A White Noise Approach to Stochastic Differential Equations Driven by Wiener and Poisson Processes

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1 Introduction

The main idea behind the white noise approach to stochastic partial differential equations (SPDEs) can be described as follows:
Recall the fact that even quite simple SPDEs, like the stochastic Poisson equation

$$
\begin{aligned}
\Delta F(x) &= -W(x)\,; \qquad x \in D \subset \mathbb{R}^d\,, \quad d > 3 \\
F(x) &= 0\,; \qquad x \in \partial D
\end{aligned}
\tag{1.1}
$$

(where $\Delta = \sum_{i=1}^d \partial^2/\partial x_i^2$ is the Laplacian and W is Gaussian white noise), do not allow solutions which are ordinary stochastic processes/fields $F(x,\omega)\colon \overline{D} \times \Omega \to \mathbb{R}$. However, Walsh [9] proved that solutions exist in the distribution sense:

(1.2) For almost all ω the map $x \mapsto F(x,\omega)$ is a distribution in x.

This approach works fine for linear equations like (1.1), but requires more work in the nonlinear case. It is, however, possible also to handle some such cases by introducing Colombeau random generalized functions. See [7] and the references therein.
The *white noise approach* looks for generalized solutions in the following sense:

(1.3) For all x the map $\omega \mapsto F(x,\omega)$ is a *stochastic* distribution in ω.

Since there is a natural product in the stochastic distribution spaces (the Wick product), this approach has the advantage of extending easily to many nonlinear SPDEs, provided they are interpreted in the Wick sense. Moreover, there is an efficient solution algorithm for such equations, based on the Hermite transform (see, e.g., [6, Theorem 4.1.1], and Section 3 in this paper).

The theory of *Gaussian* white noise and its applications to stochastic differential equations (SDEs), both ordinary and partial, are now well-known. See for example the presentations in [5] and [6]. It is also known—although not so widely—that a similar type of theory exists for Poissonian white noise. Moreover, this Poissonian white noise theory has similar applications to SDEs driven by the Poisson process. In fact, as described in Section 2 there is even a unitary correspondence $\mathcal{U}$ between the solutions of Gaussian and Poissonian white noise equations, provided they both are of Wick type. See the exposition in [1] and the references therein. A brief survey of this connection can be found in [6, Section 4.9].

However, this connection does not immediately explain how the white noise theory can be modified to cover SDEs driven by *combinations* of Gaussian and Poissonian white noise. The purpose of this paper is to outline how this can be done and at the same time review the separate Gaussian and Poissonian cases and the relation between them in more detail.

In Section 2 we briefly review the white noise theory in the Gaussian and Poissonian cases separately and some relations between them.

Then in Section 3 we outline how to combine the two white noise theories into one theory that cover equations driven by both Gaussian and Poissonian noise.

In Section 4 we illustrate the theory by considering some examples.

2 A Review of the Gaussian and Poissonian White Noise Theories—and Relations between Them

Let $\mathcal{S}(\mathbb{R}^d)$ denote the Schwartz space of rapidly decreasing smooth functions on $\mathbb{R}^d$ and let $\mathcal{S}'(\mathbb{R}^d)$ be its dual (the space of tempered distributions). Then by the Bochner-Minlos theorem there exists a unique probability measure μ on $\Omega_1 := \mathcal{S}'(\mathbb{R}^d)$ defined by the property that

$$\int_{\Omega_1} e^{i\langle\omega,\phi\rangle} d\mu(\omega_1) = \exp(-\tfrac{1}{2}\|\phi\|^2)$$

for all $\phi \in \mathcal{S}(\mathbb{R}^d)$, where $\|\phi\|^2 = \int_{\mathbb{R}^d} \phi(x)^2 dx$ and $\langle\omega_1, \phi\rangle$ is the action of $\omega_1 \in \Omega_1$ on $\phi \in \mathcal{S}(\mathbb{R}^d)$. (See, e.g., [6, Section 2.1] for more details.) The measure μ is called the

Gaussian white noise probability measure.

Similarly, again by the Bochner-Minlos theorem, there exists a unique probability measure ν on $\Omega_2 := \mathcal{S}'(\mathbb{R}^d)$ defined by the property that

$$\int\limits_{\Omega_2} e^{i\langle\omega_2,\phi\rangle} d\nu(\omega_2) = \exp\left(\int\limits_{\mathbb{R}^d} (e^{i\phi(x)} - 1)dx\right)$$

for all $\phi \in \mathcal{S}(\mathbb{R}^d)$, where (again) $\langle\omega_2, \phi\rangle$ is the action of $\omega_2 \in \Omega_2$ on $\phi \in \mathcal{S}(\mathbb{R}^d)$. The measure ν is called the *Poissonian white noise probability measure.*

In the *Gaussian* case one exploits the isometry

$$E_\mu[\langle\cdot,\phi\rangle^2] = \|\phi\|^2 \ ; \qquad \phi \in S(\mathbb{R}^d)$$

to define

$$\langle\omega_1, \psi\rangle = \lim_{n\to\infty} \langle\omega_1, \phi_n\rangle,$$

with limit in $L^2(\mu)$, for all $\psi \in L^2(\mathbb{R}^d)$, where ϕ_n is a sequence in $\mathcal{S}(\mathbb{R}^d)$ such that $\phi_n \to \psi$ in $L^2(\mathbb{R}^d)$. In particular, this defines *d-parameter Brownian motion* as (the x-continuous version of)

$$B(x, \omega_1) := \langle\omega_1, \mathcal{X}_{[0,x]}\rangle \ ; \qquad x \in \mathbb{R}^d,\ \omega_1 \in \Omega_1 \ ,$$

where

$$\mathcal{X}_{[0,x]}(t_1, \dots, t_d) = \mathcal{X}_{[0,x_1]}(t_1) \cdots \mathcal{X}_{[0,x_d]}(t_d)$$

if $x = (x_1, \dots, x_d)$ with $x_j \geq 0$ for $j = 1, \dots, d$. (If $x_j < 0$, we put $\mathcal{X}_{[0,x_j]} = -\mathcal{X}_{[-x_j,0]}$.)

Similarly, in the *Poissonian* case one exploits the isometry

$$E_\nu\left[\left(\langle\cdot,\phi\rangle - \int\limits_{\mathbb{R}^d} \phi(x)dx\right)^2\right] = \|\phi\|^2 \ ; \qquad \phi \in \mathcal{S}(\mathbb{R}^d)$$

to define

$$\langle\omega_2, \psi\rangle = \lim_{n\to\infty} \langle\omega_2, \phi_n\rangle$$

with limit in $L^2(\nu)$, for all $\psi \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, where ϕ_n is a sequence in $\mathcal{S}(\mathbb{R}^d)$ converging to ψ in $L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. In particular, this defines the d-parameter Poisson process as (the right-continuous version of)

$$P(x, \omega_2) := \langle\omega_2, \mathcal{X}_{[0,x]}\rangle \ ; \qquad x \in \mathbb{R}^d,\ \omega_2 \in \Omega_2.$$

The *compensated Poisson process* is then defined by

$$Q(x,\omega_2) := P(x,\omega_2) - \prod_{j=1}^{d} x_j \; ; \qquad x=(x_1,\dots,x_d).$$

From here on one can construct stochastic integrals with respect to B and Q, respectively and the corresponding stochastic polynomials and so on. We summarize the corresponding concepts in the two cases in the list below. Unless otherwise mentioned we use the terminology of [6]. In particular, recall that $\{\eta_k\}_{k=1}^{\infty}$ is the basis of $L^2(\mathbb{R}^d)$ consisting of tensor products of the *Hermite functions*

$$\xi_n(x) = \pi^{-1/4}((n-1)!)^{-1/2} e^{-x^2/2} h_{n-1}(\sqrt{2}\,x) \; ; \qquad n\in\mathbb{N}; \quad x\in\mathbb{R},$$

where

$$h_m(y) = (-1)^m e^{y^2/2} \frac{d^m}{dy^m}(e^{-y^2/2}) \; ; \qquad m\in\mathbb{N}\cup\{0\}; \quad y\in\mathbb{R}$$

are the *Hermite polynomials.* The symbol $\mathcal{J}$ denotes the set of all multi-indices $\alpha=(\alpha_1,\dots,\alpha_n)$ where the α_i's are nonnegative integers and $n\in\mathbb{N}$.

In the following list we use the following notation. $x=(x_1,\dots,x_d)\in\mathbb{R}^d$, $\alpha=(\alpha_1,\dots,\alpha_n)\in\mathcal{J}$, $\varepsilon^{(k)}$ denotes the kth unit multi-index, namely $\varepsilon^{(k)}=(0,0,\dots,1)$ with 1 on kth place. (In [6] $\varepsilon^{(k)}$ is denoted by ε_k.) Furthermore, for any function $\xi(x)$ we define $\overline{\xi}=\int_{\mathbb{R}}\xi(x)dx$. Finally, $\hat{\otimes}$ denotes the symmetric tensor product and $\widehat{L}^2(\mathbb{R}^m)$ denotes the symmetric functions in $L^2(\mathbb{R}^m)$.

(2.1) **White noise probability measure**

GAUSSIAN: $\int_{\mathcal{S}'(\mathbb{R}^d)} e^{i\langle\omega_1,\phi\rangle} d\mu(\omega_1) = \exp(-\frac{1}{2}\|\phi\|^2)$; $\phi\in\mathcal{S}(\mathbb{R}^d)$.

POISSONIAN: $\int_{\mathcal{S}'(\mathbb{R}^d)} e^{i\langle\omega_2,\phi\rangle} d\nu(\omega_2) = \exp(\int_{\mathbb{R}^d}(e^{i\phi(x)}-1)dx)$; $\phi\in\mathcal{S}(\mathbb{R}^d)$.

(2.2) **Processes**[1]

GAUSSIAN: Brownian motion B: $B(x,\omega_1)=\langle\omega_1,\mathcal{X}_{[0,x]}\rangle$.

POISSONIAN: Poisson process P: $P(x,\omega_2)=\langle\omega_2,\mathcal{X}_{[0,x]}\rangle$. Compensated Poisson process Q: $Q(x,\omega_2)=P(x,\omega_2)-\prod_{k=1}^{d}x_k$.

(2.3) **Stochastic integrals**

[1] Note that both $\langle\omega_1,\mathcal{X}_{[0,x]}\rangle$ and $\langle\omega_2,\mathcal{X}_{[0,x]}\rangle$ are only defined almost everywhere with respect to μ and ν, respectively. Since μ and ν are not absolutely continuous with respect to each other, the two processes $B(x)$ and $P(x)$ are not the same even though they are defined by the same expression.

GAUSSIAN: $\langle\omega_1,\phi\rangle = \int_{\mathbb{R}^d}\phi(x)dB(x)$ a.s. μ for all $\phi\in L^2(\mathbb{R}^d)$.
POISSONIAN: $\langle\omega_2,\psi\rangle = \int_{\mathbb{R}^d}\psi(x)dP(x)$ a.s. ν, and
$\int_{\mathbb{R}^d}\psi(x)dQ(x) = \int_{\mathbb{R}^d}\psi(x)dP(x) - \overline{\psi}$ for all $\psi\in L^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$.

(2.4) **Stochastic polynomials**
GAUSSIAN: Hermite polynomials $H_\alpha(\omega_1) = \prod_j h_{\alpha_j}(\langle\omega_1,\eta_j\rangle)$,
POISSONIAN: Charlier polynomials $C_\alpha(\omega_2) = C_{|\alpha|}(\omega_2;\overbrace{\eta_1,\dots,\eta_1}^{\alpha_1},\dots,\overbrace{\eta_m,\dots,\eta_m}^{\alpha_m})$, where

$$C_n(\omega_2;\psi_1,\dots,\psi_n) = \frac{\partial^n}{\partial u_1\cdots\partial u_n}\exp\Big(\langle\omega_2,\ln(1+\sum_{j=1}^n u_j\psi_j)\rangle - \sum_{j=1}^n u_j\int_{\mathbb{R}^d}\psi_j(x)dx\Big)|_{u_1=\cdots=u_n=0},$$

for $\psi_j\in S(\mathbb{R}^d)$; $1\le j\le n$.

(2.4a) **Example**
GAUSSIAN: $H_{\varepsilon^{(k)}}(\omega_1) = h_1(\langle\omega_1,\eta_k\rangle) = \langle\omega_1,\eta_k\rangle = \int_{\mathbb{R}^d}\eta_k(x)dB(x)$.
POISSONIAN: $C_{\varepsilon^{(\ell)}}(\omega_2) = \langle\omega_2,\eta_\ell\rangle - \overline{\eta}_\ell = \int_{\mathbb{R}^d}\eta_\ell(x)dQ(x)$.

(2.5) **Expansion in terms of iterated integrals**
GAUSSIAN: If $f\in L^2(\mu)$, then $f = \sum_{n=0}^\infty\int_{\mathbb{R}^{nd}} f_n(x)dB^{\otimes n}(x)$, where $f_n\in\widehat{L}^2(\mathbb{R}^{nd})$. Furthermore $\|f\|^2_{L^2(\mu)} = \sum_{n=0}^\infty n!\|f_n\|^2_{L^2(\mathbb{R}^{nd})}$.
POISSONIAN: If $g\in L^2(\mu)$, then $g = \sum_{n=0}^\infty\int_{\mathbb{R}^{nd}} g_n(x)\,dQ^{\otimes n}(x)$, where $g_n\in\widehat{L}^2(\mathbb{R}^{nd})$. Furthermore $\|g\|^2_{L^2(\nu)} = \sum_{n=0}^\infty n!\|g_n\|^2_{L^2(\mathbb{R}^{nd})}$.

(2.6) **Expansion in terms of polynomials**
GAUSSIAN: If $f\in L^2(\mu)$, then $f(\omega_1) = \sum_{\alpha\in\mathcal{J}} a_\alpha H_\alpha(\omega_1)$, where $a_\alpha\in\mathbb{R}$. Furthermore $\|f\|^2_{L^2(\mu)} = \sum_{\alpha\in\mathcal{J}} a_\alpha^2\alpha!$.
POISSONIAN: If $g\in L^2(\nu)$, then $g(\omega_2) = \sum_{\beta\in\mathcal{J}} b_\beta C_\beta(\omega_2)$, where $b_\beta\in\mathbb{R}$. Furthermore $\|g\|^2_{L^2(\nu)} = \sum_{\beta\in\mathcal{J}} b_\beta^2\beta!$.

(2.7) **Relation between the two expansions**
GAUSSIAN: $\int_{\mathbb{R}^{|\alpha|d}}\eta^{\widehat{\otimes}\alpha}dB^{\otimes|\alpha|} = H_\alpha(\omega_1)$.
POISSONIAN: $\int_{\mathbb{R}^{|\alpha|d}}\eta^{\widehat{\otimes}\alpha}dQ^{\otimes|\alpha|} = C_\alpha(\omega_2)$.

(2.8) **Kondratiev norms**, $-1\le\rho\le 1$, $q\in\mathbb{Z}$
GAUSSIAN: If $F(\omega_1) = \sum_{\alpha\in\mathcal{J}} a_\alpha H_\alpha(\omega_1)$, then $\|F\|^2_{\rho,q,\mu} = \sum_\alpha a_\alpha^2(\alpha!)^{1+\rho}(2\mathbb{N})^{q\alpha}$.
POISSONIAN: if $G(\omega_2) = \sum_{\beta\in\mathcal{J}} b_\beta C_\beta(\omega_2)$, then $\|G\|^2_{\rho,q,\nu} = \sum_\beta b_\beta^2(\beta!)^{1+\rho}(2\mathbb{N})^{q\beta}$.

(2.9) **Kondratiev Hilbert spaces**
GAUSSIAN: $(\mathcal{S})_{\rho,q,\mu} := \{F; \|F\|_{\rho,q,\mu} < \infty\}$.
POISSONIAN: $(\mathcal{S})_{\rho,q,\nu} := \{G; \|G\|_{\rho,q,\nu} < \infty\}$.

(2.10) **Kondratiev stochastic test function spaces**, $0 \leq \rho \leq 1$
GAUSSIAN: $(\mathcal{S})_{\rho,\mu} = \bigcap_{r\in\mathbb{N}}(\mathcal{S})_{\rho,r,\mu}$, with projective topology.
POISSONIAN: $(\mathcal{S})_{\rho,\nu} = \bigcap_{r\in\mathbb{N}}(\mathcal{S})_{\rho,r,\nu}$, with projective topology.

(2.11) **Kondratiev stochastic distribution spaces**, $0 \leq \rho \leq 1$
GAUSSIAN: $(\mathcal{S})_{-\rho,\mu} = \bigcup_{r\in\mathbb{N}}(\mathcal{S})_{-\rho,-r,\mu}$, with inductive topology.
POISSONIAN: $(\mathcal{S})_{-\rho,\nu} = \bigcup_{r\in\mathbb{N}}(\mathcal{S})_{-\rho,-r,\nu}$, with inductive topology.

(2.12) **Hida spaces**
GAUSSIAN: $(\mathcal{S})_{\mu} := (\mathcal{S})_{0,\mu}, \qquad (\mathcal{S})^*_{\mu} := (\mathcal{S})_{-0,\mu}$.
POISSONIAN: $(\mathcal{S})_{\nu} := (\mathcal{S})_{0,\nu}, \qquad (\mathcal{S})^*_{\nu} := (\mathcal{S})_{-0,\nu}$.

(2.13) **White noise**
GAUSSIAN: $W(x,\omega_1) = \sum_{k=1}^{\infty} \eta_k(x) H_{\varepsilon^{(k)}}(\omega_1), \quad W(x,\cdot) \in (\mathcal{S})^*_{\mu}$ for all x.
POISSONIAN: $V(x,\omega_2) = \sum_{k=1}^{\infty} \eta_k(x) C_{\varepsilon^{(k)}}(\omega_2), \quad V(x,\cdot) \in (\mathcal{S})^*_{\nu}$ for all x.

(2.14) **Wick product**
GAUSSIAN: $(\sum_\alpha a_\alpha H_\alpha) \diamond (\sum_\beta b_\beta H_\beta) = \sum_{\alpha,\beta} a_\alpha b_\beta H_{\alpha+\beta} = \sum_\gamma (\sum_{\alpha+\beta=\gamma} a_\alpha b_\beta) H_\gamma$.
POISSONIAN: $(\sum_\alpha a_\alpha C_\alpha) \diamond (\sum_\beta b_\beta C_\beta) = \sum_{\alpha,\beta} a_\alpha b_\beta C_{\alpha+\beta} = \sum_\gamma (\sum_{\alpha+\beta=\gamma} a_\alpha b_\beta) C_\gamma$.

(2.15) **Properties of the Wick product**
GAUSSIAN: If $f, g \in (\mathcal{S})_{1,\mu}$, then $f \diamond g \in (\mathcal{S})_{1,\mu}$, and if $F, G \in (\mathcal{S})_{-1,\mu}$, then $F \diamond G \in (\mathcal{S})_{-1,\mu}$.
POISSONIAN: If $f, g \in (\mathcal{S})_{1,\nu}$, then $f \diamond g \in (\mathcal{S})_{1,\nu}$, and if $F, G \in (\mathcal{S})_{-1,\nu}$, then $F \diamond G \in (\mathcal{S})_{-1,\nu}$.

(2.16) **Wick products in terms of iterated integrals**
GAUSSIAN: $(\int_{\mathbb{R}^{nd}} f_n dB^{\otimes n}) \diamond (\int_{\mathbb{R}^{md}} f_m dB^{\otimes m}) = \int_{\mathbb{R}^{(n+m)d}} (f_n \widehat{\otimes} f_m) dB^{\otimes(n+m)}$.
POISSONIAN: $(\int_{\mathbb{R}^{nd}} g_n dQ^{\otimes n}) \diamond (\int_{\mathbb{R}^{md}} g_m dQ^{\otimes m}) = \int_{\mathbb{R}^{(n+m)d}} (g_n \widehat{\otimes} g_m) dQ^{\otimes(n+m)}$.

(2.17) **Wick powers and iterated integrals**[2]
GAUSSIAN: $\langle \omega_1, \phi \rangle^{\diamond n} = \int_{\mathbb{R}^{nd}} \phi^{\otimes n} dB^{\otimes n}$ for $\phi \in L^2(\mathbb{R}^d)$.
POISSONIAN: $(\langle \omega_2, \psi \rangle - \overline{\psi})^{\diamond n} = \int_{\mathbb{R}^{nd}} \psi^{\otimes n} dQ^{\otimes n}$ for $\psi \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$.

[2] These formulas follow by combining (2.3) and (2.16).

(2.18) **Wick exponentials**[3]

Gaussian: $\exp^\diamond(\langle\omega_1,\phi\rangle) := \sum_{n=0}^\infty \frac{1}{n!}\langle\omega_1,\phi\rangle^{\diamond n} = \exp(\langle\omega_1,\phi\rangle - \frac{1}{2}\|\phi\|^2)$ for all $\phi \in L^2(\mathbb{R}^d)$.

Poissonian: $\exp^\diamond(\langle\omega_2,\psi\rangle) := \sum_{n=0}^\infty \frac{1}{n!}\langle\omega_2,\psi\rangle^{\diamond n} = \exp(\langle\omega_2,\ln(1+\psi)\rangle)$ for all $\psi \in L^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$, $\psi \geq -1+\varepsilon$ for some $\varepsilon > 0$.

(2.19) **Wick powers and Wick exponentials**[4]

Gaussian: $\langle\omega_1,\phi\rangle^{\diamond n} = \frac{\partial^n}{\partial u^n}\exp^\diamond(\langle\omega_1,u\phi\rangle)|_{u=0}$
$= \frac{\partial^n}{\partial u^n}\exp(\langle\omega_1,u\phi\rangle - \frac{1}{2}u^2\|\phi\|^2)|_{u=0}$ for all $\phi \in L^2(\mathbb{R}^d)$.

Poissonian: $\langle\omega_2,\psi\rangle^{\diamond n} = \frac{\partial^n}{\partial u^n}\exp^\diamond(\langle\omega_2,u\psi\rangle)|_{u=0}$
$= \frac{\partial^n}{\partial u^n}\exp(\langle\omega_2,\ln(1+u\psi)\rangle)|_{u=0}$ for all $\psi \in L^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$.

(2.20) **Wick polynomials and Wick exponentials**[4]

Gaussian: $\langle\omega_1,\phi_1\rangle \diamond\cdots\diamond \langle\omega_1,\phi_n\rangle = \frac{\partial^n}{\partial u_1\cdots\partial u_n}\exp^\diamond(\langle\omega_1,\sum_{j=1}^n u_j\phi_j\rangle)|_{u_1=\cdots=u_n=0}$
$= \frac{\partial^n}{\partial u_1\cdots\partial u_n}\exp(\langle\omega_1,\sum_{j=1}^n u_j\phi_j\rangle - \frac{1}{2}\|\sum_{j=1}^n u_j\phi_j\|^2)|_{u_1=\cdots=u_n=0}$
for $\phi_j \in L^2(\mathbb{R}^d)$ for $1 \leq j \leq n$.

Poissonian: $\langle\omega_2,\psi_1\rangle \diamond\cdots\diamond \langle\omega_2,\psi_n\rangle = \frac{\partial^n}{\partial u_1\cdots\partial u_n}\exp^\diamond(\langle\omega_2,\sum_{j=1}^n u_j\psi_j\rangle)|_{u_1=\cdots=u_n=0}$
$= \frac{\partial^n}{\partial u_1\cdots\partial u_n}\exp(\langle\omega_2,\ln(1+\sum_{j=1}^n u_j\psi_j)\rangle)|_{u_1=\cdots=u_n=0}$
for $\psi_j \in L^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$ and $1 \leq j \leq n$.

(2.21) **Representation of stochastic polynomials in terms of Wick exponentials**[5]

Gaussian: $H_\alpha(\omega_1) = \partial^\alpha \exp^\diamond(\int_{\mathbb{R}^d}\sum_{j=1}^n u_j\eta_j(x)dB(x))|_{u_1=\cdots=u_n=0}$,
where $\partial^\alpha = \frac{\partial^{|\alpha|}}{\partial u_1^{\alpha_1}\cdots\partial u_n^{\alpha_n}}$.

Poissonian: $C_\alpha(\omega_2) = \partial^\alpha \exp^\diamond(\int_{\mathbb{R}^d}\sum_{j=1}^n u_j\eta_j(x)dQ(x))|_{u_1=\cdots=u_n=0}$.

(2.22) **Skorohod integrals**

Gaussian: If $Y(t,\omega_1) = \sum_{n=0}^\infty \int_{\mathbb{R}^n} f_n(t,\cdot)dB^{\otimes n}$, where $\sum_{n=0}^\infty (n+1)!\|\widetilde{f}_n\|_{L^2(\mathbb{R}^{n+1})} < \infty$, then $\int_{\mathbb{R}} Y(t)\delta B(t) := \sum_{n=0}^\infty \int_{\mathbb{R}^{n+1}}\widetilde{f}_n dB^{\otimes(n+1)}$. Here

[3] The Gaussian case can be found, e.g., in [6, Lemma 2.6.16]. For a proof in the Poissonian case and when $\alpha = 1$ see Corollary 2.3 below.

[4] This follows from (2.18).

[5] The Gaussian case follows from (2.4), the fact that $h_{\alpha_j}(\langle\omega_1,\eta_j\rangle) = \langle\omega_1,\eta_j\rangle^{\diamond\alpha_j}$ (which is seen from (2.7) and (2.17)) plus (2.20). To prove the Poissonian case note that by (2.3) we have, for $\psi \in L^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$, $\int_{\mathbb{R}^d}\psi(x)dQ(x) = \int_{\mathbb{R}^d}\psi(x)dP(x) - \overline{\psi} = \langle\omega_2,\psi\rangle - \overline{\psi}$. Therefore, by (2.18) we get $\exp^\diamond(\int_{\mathbb{R}^d}\psi(x)dQ(x)) = \exp^\diamond(\langle\omega_2,\psi\rangle - \overline{\psi}) = \exp(\langle\omega_2,\ln(1+\psi)\rangle - \overline{\psi})$, assuming that $\psi \geq -1+\varepsilon$ for some $\varepsilon > 0$. Applying this to appropriate approximations to $\psi_0 = \sum_{j=1}^n u_j\eta_j$ we see that (2.21) follows from the definition of C_α in (2.4).

$\widetilde{f}_n$ is the symmetrization of $f_n(t, x_1, \ldots, x_n)$ with respect to all its $n+1$ variables.
POISSONIAN: If $Z(t, \omega_2) = \sum_{n=0}^{\infty} \int_{\mathbb{R}^n} g_n(t, \cdot) dQ^{\otimes n}$, where
$\sum_{n=0}^{\infty} (n+1)! \|\widetilde{g}_n\|_{L^2(\mathbb{R}^{n+1})} < \infty$, then $\int_{\mathbb{R}} Z(t) \delta Q(t) := \sum_{n=0}^{\infty} \int_{\mathbb{R}^{n+1}} \widetilde{g}_n dQ^{\otimes(n+1)}$.

(2.23) **Connection between Skorohod integrals and Ito-Poisson integrals**[6]
GAUSSIAN: $\int_{\mathbb{R}} Y(t) \delta B(t) = \int_{\mathbb{R}} Y(t) dB(t)$ if $Y(t)$ is $\mathcal{F}_t$-adapted.
POISSONIAN: $\int_{\mathbb{R}} Z(t) \delta Q(t) = \int_{\mathbb{R}} Z(t) dQ(t)$ if $Z(t)$ is adapted to the filtration $\mathcal{G}_t$ generated by $\{P(s, \cdot)\}_{s \leq t}$.

(2.24) **White noise description of Skorohod integrals**[6]
GAUSSIAN: If $Y(t)$ is Skorohod integrable, see (2.22), then
$\int_{\mathbb{R}} Y(t) \delta B(t) = \int_{\mathbb{R}} Y(t) \diamond W(t) dt$.
POISSONIAN: If $Z(t)$ is Skorohod integrable, see (2.22), then
$\int_{\mathbb{R}} Z(t) \delta Q(t) = \int_{\mathbb{R}} Z(t) \diamond V(t) dt$.

(2.25) **Hermite transform**
GAUSSIAN: If $F(\omega_1) = \sum_{\alpha \in \mathcal{J}} a_\alpha H_\alpha(\omega_1)$, then $(\mathcal{H}F)(z) = \sum_{\alpha \in \mathcal{J}} c_\alpha z^\alpha$ for $z = (z_1, z_2, \ldots)$, where $z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \cdots$.
POISSONIAN: If $G(\omega_2) = \sum_{\beta \in \mathcal{J}} b_\beta C_\beta(\omega_2)$, then $(\mathcal{H}G)(z) = \sum_{\beta \in \mathcal{J}} b_\beta z^\beta$.

(2.26) S-**transform (definition)**, $\rho < 1$
GAUSSIAN: If $F \in (\mathcal{S})_{-\rho,\mu}$, then $(SF)(\phi) = \langle F, \exp^\diamond \int_{\mathbb{R}^d} \phi \, dB \rangle$ for all $\phi \in \mathcal{S}(\mathbb{R}^d)$.
POISSONIAN: If $G \in (\mathcal{S})_{-\rho,\nu}$, then $(SG)(\phi) = \langle G, \exp^\diamond \int_{\mathbb{R}^d} \phi dQ \rangle$ for all $\phi \in \mathcal{S}(\mathbb{R}^d)$.

(2.27) S-**transform (properties)**, $\rho < 1$
GAUSSIAN: If $F = \sum_\alpha a_\alpha H_\alpha(\omega_1) \in (\mathcal{S})_{-\rho,\mu}$, then $(SF)(\phi) = \sum_\alpha a_\alpha (\phi, \eta)^\alpha$ for all $\phi \in S(\mathbb{R}^d)$.
POISSONIAN: If $G = \sum_\beta b_\beta C_\beta(\omega_2) \in (\mathcal{S})_{-\rho,\nu}$, then $(SG)(\phi) = \sum_\beta b_\beta (\phi, \eta)^\beta$ for all $\phi \in S(\mathbb{R}^d)$.

The unitary correspondence $\mathcal{U}$
We review some other concepts and results from [1]: Define $\mathcal{U} : (\mathcal{S})_{-1,\mu} \to (\mathcal{S})_{-1,\nu}$ by

$$\mathcal{U}(\sum_\alpha a_\alpha H_\alpha) = \sum_\alpha a_\alpha C_\alpha.$$

[6] For proofs and more details in the Gaussian case see, e.g., [6, Theorem 2.5.9] and the references therein. For a proof in the Poissonian case, see [6, Theorem 5.1]. This proof is valid in the more general setting of *colored* noise.

Then $\mathcal{U}$ is a continuous bijection and $\mathcal{U}|_{(\mathcal{S})_{\rho,q,\mu}}$ is an isometry onto $(\mathcal{S})_{\rho,q,\nu}$ for all $\rho \in [-1,1]$ and all $q \in \mathbb{Z}$. Moreover, $\mathcal{U}$ preserves the Wick product:

$$\mathcal{U}(F \diamond G) = (\mathcal{U}F) \diamond (\mathcal{U}G). \tag{2.28}$$

Some examples: We have that

$$\mathcal{U}(\langle \omega_1, \psi \rangle) = \langle \omega_2, \psi \rangle - \overline{\psi}\ ; \qquad \psi \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}) \tag{2.29}$$

which follows from (2.4a). In particular,

$$\mathcal{U}(B(x, \omega_1)) = Q(x, \omega_2), \tag{2.30}$$

$$\mathcal{U}(F^{\diamond n}(\omega_1)) = (\mathcal{U}F)^{\diamond n}(\omega_2). \tag{2.31}$$

by using induction on (2.28). From the Wick-Taylor expansions and (2.31)

$$\mathcal{U}(\exp^{\diamond} F(\omega_1)) = \exp^{\diamond} \mathcal{U}F(\omega_2). \tag{2.32}$$

Recall the following fundamental result from [1]: (For notation see Corollary 4.9.7 in [6].)

Theorem 2.1 *Suppose $F(t,x) : \mathbb{R}^{d+1} \to (\mathcal{S})_{-1,\mu}$ is a solution of a Gaussian Wick type stochastic differential equation of the form*

$$A^{\diamond}(t, x, \partial_t, \nabla_x, F, W) = 0.$$

Then $G := \mathcal{U}F$ is a solution of the corresponding Poissonian Wick type equation

$$A^{\diamond}(t, x, \partial_t, \nabla_x, G, V) = 0,$$

and conversely. Here $A^{\diamond}$ denotes the Gaussian (resp. Poissonian) Wick version of a given function A.

This elegant result shows that when considering Wick type stochastic differential equations it is—at least in principle—enough to know the solution in the Gaussian case. The solution in the Poissonian case then follows by applying the map $\mathcal{U}$. We illustrate this by considering a simple example:

Example 2.2 Let $X(t)$ be the solution of the (1-parameter) SDE

$$dX(t) = \psi(t) X(t) dQ(t)\ ; \qquad t > 0 \tag{2.33}$$

where $Q(t) = Q(t, \omega_2)$ is the compensated Poisson process and $\psi(t)$ is a continuous, deterministic function, $\psi(t) > -1+\varepsilon$ for all t, for some $\varepsilon > 0$. According to the Poissonian white noise theory, the equation can be regarded as a Wick type stochastic differential equation in the (Poisson) Hida space $(\mathcal{S})^*_\nu$, namely the equation

$$\frac{dX(t)}{dt} = \psi(t)X(t) \diamond V(t) \tag{2.34}$$

where $\diamond$ denotes the Poissonian Wick product and $V(t) = dQ(t)/dt \in (\mathcal{S})^*_\nu$ is Poissonian white noise.

The corresponding Gaussian white noise differential equation is

$$\frac{dY(t)}{dt} = \psi(t)Y(t) \diamond W(t) \tag{2.35}$$

where $\diamond$ denotes the Gaussian Wick product and $W(t) = dB(t)/dt \in (\mathcal{S})^*_\mu$ is Gaussian white noise.

Using Wick calculus on equation (2.35), we get the solution

$$Y(t) = Y(0) \diamond \exp^\diamond \big(\int_0^t \psi(s)dB(s)\big) \tag{2.36}$$

(see, e.g., [6, Theorem 3.1.2] for a more general calculation).

If $Y(0) = X(0)$ is a constant, then we get

$$Y(t) = X(0) \exp^\diamond (\int_0^t \psi(s)dB(s)).$$

According to the correspondence $\mathcal{U}$ the solution $X(t)$ of (2.34) is therefore

$$X(t) = \mathcal{U}(Y(t)) = X(0) \exp^\diamond (\int_0^t \psi(s)dQ(s)). \tag{2.37}$$

In the Gaussian case we know that

$$\exp^\diamond (\int_0^t \psi(s)dB(s)) = \exp \big(\int_0^t \psi(s)dB(s) - \tfrac{1}{2} \int_0^t \psi(s)^2 ds\big),$$

but what is $\exp^\diamond(\int_0^t \psi(s)dQ(s))$ in the Poissonian case? The answer can be found using (2.18):

$$
\begin{aligned}
\exp^{\diamond}\Big(\int_0^t \psi(s)dQ(s)\Big) &= \exp^{\diamond}\Big(\int_0^t \psi(s)dP(s) - \int_0^t \psi(s)ds\Big) \\
&= \exp^{\diamond}\Big(\langle \omega_2, \mathcal{X}_{[0,t]}\psi\rangle - \int_0^t \psi(s)ds\Big) \\
&= \exp\Big(\langle \omega_2, \ln(1+\mathcal{X}_{[0,t]}\psi)\rangle - \int_0^t \psi(s)ds\Big) \\
&= \exp\Big(\int_0^t \ln(1+\psi(s))dP(s) - \int_0^t \psi(s)ds\Big).
\end{aligned}
\tag{2.38}
$$

Combining (2.37) and (2.38) we get the following expression for the solution $X(t)$ of (2.33):

$$
X(t) = X(0)\exp\Big(\int_0^t \ln(1+\psi(s))dP(s) - \int_0^t \psi(s)ds\Big).
\tag{2.39}
$$

It may be instructive to see that we can verify the formula (2.39) by using (2.34) directly as follows:

The Poisson process is described by a sequence of stopping times $0 < \tau_1 < \tau_2 < \cdots$ at which P jumps with size 1. Between these stopping times $X(t)$ satisfies the equation

$$
dX(t) = -\psi(t)X(t)dt\ ; \qquad \tau_k \le t < \tau_{k+1},
$$

since $dP(t) = dQ(t) - dt$. Hence

$$
X(t) = X(\tau_k)\exp\Big(-\int_{\tau_k}^t \psi(s)ds\Big)\ ; \qquad \tau_k \le t < \tau_{k+1}.
\tag{2.40}
$$

At the jumping times τ_k the value of X increases by

$$
\Delta X(\tau_k) := X(\tau_k) - X(\tau_k^-) = \psi(\tau_k)X(\tau_k^-).
$$

Hence

$$
X(\tau_k) = X(\tau_k^-)(1+\psi(\tau_k)) = X(\tau_k^-)\exp(\ln(1+\psi(\tau_k))).
\tag{2.41}
$$

Combining (2.40) and (2.41) we get

$$
\begin{aligned}
X(t) &= X(0)\prod_{\tau_k \le t} \exp(\ln(1+\psi(\tau_k)))\exp\Big(-\int_{\tau_k}^{t\wedge\tau_{k+1}} \psi(s)ds\Big) \\
&= X(0)\exp\Big(\sum_{\tau_k \le t} \ln(1+\psi(\tau_k)) - \int_{\tau_k}^{t\wedge\tau_{k+1}} \psi(s)ds\Big) \\
&= X(0)\exp\Big(\int_0^t \ln(1+\psi(s))dP(s) - \int_0^t \psi(s)ds\Big),
\end{aligned}
$$

which is (2.39). This ends Example 2.2.

Since (2.39) is equivalent to (2.37), the above argument proves the following result:

Corollary 2.3 *Suppose ψ is continuous, $\psi > -1$ and*

$$\int_0^t (\ln(1+\psi(s)))^2 ds < \infty. \tag{2.42}$$

Then

$$\exp^{\diamond}\Big(\int_0^t \psi(s)dQ(s)\Big) = \exp\Big(\int_0^t \ln(1+\psi(s))dP(s) - \int_0^t \psi(s)ds\Big). \tag{2.43}$$

Remark The above argument was carried out under the assumption that $\psi \geq -1+\varepsilon$ for some $\varepsilon > 0$. However, the identity (2.43) extends to the case (2.42) because

$$\begin{aligned} E\Big[\Big(\int_0^t \ln(1+\psi(s))dP(s)\Big)^2\Big] &= E\Big[\Big(\int_0^t \ln(1+\psi(s))dQ(s) + \int_0^t \ln(1+\psi(s))ds\Big)^2\Big] \\ &= \int_0^t \ln(1+\psi(s))^2 ds + \Big(\int_0^t \ln(1+\psi(s))ds\Big)^2. \end{aligned}$$

3 The Mixed Gaussian-Poissonian Case

Now suppose there are m noise sources, of which k are Gaussian white noises and the remaining $m-k$ are Poissonian. Then we define the *combined Gaussian-Poissonian white noise probability measure* θ on $\Omega := \mathcal{S}'(\mathbb{R}^d) \times \cdots \times \mathcal{S}'(\mathbb{R}^d)$ (m factors) by

$$d\theta(\omega_1, \ldots, \omega_m) = d\mu(\omega_1)\cdots d\mu(\omega_k)d\nu(\omega_{k+1})\cdots d\nu(\omega_m) \tag{3.1}$$

where $\omega = (\omega_1, \ldots, \omega_m) \in \Omega$.
For notational simplicity we will from now on assume that $m = 2$ and $k = 1$. For $\omega = (\omega_1, \omega_2) \in \Omega = \mathcal{S}'(\mathbb{R}^d) \times \mathcal{S}'(\mathbb{R}^d)$ and $\phi = (\phi_1, \phi_2) \in \mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$ we define

$$\langle \omega, \phi \rangle = \langle \omega_1, \phi_1 \rangle + \langle \omega_2, \phi_2 \rangle. \tag{3.2}$$

Then by (2.1) we see that

$$\int_\Omega e^{i\langle \omega, \phi \rangle} d\theta(\omega) = \exp\Big(-\tfrac{1}{2}\|\phi_1\|^2 + \int_{\mathbb{R}^d} (e^{i\phi_2(x)} - 1)dx \Big). \tag{3.3}$$

Let $\Gamma = \mathcal{J} \times \mathcal{J}$ be the set of all pairs $\gamma = (\alpha, \beta)$ of multi-indices $\alpha, \beta \in \mathcal{J}$, and define

$$(3.4) \qquad M_\gamma(\omega_1, \omega_2) = H_\alpha(\omega_1) G_\beta(\omega_2) \; ; \qquad \gamma = (\alpha, \beta) \in \Gamma.$$

For example, this gives (using (2.4a))

$$(3.5) \qquad M_{(\varepsilon^{(k)},0)}(\omega_1, \omega_2) = H_{\varepsilon^{(k)}}(\omega_1) = \langle \omega_1, \eta_k \rangle,$$

$$(3.6) \qquad M_{(0,\varepsilon^{(\ell)})}(\omega_1, \omega_2) = C_{\varepsilon^{(\ell)}}(\omega_2) = \langle \omega_2, \eta_\ell \rangle - \overline{\eta}_\ell.$$

From tensor product considerations, we see that we have the following analogue of (2.6): Any $f \in L^2(\theta)$ can be written (uniquely)

$$(3.7) \qquad f(\omega_1, \omega_2) = \sum_{\gamma \in \Gamma} c_\gamma M_\gamma(\omega_1, \omega_2)$$

where $c_\gamma \in \mathbb{R}$. Moreover, we have the isometry

$$(3.8) \qquad \|f\|^2_{L^2(\theta)} = \sum_{\gamma \in \Gamma} c_\gamma^2 \gamma!$$

where $\gamma! = \alpha! \, \beta!$ if $\gamma = (\alpha, \beta) \in \Gamma$.

We proceed as in the list in Section 2. For $\rho \in [-1, 1]$ and $q \in \mathbb{Z}$ define the *Kondratiev norms*

$$(3.9) \qquad \|F\|^2_{\rho,q,\theta} = \sum_{\gamma \in \Gamma} c_\gamma^2 (\gamma!)^{1+\rho} (2\mathbb{N})^{q\gamma} \text{ if } F = \sum_{\gamma \in \Gamma} c_\gamma M_\gamma(\omega) \; ,$$

where

$$(2\mathbb{N})^{q\gamma} = (2\mathbb{N})^{q\alpha} (2\mathbb{N})^{q\beta}, \qquad \gamma = (\alpha, \beta) \in \Gamma$$

and the *Kondratiev Hilbert spaces*

$$(3.10) \qquad (\mathcal{S})_{\rho,q,\theta} = \{F; \|F\|_{\rho,q,\theta} < \infty\}.$$

For $0 \leq \rho \leq 1$ we define the *Kondratiev stochastic test function spaces*

$$(3.11) \qquad (\mathcal{S})_{\rho,\theta} = \bigcap_{r \in \mathbb{N}} (\mathcal{S})_{\rho,r,\theta} \qquad \text{with projective topology}$$

and the *Kondratiev stochastic distribution spaces*

$$(3.12) \qquad (\mathcal{S})_{-\rho,\theta} = \bigcup_{r \in \mathbb{N}} (\mathcal{S})_{-\rho,-r,\theta} \qquad \text{with inductive topology.}$$

Furthermore, we define the *Hida test function space*

$$(\mathcal{S})_\theta := (\mathcal{S})_{0,\theta}, \tag{3.13}$$

and the *Hida distribution space*

$$(\mathcal{S})^*_\theta := (\mathcal{S})_{-0,\theta}. \tag{3.14}$$

The *combined Gaussian-Poissonian white noise* $U(x,\omega)$ is defined by

$$U(x,\omega) = (W(x,\omega_1), V(x,\omega_2)), \tag{3.15}$$

where

$$W(x,\omega_1) = \sum_{k=1}^{\infty} \eta_k(x) H_{\varepsilon^{(k)}}(\omega_1), \qquad V(x,\omega_2) = \sum_{\ell=1}^{\infty} \eta_\ell(x) C_{\varepsilon^{(\ell)}}(\omega_2)$$

is Gaussian and Poissonian white noise, respectively. If

$$F(\omega) = \sum_{\lambda\in\Gamma} a_\lambda M_\lambda(\omega) \in (\mathcal{S})_{-1,\theta},$$

and

$$G(\omega) = \sum_{\delta\in\Gamma} b_\delta M_\delta(\omega) \in (\mathcal{S})_{-1,\theta},$$

we define their *Wick product* $F \diamond G$ by

$$(F\diamond G)(\omega) = \sum_{\lambda,\delta\in\Gamma} a_\lambda b_\delta M_{\lambda+\delta}(\omega) = \sum_{\gamma\in\Gamma}\Big(\sum_{\lambda+\delta=\gamma} a_\lambda b_\delta\Big) M_\gamma(\omega). \tag{3.16}$$

As in the Gaussian case one can prove that

$$F, G \in (\mathcal{S})_{-1,\theta} \Rightarrow F\diamond G \in (\mathcal{S})_{-1,\theta}, \tag{3.17}$$

$$f, g \in (\mathcal{S})_{1,\theta} \Rightarrow f\diamond g \in (\mathcal{S})_{1,\theta} \tag{3.18}$$

Examples

a) Using (3.5) and (3.6) we get

$$\begin{aligned}
\langle\omega_1,\eta_k\rangle \diamond \langle\omega_2,\eta_\ell\rangle &= M_{(\varepsilon^{(k)},0)}(\omega) \diamond (M_{(0,\varepsilon^{(\ell)})}(\omega) + \overline{\eta}_\ell) \\
&= M_{(\varepsilon^{(k)},\varepsilon^{(\ell)})}(\omega) + \overline{\eta}_\ell M_{(\varepsilon^{(k)},0)}(\omega) \\
&= H_{\varepsilon^{(k)}}(\omega_1) C_{\varepsilon^{(\ell)}}(\omega_2) + \overline{\eta}_\ell H_{\varepsilon^{(k)}}(\omega_1) \\
&= \langle\omega_1,\eta_k\rangle(\langle\omega_2,\eta_\ell\rangle - \overline{\eta}_\ell) + \overline{\eta}_\ell\langle\omega,\eta_k\rangle = \langle\omega_1,\eta_k\rangle\cdot\langle\omega_2,\eta_\ell\rangle.
\end{aligned} \tag{3.19}$$

b) The Wick product of Gaussian and Poissonin white noise reads

$$W(x,\omega_1) \diamond V(y,\omega_2) = \Big(\sum_k \eta_k(x) H_{\varepsilon^{(k)}}(\omega_1)\Big) \diamond \Big(\sum_\ell \eta_\ell(y) C_{\varepsilon^{(\ell)}}(\omega_2)\Big)$$

$$= \sum_{k,\ell} \eta_k(x)\eta_\ell(y) H_{\varepsilon^{(k)}}(\omega_1) C_{\varepsilon^{(\ell)}}(\omega_2), \tag{3.20}$$

which can be written

$$\sum_{k,\ell} \eta_k(x)\eta_\ell(y) M_{(\varepsilon^{(k)},\varepsilon^{(\ell)})}(\omega)$$

and therefore

$$\begin{aligned}\|W(x)\diamond V(y)\|^2_{0,-r,\theta} &= \sum_{k,\ell} \eta_k(x)^2\eta_\ell(y)^2 \varepsilon^{(k)}!\varepsilon^{(\ell)}!(2\mathbb{N})^{-r(\varepsilon^{(k)},\varepsilon^{(\ell)})}\\ &= \sum_{k,\ell} \eta_k(x)^2\eta_\ell(y)^2(2k)^{-r}(2\ell)^{-r}\\ &= \Big(\sum_k \eta_k(x)^2(2k)^{-r}\Big)\Big(\sum_\ell \eta_\ell(y)^2(2\ell)^{-r}\Big) < \infty \qquad \text{for } r>1\end{aligned}$$

because

$$\sup_{x,k} |\eta_k(x)| < \infty. \tag{3.21}$$

Hence

$$W(x)\diamond V(y) \in (\mathcal{S})^*_\theta \qquad \text{for all } x, y.$$

c) The preceding example illustrates a general result: If

$$F(\omega) = F(\omega_1) = \sum_{\alpha\in\mathcal{J}} a_\alpha H_\alpha(\omega_1) \in (\mathcal{S})_{-1,\theta},$$

$$G(\omega) = G(\omega_2) = \sum_{\beta\in\mathcal{J}} b_\beta C_\beta(\omega_2) \in (\mathcal{S})_{-1,\theta},$$

then

$$(F\diamond G)(\omega_1,\omega_2) = F(\omega_1)\cdot G(\omega_2). \tag{3.22}$$

The proof is an immediate consequence of the definition (3.16):

$$\begin{aligned}(F\diamond G)(\omega_1,\omega_2) &= \sum_{\alpha,\beta\in\mathcal{J}} a_\alpha b_\beta M_{(\alpha,\beta)}(\omega_1,\omega_2) = \sum_{\alpha,\beta\in\mathcal{J}} a_\alpha b_\beta H_\alpha(\omega_1) C_\beta(\omega_2)\\ &= \Big(\sum_{\alpha\in\mathcal{J}} a_\alpha H_\alpha(\omega_1)\Big)\cdot\Big(\sum_{\beta\in\mathcal{J}} b_\beta C_\beta(\omega_2)\Big) = F(\omega_1)\cdot G(\omega_2).\end{aligned}$$

The *Hermite transform* of $F(\omega) = \sum_{\gamma \in \Gamma} c_\gamma M_\gamma(\omega) \in (\mathcal{S})_{-1,\theta}$ is the function $\mathcal{H}F\colon \mathbb{C}_0^{\mathbb{N}} \times \mathbb{C}_0^{\mathbb{N}} \to \mathbb{C}$ defined by

$$\widetilde{F}(z,w) := \mathcal{H}F(z,w) := \sum_{\gamma \in \Gamma} c_\gamma z^\alpha w^\beta\ ; \qquad \gamma = (\alpha, \beta), \tag{3.23}$$

where

$$z = (z_1, z_2, \dots) \in \mathbb{C}_0^{\mathbb{N}}\ , \qquad w = (w_1, w_2, \dots) \in \mathbb{C}_0^{\mathbb{N}},$$

and

$$z^\alpha w^\beta = z_1^{\alpha_1} z_2^{\alpha_2} \cdots w_1^{\beta_1} w_2^{\beta_2} \cdots$$

if $\alpha = (\alpha_1, \alpha_2 \dots)$, $\beta = (\beta_1, \beta_2, \dots)$.

The Hermite transform solution scheme

Consider a Gaussian-Poissonian white noise differential equation of Wick type in the unknown $F\colon \mathbb{R}^d \to (\mathcal{S})_{-1,\theta}$:

$$A^\diamond(t, x, \partial_t, \nabla_x, F, U) = 0 \tag{3.24}$$

where $A^\diamond$ is the Wick version of a given function A. We apply the Hermite transform to this equation and obtain

$$\widetilde{A}(t, x, \partial_t, \nabla_x, f(z,w), \widetilde{U}(z,w)) = 0, \tag{3.25}$$

where $f = \widetilde{F}$. Suppose that for each z, w we can solve the deterministic differential equation (3.25) for $f = f(t,x,z,w)$. Then under certain conditions we get that $f(t,x,z,w)$ is the Hermite transform of some $F : \mathbb{R}^d \to (\mathcal{S})_{-1,\theta}$. Moreover, F solves (3.24). Sufficient conditions for this are given in [6, Theorem 4.1.1] in the Gaussian case and a similar result can easily be proved in the mixed Gaussian-Poissonian case. We omit the details.

4 Applications to Gaussian-Poissonian SDEs

We now illustrate the theory by applying it to some stochastic (ordinary and partial) differential equations driven by a Brownian motion and (an independent) Poisson process.

Example 4.1 (Geometric Brownian-Poissonian motion) Consider the equation

$$dX(t) = a(t)X(t)dt + \phi(t)X(t)dB(t) + \psi(t)X(t)dQ(t) \tag{4.1}$$

where $a(t)$, $\phi(t)$, and $\psi(t)$ are deterministic functions, ψ is continuous, $\psi > -1$ and

$$\int_0^T (|a(t)| + \phi^2(t) + (\ln(1+\psi(t)))^2)dt < \infty \ .$$

In view of (2.24) we can rewrite the equation as

$$\begin{aligned} \frac{dX(t)}{dt} &= X(t) \diamond [a(t) + \phi(t)W(t) + \psi(t)V(t)] \\ &= X(t) \diamond [a(t) + (\phi(t), \psi(t)) \cdot U(t)]. \end{aligned}$$

Applying the usual trick of Wick-multiplying the equation by the integrating factor

$$\exp^{\diamond}\left[- \int_0^t (a(s) + \phi(s)W(s) + \psi(s)V(s))ds \right]$$

gives us the solution

$$X(t) = X(0) \diamond \exp^{\diamond}\left[\int_0^t a(s)ds + \int_0^t \phi(s)dB(s) + \int_0^t \psi(s)dQ(s) \right]. \tag{4.2}$$

If $X(0)$ is constant, this may be written (using (2.18) and Corollary 2.3)

$$\begin{aligned} X(t) = X(0) \exp\Bigg[&\int_0^t (a(s) - \tfrac{1}{2}\phi(s)^2 - \psi(s))ds \\ &+ \int_0^t \phi(s)dB(s) + \int_0^t \ln(1+\psi(s))dP(s) \Bigg]. \end{aligned} \tag{4.3}$$

Example 4.2 Consider the Poisson equation in a Gaussian-Poissonian stochastic medium

$$\begin{aligned} \Delta F(x) &= -\phi(s)W(x) - \psi(s)V(x) \ = -(\phi(x), \psi(x)) \cdot U(x) \ ; \quad x \in D \\ F(x) &= 0 \ ; \qquad x \in \partial D \end{aligned} \tag{4.4}$$

where $\phi(x)$, $\psi(x)$ are bounded, continuous, deterministic functions on a bounded domain $D \subset \mathbb{R}^d$ with regular boundary ∂D.
To solve this equation we proceed as in the purely Gaussian case (see [6, Section 4.2]): We first take the Hermite transform and obtain the equation

$$\begin{aligned} \Delta\widetilde{F}(x,z,w) &= -\phi(s)\widetilde{W}(x,z) - \psi(x)\widetilde{V}(x,w) \ ; \quad x \in D \\ \widetilde{F}(x,z,w) &= 0 \ ; \quad x \in \partial D \end{aligned} \tag{4.5}$$

with $(z,w)\in\mathbb{C}_0^{\mathbb{N}}\times\mathbb{C}_0^{\mathbb{N}}$ as parameters. For each fixed (z,w) this equation has the solution

$$\widetilde{F}(x,z,w)=\int_D G(x,y)[\phi(y)\widetilde{W}(y,z)+\psi(y)\widetilde{V}(y,w)]dy \tag{4.6}$$

where G is the classical Green function in D. We recognize (4.6) as the Hermite transform of

$$F(x)=F(x,\omega)=\int_D G(x,y)[\phi(y)W(y,\omega_1)+\psi(y)V(y,\omega_2)]dy\ ; \qquad \omega=(\omega_1,\omega_2). \tag{4.7}$$

This the solution of (4.4), in virtue of the Hermite transform solution scheme. Note that $F(x,\cdot)\in(\mathcal{S})^*_\theta$ for all x, for all dimensions d. Moreover, for $d\le 3$ we actually have $F(x,\cdot)\in L^2(\theta)$ because then $G(x,\cdot)\in L^2(dy)$ (see [6, p. 5]). And then the solution can be written in the more familiar form

$$F(x,\omega)=\int_D G(x,y)[\phi(y)dB(y,\omega_1)+\psi(y)dQ(y,\omega_2)] \qquad (d\le 3). \tag{4.8}$$

Similarly the reader can verify that the discussions of the other Wick type stochastic partial differential equations in [6, Chapter 4] carry over to the mixed Gaussian-Poissonian case.

The Gjessing Formula and the Benth-Gjessing Method in the Mixed Gaussian-Poissonian Case

By inspecting the proof of the Gjessing formula [3] for Gaussian white noise (see, e.g., [6, Theorem 2.10.7], we see that it carries over to the mixed Gaussian-Poissonian case as follows:

Theorem 4.4 (The Gjessing formula, mixed Gaussian-Poissonian case) *Suppose* $\int_{\mathbb{R}^d}(\phi(x)^2+|\psi(x)|+\ln(1+\psi(x))^2)dx<\infty$ *and* $X\in L^p(\theta)$ *for some* $p>1$. *Then* $X\diamond\exp^\diamond(\int_{\mathbb{R}^d}\phi dB+\int_{\mathbb{R}^d}\psi dQ)\in L^\rho(\theta)$ *for all* $\rho<p$ *and almost surely we have*

$$\begin{aligned}\Big(X\diamond\exp^\diamond\big(\int_{\mathbb{R}^d}\phi dB+\int_{\mathbb{R}^d}\psi dQ\big)\Big)(\omega_1,\omega_2)&\\ =X(\omega_1-\phi,\omega_2-\psi)\cdot\exp^\diamond\Big(\int_{\mathbb{R}^d}\phi dB+\int_{\mathbb{R}^d}\psi dQ\Big)(\omega_1,\omega_2).&\end{aligned} \tag{4.9}$$

In the Gaussian case Benth and Gjessing [2] used the Gjessing formula to solve a class of quasi-linear stochastic differential equations. (See also the exposition in [6, Section 4.8].) This method carries over to the mixed Gaussian-Poissonian case. To illustrate this we discuss the following example:

Example 4.5 Consider the stochastic differential equation

$$dX(t) = a(t)X(t)^{\lambda}dt + \phi(t)X(t)dB(t) + \psi(t)X(t)dQ(t) \tag{4.10}$$

with $a(t) \geq 0$, $\phi(t), \psi(t)$ as in Example 4.1, where $\lambda \in \mathbb{R}$ is a constant and we assume $X(0) > 0$. The case $\lambda = 1$ is covered in Example 4.1, so we assume $\lambda \neq 1$. The noise interpretation of this equation is (see (2.24))

$$\frac{dX(t)}{dt} = a(t)X(t)^{\lambda} + X(t) \diamond [\phi(t)W(t) + \psi(t)V(t)].$$

This equation can be written

$$\frac{d}{dt}[J(t) \diamond X(t)] = a(t)J(t) \diamond X(t)^{\lambda}\ , \tag{4.11}$$

where

$$J(t) = \exp^{\diamond}\Big(-\int_0^t \phi(s)W(s)ds - \int_0^t \psi(s)V(s)ds \Big). \tag{4.12}$$

Define

$$Y(t,\omega) = (J(t) \diamond X(t))(\omega) = J(t,\omega) \cdot X(t, \omega_1 + \phi^{(t)}, \omega_2 + \psi^{(t)}) \tag{4.13}$$

where $\phi^{(t)}(s) = \mathcal{X}_{[0,t]}(s)\phi(s)$, $\psi^{(t)}(s) = \mathcal{X}_{[0,t]}(s)\psi(s)$. Then by (4.11), (4.9) and (4.13)

$$\begin{aligned} \frac{dY(t,\omega)}{dt} &= a(t)(J(t) \diamond X(t)^{\lambda})(\omega) \\ &= a(t)(J(t,\omega) \cdot X(t, \omega_1 + \phi^{(t)}, \omega_2 + \psi^{(t)})^{\lambda}) \\ &= a(t)J(t,\omega)[J^{-1}(t,\omega)Y(t,\omega)]^{\lambda} \\ &= a(t)J(t,\omega)^{1-\lambda}Y(t,\omega)^{\lambda}. \end{aligned} \tag{4.14}$$

For fixed ω this is a (deterministic) separable differential equation in $Y(t,\omega)$, with solution

$$Y(t,\omega) = [Y(0)^{1-\lambda} + (1-\lambda)\int_0^t a(s)J(s,\omega)^{1-\lambda}ds]^{\frac{1}{1-\lambda}}. \tag{4.15}$$

Finally, we use (4.13) to find X:

$$X(t, \omega_1, \omega_2) = J(t, \omega_1 - \phi^{(t)}, \omega_2 - \psi^{(t)}) Y(t, \omega_1 - \phi^{(t)}, \omega_2 - \psi^{(t)}), \tag{4.16}$$

where, by (4.12), (2.18) and (2.3),

$$J(t, \omega_1, \omega_2) = \exp\Big(- \int_0^t \phi dB - \int_0^t \ln(1 + \psi) dP - \int_0^t (\tfrac{1}{2}\phi^2 - \psi) ds \Big). \tag{4.17}$$

From (4.15) we conclude that the solution $X(t)$ exists for all $t \geq 0$ if $\lambda < 1$ (since $a \geq 0$). However, if $\lambda > 1$ we may have explosions: $Y(t, \omega)$ only exists up to the first time t such that

$$X(0)^{1-\lambda} + (1 - \lambda) \int_0^t a(s) J(s, \omega)^{1-\lambda} ds = 0.$$

For extensions of white noise theory beyond the mixed Gaussian-Poissonian case we refer to [4] and [8].

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References

[1] Benth, F.E., and Gjerde, J.: A remark on the equivalence between Poissonian and Gaussian stochastic partial differential equations, Potential Analysis 8 (1998), 179–193.

[2] Benth, F.E., and Gjessing, H.: A non-linear parabolic equation with noise. A reduction method, Manuscripte Nr. 179 (1994), Universität Mannheim.

[3] Gjessing, H.: Anticipating Calculus and Wick Products, Dr. Scient. Thesis, University of Bergen, Norway, 1995.

[4] Kondratiev, Y., Streit, L., Westerkamp, W., and Yan, J.: Generalized functions in infinite dimensional analysis, Manuscript, Universität Bielefeld 1995.

[5] Hida, T., Kuo, H.-H., Potthoff, J., and Streit, L.: White Noise Analysis, Kluwer 1993.

[6] Holden, H., Øksendal, B., Ubøe, J., and Zhang, T.: Stochastic Partial Differential Equations, Birkhäuser 1996.

[7] Oberguggenberger, M., Russo, F.: Nonlinear SPDEs: Colombeau solutions and pathwise limits, In Decreusefond et al. (eds.): Stochastic Analysis and Related Topics VI, Birkhäuser 1998, pp. 319–332.

[8] Us, G.: Towards a coloured noise analysis, Manuscript, Kiev University 1997.

[9] Walsh, J.: An introduction to stochastic partial differential equations, In R. Carmona et al. (eds.): École d'Été de Probabilités de Saint-Flour XIX–1984, Springer Lecture Notes in Mathematics, vol. 1180 (1986), 265–437.

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