left(1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 5z^{-4} + \cdots\right)$$

$$zF(z) - F(z) = 3\left(1 + z^{-1} + z^{-2} + z^{-3} + \cdots\right) = (z-1)F(z) = G(z)$$

$$zG(z) = 3\left(z + 1 + z^{-1} + z^{-2} + z^{-3} + \cdots\right)$$

$$zG(z) - G(z) = 3z = (z-1)G(z) = (z-1)^2 F(z)$$

$$\Rightarrow \underline{F(z) = \frac{3z}{(z-1)^2}}$$

Avsnitt 22.11, s.689, oppg.1:
Finn invers Z-transformasjon til følgende funksjoner:

a) $F(z)=\dfrac{4z}{z-4}$ Denne finner vi direkte fra tabell s.670:

$F(z)=4\cdot\dfrac{z}{z-4}\Rightarrow \underline{f[k]=4\cdot 4^k=4^{k+1}}$

b) $F(z)=\dfrac{z^2+2z}{3z^2-4z-7}=z\left(\dfrac{z+2}{3(z+1)(z-\frac{7}{3})}\right)\equiv z\left(\dfrac{A}{z+1}+\dfrac{B}{z-\frac{7}{3}}\right)=z\left(\dfrac{3A(z-\frac{7}{3})+3B(z+1)}{3(z+1)(z-\frac{7}{3})}\right)$

$F(z)=z\left(\dfrac{z(3A+3B)+3B-7A}{3(z+1)(z-\frac{7}{3})}\right)\Rightarrow\begin{cases}3A+3B=1\\3B-7A=2\end{cases}\Rightarrow \underline{A=-\frac{1}{10}\ B=\frac{13}{30}}$

$\Rightarrow F(z)=\frac{13}{30}\cdot\dfrac{z}{z-\frac{7}{3}}-\frac{1}{10}\cdot\dfrac{z}{z+1}$ Nå finner vi i tabellen:

$\underline{f[k]=\frac{13}{30}(\frac{7}{3})^k-\frac{1}{10}(-1)^k}$

c) $F(z)=\dfrac{z+1}{z^2(z-3)}=z^{-2}\dfrac{z}{z-3}+z^{-3}\dfrac{z}{z-3}$

$Z^{-1}\left\{\dfrac{z}{z-3}\right\}=3^k$ Dette kombinerer vi nå med det *andre shiftteoremet*:

$Z\{f[k-i]u[k-i]\}=z^{-i}F(z)$

$\Rightarrow \underline{F(z)=3^{k-2}u[k-2]+3^{k-3}u[k-3]}$

d) $F(z)=\dfrac{2z^3+z}{(z-3)^2(z-1)} \Rightarrow \dfrac{F(z)}{z}=\dfrac{2z^2+1}{(z-3)^2(z-1)}=G(z)$

hvor $\underline{G(z)=\dfrac{2z^2+1}{(z-3)^2(z-1)} \equiv \dfrac{A}{z-3}+\dfrac{B}{(z-3)^2}+\dfrac{C}{z-1}}$

$$G(z)=\frac{A(z-3)(z-1)+B(z-1)+C(z-3)^2}{(z-3)^2(z-1)}$$

$$G(z)=\frac{A(z^2-4z+3)+B(z-1)+C(z^2-6z+9)}{(z-3)^2(z-1)}$$

$$\underline{G(z)=\frac{z^2(A+C)+z(-4A+B-6C)+3A-B+9C}{(z-3)^2(z-1)}}$$

Dette gir tre likninger:

$A+C=2 \qquad -4A+B-6C=0 \qquad 3A-B+9C=1$ som gir:

$\underline{A=\frac{5}{4}} \qquad \underline{B=\frac{19}{2}} \qquad \underline{C=\frac{3}{4}}$

$$\Rightarrow G(z)=\tfrac{19}{2}\cdot\frac{1}{(z-3)^2}+\tfrac{5}{4}\cdot\frac{1}{z-3}+\tfrac{3}{4}\cdot\frac{1}{z-1}$$

$$\underline{F(z)} = zG(z)=\tfrac{19}{2}\cdot\frac{z}{(z-3)^2}+\tfrac{5}{4}\cdot\frac{z}{z-3}+\tfrac{3}{4}\cdot\frac{z}{z-1}=\underline{\tfrac{19}{6}\cdot\frac{3z}{(z-3)^2}+\tfrac{5}{4}\cdot\frac{z}{z-3}+\tfrac{3}{4}\cdot\frac{z}{z-1}}$$

Vi bruker nå tabellen s.670 sammen med første skiftteorem og får:

$\underline{f[k]=\tfrac{19k}{6}\cdot 3^k+\tfrac{5}{4}\cdot 3^k+\tfrac{3}{4}u[k]}$

Avsnitt 22.12, s.691, oppg.1(a,b,c):

Bruk Z-transformasjonen til å løse følgende differens-likninger:

a) $x[k+1]-3x[k]=-6\,,\quad x[0]=1$

Vi skriver likningen på Z-transformert form:

$$zX(z)-zx[0]-3X(z)=\frac{-6z}{z-1}\Rightarrow(z-3)X(z)=z-\frac{6z}{z-1}=\frac{z^2-7z}{z-1}$$

$$X(z)=\frac{z^2-7z}{(z-1)(z-3)}\Rightarrow\frac{X(z)}{z}=\frac{z-7}{(z-1)(z-3)}\equiv\frac{A}{z-1}+\frac{B}{z-3}$$

$$\frac{X(z)}{z}=\frac{A(z-3)+B(z-1)}{(z-1)(z-3)}=\frac{z(A+B)-(3A+B)}{(z-1)(z-3)}\equiv\frac{z-7}{(z-1)(z-3)}$$

Dette gir likningene:

$A+B=1\qquad 3A+B=7\quad$ som gir: $\underline{A=3}\quad\underline{B=-2}$

$$\frac{X(z)}{z}=\frac{3}{z-1}-\frac{2}{z-3}\Rightarrow X(z)=\frac{3z}{z-1}-\frac{2z}{z-3}\Rightarrow\underline{3u[k]-2(3^k)}$$

b) $2x[k+1]-x[k]=2^k\,,\quad x[0]=2$

Vi skriver likningen på Z-transformert form:

$$2zX(z)-2z\cdot 2-X(z)=\frac{z}{z-2}\Rightarrow(2z-1)X(z)=\frac{z}{z-2}+4z=\frac{4z^2-7z}{z-2}$$

$$X(z)=\frac{4z^2-7z}{(z-2)(2z-1)}=\frac{2z^2-\frac{7}{2}z}{(z-2)(z-\frac{1}{2})}\Rightarrow\frac{X(z)}{z}=\frac{2z-\frac{7}{2}}{(z-2)(z-\frac{1}{2})}$$

$$\frac{X(z)}{z}=\frac{2z-\frac{7}{2}}{(z-2)(z-\frac{1}{2})}\equiv\frac{A}{z-2}+\frac{B}{z-\frac{1}{2}}\Rightarrow\underline{A=\tfrac{1}{3}}\quad\underline{B=\tfrac{5}{3}}$$

$$\frac{X(z)}{z}=\tfrac{1}{3}\frac{1}{z-2}+\tfrac{5}{3}\frac{1}{z-\frac{1}{2}}\Rightarrow X(z)=\tfrac{1}{3}\frac{z}{z-2}+\tfrac{5}{3}\frac{z}{z-\frac{1}{2}}\Rightarrow\underline{x[k]=\frac{2^k}{3}+\frac{5}{3\cdot 2^k}}$$

c) $x[k+1]+x[k]=2k+1\,,\quad x[0]=0$

$$zX(z)+X(z)=\frac{2z}{(z-1)^2}+\frac{z}{z-1}=\frac{2z+z(z-1)}{(z-1)^2}=\frac{z^2+z}{(z-1)^2}$$

$$X(z)=\frac{z^2+z}{(z+1)(z-1)^2}=\frac{z}{(z-1)^2}\Rightarrow\underline{x[k]=k}$$