EQUIVARIANT DEFORMATIONS OF HOMOGENEOUS SPACES

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A C^*-algebraic deformation of a locally compact space X (or the algebra $C_0(X)$) is a C^*-algebra obtained by completing a *-algebra whose underlying vector space is a dense subspace of $C_0(X)$. It is important in applications that extra structure possessed by X can be carried over to the deformation: thus, for example, one might expect deformations of groups to have comultiplications – that is, to be quantum groups. Here we consider deformations of homogeneous spaces G/Γ, and we look for ones which preserve the natural left action of G. Thus we want deformations which are *equivariant* in the sense that the underlying vector space V is invariant under the action of G by left translation on $C_0(G/\Gamma)$ and the resulting action of G on V extends to the C^*-completion. Our algebras are based on the equivariant deformations of $C_0(G)$ constructed in our previous paper [LR], and the motivating examples are the Heisenberg manifolds, where our construction yields the deformations obtained by Rieffel [R1].

The deformations of $C_0(G)$ in [LR] have as underlying vector space the Fourier algebra $A(G)$, and the multiplication is the usual one twisted by a dual cocycle ω on G. For our present purposes it is enough to consider cocycles inflated from dual cocycles on an abelian subgroup H of G, which are the Fourier transforms of ordinary 2-cocycles on the dual group $\widehat{H}$. To obtain such 2-cocycles on $\widehat{H}$, we fix a compact subgroup K of H and a homomorphism ρ of $\widehat{K}$ into H. A precise description of the resulting cocycle is given in [LR, §4], but we shall not need it since we only need to know the formulas for the operations in $A(G,\omega)$. For these, let $\widehat{f} : G \times \widehat{K} \to \mathbf{C}$ be the partial Fourier transform defined by $\widehat{f}(x,s) := \int_K \overline{\langle k,s\rangle} f(xk)\,dk$. Then for $f,g \in A(G,\omega)$, we have from [LR, Theorem 4.3] that

$$(*)\qquad f * g(x) = \sum_{s,t} \widehat{f}(x\rho(t),s)\widehat{g}(x\rho(-s),t), \quad f^*(x) = \overline{f(x)}.$$

To obtain deformations of $C_0(G/\Gamma)$, we would like to use the same formulas on appropriate functions in $C_0(G/\Gamma) \subset C_b(G)$. There are three main problems. First, for the noncompact Γ of interest to us, there will be no functions in $A(G)$ or $C_0(G)$ which are constant on Γ-cosets. With this problem in mind, we proved in [LR, Proposition 5.2] that

$$C_{b,1}(G) := \Big\{ f \in C_b(G) : \|f\|_{\infty,1} := \sum_s \sup_x |\widehat{f}(x,s)| < \infty \Big\}$$

is a Banach *-algebra under the operations $(*)$, and this algebra will be large enough for our purposes. Second, it is not obvious that the action of Γ by right translation on $C_b(G)$

preserves the multiplication on $C_{b,1}(G)$, so $C_1(G/\Gamma,\rho) := C(G/\Gamma) \cap C_{b,1}(G,\rho)$ need not be a *-subalgebra. This problem does not arise in our motivating example; in general, we have to impose conditions on the structure of G, Γ and ρ to ensure that $C_1(G/\Gamma,\rho)$ is a *-subalgebra of $C_{b,1}(G,\rho)$. Finally, we have to complete $C_1(G/\Gamma,\rho)$, and for this we need a regular representation of $C_{b,1}(G)$. We constructed such a representation of $A(G,\omega)$ on $L^2(G)$ in [LR], and it is relatively straightforward to work out a detailed formula for this representation and check that it extends to a representation μ of $C_{b,1}(G,\rho)$. We can then define $C_r^*(\widehat{G}/\Gamma,\rho)$ to be the closure of $\mu(C_1(G/\Gamma,\rho))$.

Our main goal, then, is to describe a general procedure for constructing deformations which covers both known and new examples. Our procedure has some potentially useful features. We work in a concrete space of functions $C_{b,1}(G)$, which allows us to easily identify and manipulate elements with special properties (see, for example, Proposition 3.6 below). This should be helpful in calculating the K-theory of deformations (which has already been done for the Heisenberg manifolds by Abadie [A1, A2], and in other cases by Rieffel [R4]). Because our spaces are large, it is always possible to restrict to subalgebras with appropriate support or smoothness properties: for example, if one wishes to discuss Poisson brackets, one can work with the same formulas in $C^\infty(G/\Gamma) \subset C_1(G/\Gamma,\rho)$. We have not followed up all possible applications: for example, it seems clear that our techniques could be used to deform double coset spaces $\Gamma_1\backslash G/\Gamma_2$.

We begin in §1 with a discussion of $C_{b,1}(G,\rho)$, and its regular representation μ on $L^2(G)$. The right regular representation R of G implements the action α of G by left translation on $C_{b,1}(G,\rho)$, so any deformations we form by completing α-invariant subspaces in the norm pulled back from $B(L^2(G))$ will be equivariant. In §2 we give an immediate application, in which we mod out by the left action of a compact subgroup Γ to obtain (nonequivariant) deformations of $\Gamma\backslash G$. When $G = U_2(\mathbf{C})$, we obtain the noncommutative 3-spheres of [MT, M, NO].

In §3, we discuss deformations of the Heisenberg manifold. The results are a special case of those in §4, but we have given a complete treatment because it motivates our constructions in §4, and is technically substantially easier. In this example, the right translations by elements of the subgroup Γ extend to *-automorphisms of $C_{b,1}(G,\rho)$, and hence the algebras $C_1(G/\Gamma,\rho) := C(G/\Gamma) \cap C_{b,1}(G,\rho)$ have completions which are equivariant deformations $C_r^*(\widehat{G}/\Gamma,\rho)$ of the Heisenberg manifold G/Γ. These deformations were previously constructed by Rieffel [R1]; as evidence that our approach gives new insight, we give a direct, self-contained proof of Rieffel's criterion for simplicity of the algebras $C_r^*(\widehat{G}/\Gamma,\rho)$.

Our main constructions are in §4. As we mentioned above, we need to determine conditions under which Γ acts by right translation on $C_{b,1}(G, \rho)$. The crucial issue here is how Γ interacts with the abelian subgroup $H = \overline{\rho(\widehat{K})K}$ on which the dual cocycle lives. Our examples suggest it is reasonable to assume that Γ commutes with K, and then we have to insist that each $x \in \Gamma$ normalises H in such a way that $B_x(s) := x\rho(s)x^{-1}\rho(-s)$ belongs to K and the form $(s,t) \mapsto \langle B_x(s), t\rangle$ is symmetric on $\widehat{K} \times \widehat{K}$. For a subgroup Γ satisfying these conditions, $C_{0,1}(G/\Gamma, \rho) := C_0(G/\Gamma) \cap C_{b,1}(G, \rho)$ is a Banach *-algebra which can be completed to give equivariant deformations of $C_0(G/\Gamma)$. We study the structure of these algebras when G/Γ is compact and $\Gamma \cap K = \{e\}$, and show that Rieffel's simplicity results can be generalised to this setting: we construct a particular subgroup G_ρ of G containing Γ such that the center of $C_r^*(\widehat{G}/\Gamma, \rho)$ is isomorphic to $C(G/G_\rho)$, and such that $C_r^*(\widehat{G}/\Gamma, \rho)$ is simple if and only if $G = G_\rho$. In view of this theorem, it is interesting to decide when $G = G_\rho$, and we show that in quite general circumstances this property forces G to be nilpotent. Thus Rieffel's examples are typical if we seek deformations which are simple C^*-algebras.

In the last two sections, we discuss the relationship between our results and those of Rieffel. In §5, we show that $C_r^*(\widehat{G}/\Gamma, \rho)$ is the generalised fixed-point algebra for the action β of Γ by right translation, in the sense of [R3], and is therefore Morita equivalent to the crossed product $C_r^*(\widehat{G}) \times_\beta \Gamma$. In §6, we discuss some minor variations on our theme which yield the noncommutative spaces studied in [BEEK, FW], and what happens if we try to relax the hypotheses we imposed in §4.

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§1. Preliminaries.

We start with a short description of the equivariant deformations of $C_0(G)$ of [LR] appropriate to the situations we want to treat. The first standard assumption we make throughout this article is:

(S1) *K is a compact abelian subgroup of G and $\rho : \widehat{K} \mapsto G$ is a homomorphism such that $\rho(s)k = k\rho(s)$ for $s \in \widehat{K}$ and $k \in K$.*

In [LR, Theorem 4.3], we described how to obtain from such data a dual cocycle on the abelian subgroup $\overline{\rho(\widehat{K})K}$ of G. (It was not necessary in the proof of [LR, Theorem 4.3] to assume that $\rho(\widehat{K}) \cap K = \{e\}$.) We shall need an explicit formula for the corresponding inflated dual cocycle ω on G. (Warning: this was denoted $\operatorname{Inf}\omega$ in [LR].)

Definition 1.1. Let $\{P_s \mid s \in \widehat{K}\}$ be the projections corresponding to the left regular representation $L^G|_K$ of K in $\mathcal{L}(G)$, i.e.

$$P_s = \int_K \overline{\langle k, s\rangle} L_k \, dk \tag{1.1}$$

Then define a dual cocycle on G by

$$\omega = \omega_0 \sigma(\omega_0)^* \quad \text{where} \quad \omega_0 = \sum_s P_s \otimes L_{\rho(s)}, \tag{1.2}$$

and σ is the flip on $B(L^2(G) \otimes L^2(G))$ (which was denoted Σ in [LR]). Note that the partial Fourier transform

$$\widehat{f}(x, s) = \int_K \overline{\langle k, s\rangle} f(xk) \, dk \tag{1.3}$$

used in [LR] satisfies

$$\widehat{f}(x, s) = \langle L_x P_s, f\rangle \text{ for } f \in A(G). \tag{1.4}$$

Proposition 1.2. *The dual cocycle ω defined by (1.2) corresponds to the one in [LR, Theorem 4.3], i.e.*

$$f *_\omega g(x) = \langle \delta_G(L_x)\omega^*, f \otimes g\rangle = \sum_{s,t} \widehat{f}(x\rho(t), s)\widehat{g}(x\rho(-s), t). \tag{1.5}$$

Proof. From (1.4) and Theorem 4.3 in [LR] we have

$$\begin{aligned} f *_\omega g(x) &= \sum_{s,t} \widehat{f}(x\rho(t), s)\widehat{g}(x\rho(-s), t) \\ &= \sum_{s,t} \langle L_{x\rho(t)} P_s, f\rangle \langle L_{x\rho(-s)} P_t, g\rangle \\ &= \Big\langle \delta_G(L_x)\Big(\sum_{s,t} L_{\rho(t)} P_s \otimes L_{\rho(-s)} P_t\Big), f \otimes g\Big\rangle \\ &= \langle \delta_G(L_x)\sigma(\omega_0)\omega_0^*, f \otimes g\rangle = \langle \delta_G(L_x)\omega^*, f \otimes g\rangle. \ \square \end{aligned}$$

From now on, we shall largely forget about dual cocycles, and recall from [LR, Proposition 5.2] that the right hand side of (1.5) defines a product also on

$$C_{b,1}(G) := \Big\{ f \in C_b(G) : \|f\|_{\infty,1} := \sum_s \sup_x |\widehat{f}(x,s)| < \infty \Big\}. \tag{1.6}$$

Thus $C_{b,1}(G)$ is a Banach *-algebra with norm $\|\cdot\|_{\infty,1}$ and operations

$$f * g(x) = \sum_{s,t} \widehat{f}(x\rho(t), s)\widehat{g}(x\rho(-s), t), \tag{1.7}$$

$$f^*(x) = \overline{f(x)}. \tag{1.8}$$

We shall work in $C_{b,1}(G)$ because it has a large intersection with $C(G/\Gamma)$; Example 5.4 in [LR] shows that other choices, such as $B(G) \cap C(G/\Gamma)$, can be too small. We shall write $C_{b,1}(G,\rho)$ when we want to emphasize that we are looking at $C_{b,1}(G)$ with the multiplication in (1.7). Similarly we shall write $A(G,\rho)$ for $A(G,\omega)$ with the multiplication in (1.5).

The following formula will be useful, and is easily checked:

$$(f * g)\widehat{\ }(x,s) = \sum_t \widehat{f}(x\rho(s-t), t)\widehat{g}(x\rho(-t), s-t). \tag{1.9}$$

In order to obtain C^*-algebras we need *-representations of these algebras. The regular *-representation of $A(G,\rho)$ defined in [LR, 1.15] is given by $\mu(f) = S_f(V)$, where S_f is the slice map corresponding to the functional $f \in A(G) = \mathcal{L}(G)_*$, and $V = W_G^*\sigma(\omega^*) = W_G^*\omega$. It will be crucial that this can be extended to a *-representation of $C_{b,1}(G,\rho)$:

Proposition 1.3. *Suppose K and ρ satisfy (S1). The regular representation of $A(G,\rho)$ is given by*

$$\mu(f) = \sum_{s,t} L_{\rho(s)} M(\widehat{f}(\cdot,t)) L_{\rho(-s-t)} P_s = \sum_{s,t} P_t L_{\rho(s)} M(f) L_{\rho(-t)} P_s, \tag{1.10}$$

where M is the representation of $C_b(G)$ on $L^2(G)$ given by $M(f)\xi(x) = f(x^{-1})\xi(x)$. Hence μ extends to a bounded, faithful *-representation of $C_{b,1}(G,\rho)$, and then (μ, R) is covariant for the action α of G by left translation on $C_{b,1}(G,\rho)$. (R is the right regular representation of G on $L^2(G)$.)

Proof. For the vector functional $f = \omega_{\phi,\psi} \in A(G)$ we have

$$\begin{aligned}
\langle \mu(f)\xi \mid \eta \rangle &= \langle W_G^* \omega(\xi \otimes \phi) \mid \eta \otimes \psi \rangle \\
&= \sum_{s,t} \big\langle W_G^* L_{\rho(-t)} P_s \xi \otimes L_{\rho(s)} P_t \phi \mid \eta \otimes \psi \big\rangle \\
&= \sum_{s,t} \int L_{\rho(-t)} P_s \xi(x) L_{\rho(s)} P_t \phi(xy) \overline{\eta(x)\psi(y)}\, dxdy \\
&= \sum_{s,t} \int L_{\rho(-t)} P_s \xi(x) \overline{\langle k, t \rangle} \phi(k^{-1}\rho(-s)xy) \overline{\eta(x)\psi(y)}\, dkdxdy \\
&= \sum_{s,t} \int f(x^{-1}\rho(s)k) \overline{\langle k, t \rangle} L_{\rho(-t)} P_s \xi(x) \overline{\eta(x)}\, dkdx \\
&= \sum_{s,t} \int f(x^{-1}k) \overline{\langle k, t \rangle} L_{\rho(-s-t)} P_s \xi(x) \overline{L_{\rho(-s)}\eta(x)}\, dkdx
\end{aligned}$$

Now on the one hand, this equals

$$\begin{aligned}
&= \sum_{s,t} \int \widehat{f}(x^{-1}, t) L_{\rho(-s-t)} P_s \xi(x) \overline{L_{\rho(-s)}\eta(x)}\, dx \\
&= \big\langle \sum_{s,t} L_{\rho(s)} M(\widehat{f}(\cdot, t)) L_{\rho(-s-t)} P_s \xi \mid \eta \big\rangle,
\end{aligned}$$

and on the other, it equals

$$\begin{aligned}
&= \sum_{s,t} \int f(x^{-1}) \overline{\langle k, t \rangle} L_{\rho(-s-t)} P_s \xi(kx) \overline{L_{\rho(-s)}\eta(kx)}\, dkdx \\
&= \sum_{s,t} \int f(x^{-1}) \overline{\langle k, s+t \rangle} L_{\rho(-s-t)} P_s \xi(x) \overline{L_{k^{-1}\rho(-s)}\eta(x)}\, dkdx \\
&= \sum_{s,t} \langle M(f) L_{\rho(-s-t)} P_s \xi \mid P_{s+t} L_{\rho(-s)} \eta \rangle \\
&= \sum_{s,t} \langle P_t L_{\rho(s)} M(f) L_{\rho(-t)} P_s \xi \mid \eta \rangle.
\end{aligned}$$

To make sure that there are no problems in interchanging sums and integrations, one should first take ξ, η such that $\sum_{s,t}$ is finite. Since such vectors are dense in $L^2(G)$, this will prove the formulas for $f \in A(G)$. An easy computation shows that

$$P_t M(f) P_s = M(\widehat{f}(\cdot, t-s)) P_s. \tag{1.11}$$

Now to see that (1.10) actually defines a bounded operator $\mu(f)$ for any $f \in C_{b,1}(G)$, we do the following estimate:

$$\begin{aligned}
|\langle \mu(f)\xi \mid \eta\rangle| &= |\big\langle \sum_{s,t} L_{\rho(s)} M(\widehat{f}(\cdot, t-s)) L_{\rho(-t)} P_s\xi \mid P_t\eta\big\rangle| \\
&\leq \sum_{s,t} \sup_x |\widehat{f}(x, t-s)| \, \|P_s\xi\| \, \|P_t\eta\| \\
&= \sum_{s,t} \sup_x |\widehat{f}(x,t)| \, \|P_s\xi\| \, \|P_{s+t}\eta\| \\
&\leq \sum_t \sup_x |\widehat{f}(x,t)| \Big(\sum_s \|P_s\xi\|^2\Big)^{1/2} \Big(\sum_q \|P_{q+t}\eta\|^2\Big)^{1/2} \\
&= \|f\|_{\infty,1} \|\xi\| \, \|\eta\|.
\end{aligned}$$

Thus μ extends to a continuous linear map of $C_{b,1}(G)$ into $B(L^2(G))$. That μ is multiplicative follows from

$$\begin{aligned}
\mu(f * g) &= \sum_{s,t} L_{\rho(s)} M[(f*g)\widehat{\ }(\cdot, t)] L_{\rho(-s-t)} P_s \\
&= \sum_{s,t,u} L_{\rho(s)} M[\widehat{f}(\cdot\rho(t-u), u)\widehat{g}(\cdot\rho(-u), t-u)] L_{\rho(-s-t)} P_s \\
&= \sum L_{\rho(s+t-u)} M(\widehat{f}(\cdot, u)) L_{\rho(-t)} M(\widehat{g}(\cdot, t-u)) L_{\rho(u-s-t)} P_s \\
&= \sum L_{\rho(t)} M(\widehat{f}(\cdot, u)) L_{\rho(-t-u+s)} M(\widehat{g}(\cdot, t-s)) L_{\rho(-t)} P_s \\
&= \sum L_{\rho(t)} M(\widehat{f}(\cdot, u)) L_{\rho(-t-u+s)} P_t M(g) L_{\rho(-t)} P_s \\
&= \sum_{t,u} L_{\rho(t)} M(\widehat{f}(\cdot, u)) L_{\rho(-t-u)} P_t \mu(g) \\
&= \mu(f)\mu(g).
\end{aligned}$$

Directly from (1.10) it follows that $\mu(f^*) = \mu(f)^*$. To see that μ is faithful on $C_{b,1}(G)$, note using (1.11) that

$$\begin{aligned}
\mu(f) = 0 \implies& L_{\rho(s)} P_t M(f) P_s L_{\rho(-t)} = P_t \mu(f) P_s = 0 \text{ for all } s,t \\
\implies& P_t M(f) P_s = 0 \text{ for all } s,t \implies M(f) = 0 \implies f = 0.
\end{aligned}$$

That (μ, R) is a covariant representation now follows from (1.10), because each R_y commutes with L_x and P_t.

§2. Deformations of $C_0(\Gamma\backslash G)$.

Since the left action of G is preserved under our deformations of $C_0(G)$, it will be easy to obtain corresponding deformations of $C_0(\Gamma\backslash G)$ when Γ is a closed subgroup of G.

Proposition 2.1. *If Γ is a closed subgroup of G, then $C_{0,1}(\Gamma\backslash G) := C_0(\Gamma\backslash G) \cap C_{b,1}(G)$ is a closed *-subalgebra of $C_{b,1}(G,\rho)$. By taking the C^*-closure of $\mu(C_{0,1}(\Gamma\backslash G))$ we obtain a deformation of $C_0(\Gamma\backslash G)$, which we denote by $C^*_r(\Gamma\backslash\widehat{G},\rho)$.*

Proof. Omitted.

The structure theorem in [LR, §3] can now be used to describe $C^*_r(\Gamma\backslash\widehat{G},\rho)$, at least when Γ is compact. Take $H = \overline{\rho(\widehat{K})K}$, and let $\mathcal{B} = C^*_r(\widehat{H},\rho)$ be the C^*-algebra obtained in the same way from $C_{0,1}(H,\rho)$. The corresponding action of H by left translation on $\mathcal{B}$ is denoted α^H. [LR, Theorem 3.3] then says that the covariant system $(C^*_r(\widehat{G},\rho),G,\alpha)$ is equivalent to the induced system $(\mathrm{Ind}_H^G(\mathcal{B},\alpha^H),G,\alpha)$, in which

$$\mathrm{Ind}_H^G(\mathcal{B},\alpha^H) := \left\{ f \in C_b(G,\mathcal{B}) \,\middle|\, \begin{array}{c} f(xh) = \alpha^H_{h^{-1}}(f(x)) \text{ for } x \in G, h \in H, \\ \text{and } xH \mapsto \|f(x)\| \text{ is in } C_0(G/H) \end{array} \right\}$$

and $\alpha_y(f)(x) = f(y^{-1}x)$, c.f. [LR, §2]. This gives:

Proposition 2.2. *If Γ is a compact subgroup of G, then with $(\mathcal{B},H,\alpha^H)$ as above*

$$C^*_r(\Gamma\backslash\widehat{G},\rho) \cong \left\{ f \in C_b(\Gamma\backslash G,\mathcal{B}) \,\middle|\, \begin{array}{c} f(xh) = \alpha^H_{h^{-1}}(f(x)) \text{ for } x \in G, h \in H, \\ \text{and } \Gamma xH \mapsto \|f(x)\| \text{ is in } C_0(\Gamma\backslash G/H) \end{array} \right\}$$

Proof. The isomorphism Φ in [LR, Theorem 3.3] of $C^*_r(\widehat{G},\rho)$ onto $\mathrm{Ind}_H^G(\mathcal{B},\alpha^H)$ identifies the right-hand side with

$$C^*_r(\widehat{G},\rho)^\Gamma = \{a \in C^*_r(\widehat{G},\rho) \mid R_y a R_y^* = a \text{ for } y \in \Gamma\}.$$

$C^*_r(\widehat{G},\rho)$ is generated by $\mu(C_{0,1}(G))$, and since Γ is compact $\mu(C_{0,1}(\Gamma\backslash G))$ will generate $C^*_r(\widehat{G},\rho)^\Gamma$.

Example 2.3. As in [LR, 3.6.3], we let G be the unitary group $U_2(\mathbf{C})$. We fix some $q \in \mathbf{T}$, take

$$K = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} : |t| = 1 \right\} \cong \mathbf{T}, \text{ and define } \rho(n) = \begin{pmatrix} q^n & 0 \\ 0 & 1 \end{pmatrix}.$$

Then $\overline{\rho(\widehat{K})K}$ is contained in the maximal torus H of diagonal matrices, and $\mathcal{B}$ is a rotation algebra. With $\Gamma = K$, the map

$$\begin{pmatrix} z & w \\ -t\overline{w} & t\overline{z} \end{pmatrix} \mapsto (z,w)$$

of G into $\mathbf{C}^2$ induces a homeomorphism of $\Gamma\backslash G$ onto the 3-dimensional sphere which converts the right action of $H = \mathbf{T}^2$ on $\Gamma\backslash G$ into the action $(z,w)\cdot(s,t) = (zs,wt)$. So we obtain the following deformations of $C(\Gamma\backslash G)$:

$$C^*_r(\Gamma\backslash\widehat{G},\rho) = \{f \in C_b(\Gamma\backslash G,\mathcal{B}) |\, f(zs,wt) = (\alpha^H_{s,t})^{-1}(f(z,w)) \ \text{ for } s,t \in \mathbf{T}\}.$$

The product in $C_1(\Gamma\backslash G,\rho)$ is then described as follows: the partial Fourier transform (1.3) will here be

$$\widehat{f}(z,w,n)=\int_{\mathbf{T}} t^{-n}f(z,tw)dt,$$

and the product (1.7) will be

$$f*g(z,w)=\sum_{m,n}\widehat{f}(q^n z,w,m)\widehat{g}(q^{-m}z,w,n).$$

If we take $\phi(z,w)=z$ and $\psi(z,w)=w$, we leave it to the reader to show that $\widehat{\phi}(\cdot,n)=0$ for $n\neq 0$ and $\widehat{\psi}(\cdot,n)=0$ for $n\neq 1$, and that therefore ϕ and ψ are normal elements which generate $C^*_r(\Gamma\backslash\widehat{G},\rho)$ and satisfy

$$|\phi|^2+|\psi|^2=1 \quad\text{and}\quad \phi*\psi=q^2\psi*\phi.$$

These non-commutative spheres are studied in more detail in [M, MT and NO].

§3. The Heisenberg manifold.

Before we show how to define $C^*_r(\widehat{G}/\Gamma,\rho)$ in general, we look in some detail at the non-commutative Heisenberg manifolds of Rieffel [R1], which inspired our work on equivariant deformations. Our treatment of the Heisenberg manifold will motivate the definitions and proofs in §4.

As in [LR, Example 4.8 and 3.6.2], we let $G=\mathbf{R}^2\times\mathbf{T}$ with the product

$$(x_1,y_1,z_1)(x_2,y_2,z_2)=(x_1+x_2,y_1+y_2,z_1z_2\exp(2\pi i y_1x_2)). \tag{3.1}$$

We set $K=\{(0,0,t)|t\in\mathbf{T}\}$; we shall identify $t\in\mathbf{T}$ with $(0,0,t)\in K$. We define $\rho:\widehat{K}\mapsto G$ by

$$\rho(n)=(\mu n,\nu n,\exp(i\pi\mu\nu n^2))\quad\text{for some fixed } \mu,\nu\in\mathbf{R}. \tag{3.2}$$

(Note that the $-$ sign in $\exp(-\pi i\mu\nu s^2)$ in [LR, Example 4.8] should be a $+$ sign.) As before, we define the partial Fourier transform of $f\in C_b(G)$ by

$$\widehat{f}(x,n)=\int_{\mathbf{T}} t^{-n}f(xt)\,dt. \tag{3.3}$$

We shall need the following formula for the multiplication in $C_{b,1}(G)$:

$$(f*g)\widehat{\ }(x,m)=\sum_n \widehat{f}(x\rho(m-n),n)\widehat{g}(x\rho(-n),m-n). \tag{3.4}$$

Lemma 3.1. *For every $x \in G$, $\beta_x f(y) := f(yx)$ defines a *-automorphism of $C_{b,1}(G,\rho)$. Thus for any closed subgroup Γ of G, $C_{0,1}(G/\Gamma,\rho) := C_0(G/\Gamma) \cap C_{b,1}(G,\rho)$ is a closed *-subalgebra of $C_{b,1}(G,\rho)$.*

Proof. First note that since K is central in G we have

$$\beta_x(f)\widehat{\ }(y,n) = \widehat{f}(yx,n).$$

By a direct computation or looking at [LR, Example 4.8], we have

$$(s,t,z)\rho(n) = \rho(n)(s,t,z)u^n \quad \text{where} \quad u = (0,0,\exp 2\pi i(\mu t - \nu s)),$$

and applying this to $x = (s,t,z)$ gives

$$\begin{aligned}
\beta_x(f*g)(y) = f*g(yx) &= \sum_{m,n} \widehat{f}(yx\rho(n),m)\widehat{g}(yx\rho(-m),n) \\
&= \sum_{m,n} \widehat{f}(y\rho(n)xu^n,m)\widehat{g}(y\rho(-m)xu^{-m},n) \\
&= \sum_{m,n} \widehat{f}(y\rho(n)x,m)u^{nm}\widehat{g}(y\rho(-m)x,n)u^{-mn} \\
&= \sum_{m,n} \beta_x(f)\widehat{\ }(y\rho(n),m)\beta_x(g)\widehat{\ }(y\rho(-m),n) = \beta_x(f)*\beta_x(g)(y).
\end{aligned}$$

That $\beta_x(f)^* = \beta_x(f^*)$ is obvious, and we deduce that $C_b(G/\Gamma) \cap C_{b,1}(G,\rho)$ is a *-subalgebra of $C_{b,1}(G,\rho)$. If $f \in C_0(G/\Gamma)$ then $\widehat{f}(\cdot,m) \in C_0(G/\Gamma)$, so the uniform convergence of the formula (1.7) for the product shows that $C_0(G/\Gamma) \cap C_{b,1}(G)$ is a closed *-subalgebra of $C_{b,1}(G,\rho)$.

Definition 3.2. It follows from Proposition 1.3, or by a direct calculation, that (1.10) defines a faithful *-representation V of $C_{b,1}(G,\rho)$ on $L^2(G)$. (We use V instead of μ in this chapter because we want μ,ν to have the same meaning as in [R1].) For a closed subgroup Γ of G we define $C^*_r(\widehat{G}/\Gamma,\rho)$ to be the C^*-closure of $V(C_{0,1}(G/\Gamma))$. From Proposition 1.3 it follows that $\operatorname{Ad} R_x$ is a *-automorphism of $C^*_r(\widehat{G}/\Gamma,\rho)$ satisfying

$$R_x V(f) R_x^* = V(\alpha_x(f)). \tag{3.5}$$

Lemma 3.3. *The maps β_k for $k \in K$ extend to *-automorphisms of $C^*_r(\widehat{G}/\Gamma,\rho)$ such that*

$$C^*_r(\widehat{G}/\Gamma,\rho)^K := \{a \in C^*_r(\widehat{G}/\Gamma,\rho) \mid \beta_k(a) = a \text{ for all } k \in K\} = V(C_0(G/K\Gamma)).$$

Proof. Since $k \in K$ is central it follows from (1.7) that β_k is a *-automorphism. We are taking the fixed point algebra under an action of a compact group, hence $V(C_{0,1}(G/\Gamma))^K =$

$V(C_0(G/K\Gamma))$ is dense in $C_r^*(\widehat{G}/\Gamma, \rho)^K$. On $C_0(G/K\Gamma)$ the product (1.7) is the ordinary pointwise product, so V is a *-isomorphism on the commutative C*-algebra $C_0(G/K\Gamma)$ and the two algebras must be equal.

The homogeneous manifolds studied in [R1] were obtained by taking the subgroups

$$\Gamma_c = \{(m, n, z) \mid m, n \in \mathbf{Z},\ z^c = 1\}$$

where $c \in \mathbf{Z}$. We shall only consider $c = 1$. In this case G/Γ is compact and $C_r^*(\widehat{G}/\Gamma, \rho)$ will be generated by $V(C_1(G/\Gamma, \rho))$. Note that in this case $C(G/K\Gamma) \cong C(\mathbf{T}^2)$, and that the usual 2nd derivative trick shows that $C^\infty(G/\Gamma)$ is a dense *-subalgebra of $C_1(G/\Gamma)$.

We shall start computing the product for some particular functions.

Lemma 3.4. *Let $\phi, \psi \in C_1(G/\Gamma)$ be defined by*

$$\phi(x, y, z) = \exp 2\pi i x \quad \text{and} \quad \psi(x, y, z) = \exp 2\pi i y.$$

Then for all $f \in C_1(G/\Gamma)$ we have

$$\phi * f * \phi^*(x, y, z) = f(x, y, z \exp 4\pi i\mu) \tag{3.6}$$

$$\psi * f * \psi^*(x, y, z) = f(x, y, z \exp 4\pi i\nu) \tag{3.7}$$

Proof. Writing u for (x, y, z), and noting that $\widehat{\phi}(u, n) = 0$ for $n \neq 0$, we have from (1.7) and (3.4) that

$$\begin{aligned}
\phi * f * \phi^*(u) &= \sum_m \phi(u\rho(m))(f * \phi^*)\widehat{\ }(u, m) \\
&= \sum_m \phi(u\rho(m))\widehat{f}(u, m)\overline{\phi(u\rho(-m))} \\
&= \sum_m \exp(4\pi i\mu m)\widehat{f}(u, m) \\
&= f(u \exp 4\pi i\mu).
\end{aligned}$$

The computations for ψ are similar.

Corollary 3.5. *Every ideal of $C_r^*(\widehat{G}/\Gamma, \rho)$ is invariant under β_t for $t = \exp 4\pi i(m\mu + n\nu)$ with $m, n \in \mathbf{Z}$.*

Proposition 3.6. *There are functions $f, g \in C_1(G/\Gamma)$ with*

(3.8) $\beta_t(f) = tf$ *and* $\beta_t(g) = tg$ *for all* $t \in \mathbf{T}$, *and*

(3.9) $f * h * f^*(u) + g * h * g^*(u) = h(u\rho(-2))$ *for all* $h \in C(G/K\Gamma) \cong C(\mathbf{T}^2)$.

Proof. (One will find similar functions in [A1, Proposition 2.1].) Take

$$f(x,y,z) = z|\sin(\pi x)|\exp(-2\pi i[x]y)$$
$$g(x,y,z) = f(x+\tfrac{1}{2},y,z),$$

where $[x]$ denotes the integer part of x. The function f is obviously continuous at any point where $x \notin \mathbf{Z}$, and since $|f(x,y,z)| = |\sin(\pi x)|$, f is also continuous at points where $x \in \mathbf{Z}$. It is easily checked that $\beta_t(f) = tf$, and similarly for g; thus the partial Fourier series of both f and g have just one term and both belong to $C_1(G)$. For $m,n \in \mathbf{Z}$, we have

$$\begin{aligned} f((x,y,z)(m,n,1)) &= f(x+m,y+n,z\exp 2\pi imy) \\ &= z|\sin(\pi(x+m))|\exp 2\pi i(my-[x+m](y+n)) \\ &= z|\sin(\pi x)|\exp(-2\pi i[x]y) = f(x,y,z) \\ g((x,y,z)(m,n,1)) &= g(x+m,y+n,z\exp 2\pi imy) \\ &= f(x+\tfrac{1}{2}+m,y+n,z\exp 2\pi imy) \\ &= f(x+\tfrac{1}{2},y,z) = g(x,y,z), \end{aligned}$$

and hence $f,g \in C_1(G/\Gamma)$. Because $f = \widehat{f}(\cdot,1)$, we deduce from (1.7) and (3.4) that

$$\begin{aligned} f*h*f^*(u) &= \widehat{f}(u\rho(-1),1)(h*f^*)\widehat{\ }(u\rho(-1),-1) \\ &= f(u\rho(-1))\widehat{h}(u\rho(-1)\rho(-1),0)\widehat{f^*}(u\rho(-1),-1) \\ &= |f(u\rho(-1))|^2 h(u\rho(-2)). \end{aligned}$$

A similar computation holds for g, and since $|f|^2+|g|^2 = 1$ the conclusion follows.

Theorem 3.7. *The C^*-algebra $C^*_r(\widehat{G}/\Gamma,\rho)$ is simple if and only if $\{\mu,\nu,1\}$ is independent over $\mathbf{Q}$.*

Proof. $\Longleftarrow$: Suppose $\mathcal{J}$ is a non-zero 2-sided ideal. Then by Corollary 3.5 $\mathcal{J}$ is invariant under β_t for all $t \in \mathbf{T}$, and

$$\mathcal{J}_0 = \{a \in \mathcal{J} \mid \beta_t(a) = a \text{ for all } t \in \mathbf{T}\}$$

is a closed non-zero ideal in $C^*_r(\widehat{G}/\Gamma,\rho)^K = C(G/K\Gamma) \cong C(\mathbf{T}^2)$. (3.9) shows that $\mathcal{J}_0$ is invariant under translation by $\rho(2)$, which corresponds to translation by the element $(\exp 4\pi i\mu, \exp 4\pi i\nu)$ on $C(\mathbf{T}^2)$. Since such elements generate a dense subgroup of $\mathbf{T}^2$, we have $\mathcal{J}_0 = C(\mathbf{T}^2)$. Thus $1 \in \mathcal{J}_0 \subset \mathcal{J}$, which implies that $\mathcal{J} = C^*_r(\widehat{G}/\Gamma,\rho)$.

$\Longrightarrow$: Suppose $\mu p + \nu q = r$ with $p,q,r \in \mathbf{Z}$, not all 0, and define $h(x,y,z) = \exp 2\pi i(xp+yq)$. Then $h \in C(G/K\Gamma)$ and $h(u\rho(m)) = h(u)$ for all $m \in \mathbf{Z}$. Thus by

(1.7) h is a non-constant element of the center of $C_r^*(\widehat{G}/\Gamma,\rho)$, so this C*-algebra cannot be simple.

Remarks. It is a tedious but straightforward exercise to show that $C_1(G/\Gamma)$ and the algebra defined in [R1, Theorem 5.5] have isomorphic dense subalgebras, and hence they generate the same C^*-algebra under the regular representation.

It is not clear whether the algebras $C_r^*(\widehat{G}/\Gamma,\rho)$ are new simple C^*-algebras, or if they for instance are isomorphic to some rotation algebra $C_r^*(\mathbf{Z}^3,\rho)$. However, one should note that β cannot be a dual action, because in [AEE, Example 3.3] it is shown by a winding number argument that there are no unitaries f or g satisfying (3.8), and a spectral subspace for a dual action always contains a unitary. This also shows that at least two functions f,g are needed in Proposition 3.6.

§4. Deformations of $C_0(G/\Gamma)$.

We shall now look at how to deform $C_0(G/\Gamma)$ in general. In [LR, 4.5 and 4.7] we showed that

$$\beta_x \in \operatorname{Aut}(C_r^*(\widehat{G},\rho)) \iff (L_x\otimes L_x)\omega(L_{x^{-1}}\otimes L_{x^{-1}})=\omega, \tag{4.1}$$

so the subgroup Γ must consist of x satisfying (4.1). Also note that β_x is then unitarily implemented:

$$\begin{aligned}\langle L_y,\beta_x(f)\rangle &= \langle L_{yx},f\rangle\\ V(I\otimes L_x) &= W_G^*\sigma(\omega^*)(I\otimes L_x)=(L_x\otimes I)V(L_{x^{-1}}\otimes I)\end{aligned}$$

so $\mu(\beta_x(f))=L_xV(f)L_{x^{-1}}$.

Next we shall try to characterise those $x\in G$ satisfying (4.1), with ω given by (1.2). It is not so difficult to show that (4.1) holds if and only if x normalises $H=\overline{\rho(\widehat{K})K}$ and $\widetilde{\omega}(s\circ\operatorname{Ad}x,t\circ\operatorname{Ad}x)=\widetilde{\omega}(s,t)$, where $\widetilde{\omega}$ is the (ordinary) cocycle on $\widehat{H}\times\widehat{H}$ corresponding to ω. We shall only look at cases in which x commutes with K: in our applications K is often central, and we shall see that we still get many interesting examples.

Lemma 4.1. *Let ω be the dual cocycle defined in (1.2). If $x\in G$ and $xk=kx$ for all $k\in K$, then $(L_x\otimes L_x)\omega(L_{x^{-1}}\otimes L_{x^{-1}})=\omega$ if and only if*

$$B_x(s):=x\rho(s)x^{-1}\rho(-s)\in K \text{ for all } s\in\widehat{K} \text{ and} \tag{4.2}$$

$$\langle B_x(s),t\rangle=\langle B_x(t),s\rangle \text{ for all } s,t\in\widehat{K}. \tag{4.3}$$

Proof. Since x centralises K, we have $L_x P_t = P_t L_x$, and

$$\begin{aligned}
&(L_x \otimes L_x)\omega(L_{x^{-1}} \otimes L_{x^{-1}}) = \omega \\
&\implies L_x P_s L_{\rho(-t)x^{-1}} \otimes L_{x\rho(s)} P_t L_{x^{-1}} = P_s L_{\rho(-t)} \otimes L_{\rho(s)} P_t \\
&\implies P_s L_{x\rho(-t)x^{-1}\rho(t)} \otimes P_t L_{x\rho(s)x^{-1}\rho(-s)} = P_s \otimes P_t \\
&\implies \exists\ B(s,t) \in \mathbf{T} \text{ with } P_t L_{x\rho(s)x^{-1}\rho(-s)} = B(s,t) P_t \text{ and } B(s,t) = B(-t,s)^{-1}.
\end{aligned}$$

So $B_x(s) := x\rho(s)x^{-1}\rho(-s)$ belongs to K, and $\langle B_x(s), t\rangle$ will be a symmetric bicharacter on $\widehat{K} \times \widehat{K}$.

Conversely, if (4.2) and (4.3) hold, the above line of argument can be reversed to get $(L_x \otimes L_x)\omega(L_{x^{-1}} \otimes L_{x^{-1}}) = \omega$.

This Lemma does not describe all $x \in G$ satisfying $(L_x \otimes L_x)\omega(L_{x^{-1}} \otimes L_{x^{-1}}) = \omega$. However, the condition $xk = kx$ for all $k \in K$ implies the useful property that

$$\beta_x(f)\hat{}\,(y,s) = \widehat{f}(yx,s) \text{ for all } f \in C_{b,1}(G). \tag{4.4}$$

Definition 4.2. From now on we shall unless explicitly stated otherwise make the following *standard assumptions:*

(S1) K is a compact abelian subgroup of G and $\rho : \widehat{K} \mapsto G$ is a homomorphism such that each $\rho(s)$ commutes with K.

(S2) Γ is a closed subgroup of G such that each $x \in \Gamma$ commutes with K and satisfies

$$\begin{aligned}
&B_x(s) := x\rho(s)x^{-1}\rho(-s) \in K \textit{ for all } s \in \widehat{K} \textit{ and} \\
&\langle B_x(s), t\rangle = \langle B_x(t), s\rangle \textit{ for all } s,t \in \widehat{K}.
\end{aligned}$$

(S3) G/Γ is compact and $\Gamma \cap K = \{e\}$.

Theorem 4.3. *If (S1) and (S2) hold, $\beta_x(f)(y) := f(yx)$ defines a *-automorphism of $C_{b,1}(G,\rho)$ for all $x \in \Gamma$, and $C_{0,1}(G/\Gamma,\rho) := C_0(G/\Gamma) \cap C_{b,1}(G,\rho)$ is a closed *-subalgebra. By taking the C^*-closure of $\mu(C_{0,1}(G/\Gamma,\rho))$ we obtain a deformation of $C_0(G/\Gamma)$ which we denote by $C^*_r(\widehat{G}/\Gamma,\rho)$. The left translations α_y defined by $\alpha_y(f)(x) = f(y^{-1}x)$ extend to *-automorphisms of $C^*_r(\widehat{G}/\Gamma,\rho)$.*

Proof. (This is just a repetition of Lemma 3.1.) Note that for all $x \in \Gamma$ we have (4.4) and

$x\rho(s) = \rho(s)xB_x(s)$. So we get

$$\begin{aligned}\beta_x(f*g)(y) = f*g(yx) &= \sum_{s,t} \widehat{f}(yx\rho(t),s)\widehat{g}(yx\rho(-s),t)\\ &= \sum_{s,t} \widehat{f}(y\rho(t)xB_x(t),s)\widehat{g}(y\rho(-s)xB_x(-s),t)\\ &= \sum_{s,t} \widehat{f}(y\rho(t)x,s)\langle B_x(t),s\rangle\widehat{g}(y\rho(-s)x,t)\langle B_x(-s),t\rangle\\ &= \sum_{s,t} \beta_x(f)\widehat{\ }(y\rho(t),s)\beta_x(g)\widehat{\ }(y\rho(-s),t) = \beta_x(f)*\beta_x(g)(y).\end{aligned}$$

The same arguments as in Lemma 3.1 give that $C_{0,1}(G/\Gamma,\rho)$ is a closed *-subalgebra, and from

$$R_y\mu(f)R_y^* = \mu(\alpha_y(f)) \tag{4.5}$$

it follows that α_y extends to a *-automorphism of $C_r^*(\widehat{G}/\Gamma,\rho)$.

Lemma 4.4. *If (S1) and (S2) hold, the maps β_k for $k \in K$ extend to *-automorphisms of $C_r^*(\widehat{G}/\Gamma,\rho)$ such that*

$$C_r^*(\widehat{G}/\Gamma,\rho)^K := \{a \in C_r^*(\widehat{G}/\Gamma,\rho) \mid \beta_k(a) = a \textit{ for all } k \in K\} = \mu(C_0(G/K\Gamma)).$$

Proof. (This is the general version of Lemma 3.3.) Since $\rho(s)$ commutes with $k \in K$ it follows from (1.7) that β_k is a *-automorphism. We are taking the fixed point algebra under a compact group, hence $\mu(C_{0,1}(G/\Gamma))^K = \mu(C_0(G/K\Gamma))$ is dense in $C_r^*(\widehat{G}/\Gamma,\rho)^K$. On $C_0(G/K\Gamma)$ the product (1.7) is the ordinary pointwise product, so μ is a *-isomorphism of the commutative C*-algebra $C_0(G/K\Gamma)$ and the two algebras must be equal.

We have now defined the algebras $C_r^*(\widehat{G}/\Gamma,\rho)$ and shown that they are equivariant deformations of $C_0(G/\Gamma)$. So far we have not used the assumption (S3), but from now on it will be important. If G is a Lie group one should note as in §3 that $C^\infty(G/\Gamma)$ is a dense *-subalgebra of $C_1(G/\Gamma)$.

In order to study the structure of the algebras $C_r^*(\widehat{G}/\Gamma,\rho)$ we shall repeatedly need the following:

Lemma 4.5. *Let X be a compact space on which a compact abelian group K acts freely on the right. Then for $p \in \widehat{K}$ there are finitely many functions $\phi_i \in C(X)$ such that*

$$\phi_i(xk) = \langle k,p\rangle\phi_i(x) \textit{ for } k \in K, \textit{ and} \tag{4.6}$$

$$\sum_{i=1}^{n} |\phi_i(x)|^2 = 1. \tag{4.7}$$

Proof. If $x_0 \in X$, define a function ϕ_0 on x_0K by $\phi_0(x_0k) = \langle k, p\rangle$ and extend it by Tietze's Theorem to a continuous function on X. Then $\phi(x) = \int_K \overline{\langle k, p\rangle}\phi_0(xk)dk$ satisfies (4.6) and $\phi(x_0) = 1$.

This shows that for each $x \in X$ there is a function ϕ_x satisfying (4.6) and $\phi_x(x) = 1$. Take $U_x = \{y : |\phi_x(y)| > 0\}$; then by compactness there are $x_1, \dots, x_n \in X$ such that $X = \cup_i U_{x_i}$. Hence $\psi = \sum_i |\phi_{x_i}|^2 > 0$ everywhere, $\psi(xk) = \psi(x)$, and with $\phi_i(x) := \psi(x)^{-1/2}\phi_{x_i}(x)$, (4.6) and (4.7) are satisfied.

Corollary 4.6. *For each $p \in \widehat{K}$ there are finitely many $\phi_i \in C_1(G/\Gamma)$ with*

$$\phi_i(xk) = \langle k, p\rangle\phi_i(x) \quad \textit{and} \quad \sum_1^n \phi_i * \phi_i^* = 1.$$

Hence every $a \in C_r^*(\widehat{G}/\Gamma, \rho)$ satisfying $\beta_k(a) = \langle k, p\rangle a$ for all $k \in K$ has the form $a = \sum_1^n \mu(\phi_i) * g_i$ with $g_i \in C(G/K\Gamma)$; in particular, $a \in \mu(C_1(G/\Gamma))$.

Proof. K acts on G/Γ to the right, and since $K \cap \Gamma = \{e\}$ the action is free. Let $\phi_i \in C(G/\Gamma)$ be as in Lemma 4.5. Then $\widehat{\phi}_i(x, s) = 0$ for $s \neq p$ and

$$\sum_1^n \phi_i * \phi_i^*(x) = \sum_1^n \phi_i(x\rho(-p))\overline{\phi_i(x\rho(-p))} = 1.$$

Given a satisfying $\beta_k(a) = \langle k, p\rangle a$ for all $k \in K$, take $g_i = \mu(\phi_i^*) * a \in C_r^*(\widehat{G}/\Gamma, \rho)^K = C(G/K\Gamma)$, and the corollary follows.

Remark. Corollary 4.6 corresponds to Proposition 3.6. It can be restated as follows: the subspace of all $a \in C_r^*(\widehat{G}/\Gamma, \rho)$ satisfying $\beta_k(a) = \langle k, p\rangle a$ for all $k \in K$ is a finitely generated $C(G/K\Gamma)$-module. Thus we did not have much choice when writing down the functions in Proposition 3.6.

The proofs in §3 work because $K\Gamma$ is a normal subgroup of G with $G/K\Gamma$ abelian. This will not be the case in general, and we shall need some more subgroups. In what follows it will be instructive for the reader to try to write down the different subgroups and homomorphisms introduced when G is the Heisenberg group of §3.

Lemma 4.7. *Let $G_1 = \{\rho(2t)K\Gamma \mid t \in \widehat{K}\}^-$. Then $K\Gamma$ is a closed, normal subgroup of G_1, and $G_1/K\Gamma$ is a compact abelian group which acts freely on the right of $G/K\Gamma$.*

Proof. Since K is compact and commutes with Γ, $K\Gamma$ is a closed subgroup of G. To show that it is normal in G_1, it suffices to show that $\rho(2t)x\rho(-2t) \in K\Gamma$ for $x \in \Gamma$. But by (S2)

$$\rho(2t)x\rho(-2t) = B_x(-2t)x \in Kx \subset K\Gamma.$$

The group $G_1/K\Gamma$ is abelian because the image of $\rho(2\widehat{K})$ is dense, and G/Γ compact implies $G/K\Gamma$ compact.

Since G_1 normalises $K\Gamma$, the map $(xK\Gamma, yK\Gamma) \mapsto xyK\Gamma$ with $x \in G/K\Gamma$ and $y \in G_1/K\Gamma$ is well-defined. The action is free because $xK\Gamma yK\Gamma = xK\Gamma$ implies $y \in K\Gamma$.

Definition 4.8. The map $\widehat{K} \mapsto G_1/K\Gamma$ given by $t \mapsto \rho(2t)K\Gamma$ is a homomorphism with dense image, so the transpose map $\kappa : (G_1/K\Gamma)^\frown \mapsto K$ defined by

$$\langle\kappa(p), t\rangle = \langle\rho(2t)K\Gamma, p\rangle \tag{4.8}$$

is injective.

Lemma 4.9. *With* $K_\rho = \overline{\mathrm{range}(\kappa)}$ *we have* $K_\rho^\perp = \{t \in \widehat{K} \mid \rho(2t) \in K\Gamma\}$.

Proof. Obvious.

Definition 4.10. Since $K \cap \Gamma = \{e\}$, each element of $K\Gamma$ can be written uniquely as a product kx with $k \in K,\ x \in \Gamma$. Thus there are homomorphisms $\widetilde{\theta}$ and γ of $K_\rho^\perp$ into K and Γ such that

$$\rho(2t) = \widetilde{\theta}(t)\gamma(t) \quad \text{for} \quad t \in K_\rho^\perp. \tag{4.9}$$

Now let $\theta : \widehat{K} \mapsto K/K_\rho$ be the transpose of $\widetilde{\theta}$:

$$\langle\theta(s), t\rangle := \langle\widetilde{\theta}(t), s\rangle \quad \text{for} \quad t \in K_\rho^\perp = (K/K_\rho)^\frown,\ s \in \widehat{K}. \tag{4.10}$$

Lemma 4.11. *Let G_ρ be the closed subgroup of G generated by*

$$\{\rho(-2p)\theta(p)K_\rho\Gamma \mid p \in \widehat{K}\}.$$

Then $K_\rho\Gamma$ is a closed normal subgroup of G_ρ, and $G_\rho/K_\rho\Gamma$ is a compact abelian group.

Proof. As in Lemma 4.7, $K_\rho\Gamma$ is a closed subgroup. Note that for $x \in \Gamma$ and $s \in K_\rho^\perp$ we have

$$B_x(2s) = x\rho(2s)x^{-1}\rho(-2s) = x\widetilde{\theta}(s)\gamma(s)x^{-1}\gamma(-s)\widetilde{\theta}(-s) = x\gamma(s)x^{-1}\gamma(-s) \in K \cap \Gamma,$$

so $B_x(2s) = e$. Hence $\langle B_x(2t), s\rangle = \langle B_x(2s), t\rangle = 1$ for all $s \in K_\rho^\perp$, $t \in \widehat{K}$, and we deduce that $B_x(2t) \in K_\rho$.

Now to show that $K_\rho\Gamma$ is normal in G_ρ, it suffices to show that $\rho(2t)x\rho(-2t) \in K_\rho\Gamma$ for $x \in \Gamma$; but from (S2) we have

$$\rho(2t)x\rho(-2t) = B_x(-2t)x \in K_\rho x \subset K_\rho\Gamma.$$

That the quotient is a compact abelian group follows as in Lemma 4.7. $\square$

The next two results are the general versions of Lemma 3.4 and Corollary 3.5.

Lemma 4.12. *For $p \in (G_1/K\Gamma)\widehat{\ }$ choose by Lemmas 4.5 and 4.7 a finite set $\{\psi_i\} \subset C(G/K\Gamma)$ such that*

$$\psi_i(xy) = \langle yK\Gamma, p\rangle \psi_i(x) \quad \text{for} \quad y \in G_1. \tag{4.11}$$

and $\sum_i |\psi_i|^2 = 1$. Then

$$\sum_1^n \psi_i * f * \psi_i^*(x) = f(x\kappa(p)) \quad \text{for all} \quad f \in C_1(G/\Gamma). \tag{4.12}$$

Proof. $\{\psi_i\}$ satisfies $\sum |\psi_i(x)|^2 = 1$. Since $\widehat{\psi}_i(x,s) = 0$ for $s \neq 0$ we have for any $f \in C_1(G/\Gamma)$:

$$\begin{aligned}
\sum_i \psi_i * f * \psi_i^*(x) &= \sum_{i,s} \psi_i(x\rho(s))(f * \psi_i^*)\widehat{\ }(x,s) \\
&= \sum_{i,s} \psi_i(x\rho(s))\widehat{f}(x,s)\overline{\psi_i(x\rho(-s))} \\
&= \sum_{i,s} \psi_i(x\rho(-s))\langle \rho(2s)K\Gamma, p\rangle \widehat{f}(x,s)\overline{\psi_i(x\rho(-s))} \\
&= \sum_s \langle \kappa(p), s\rangle \widehat{f}(x,s) = \sum_s \widehat{f}(x\kappa(p), s) = f(x\kappa(p)).
\end{aligned}$$

Corollary 4.13. *Any two-sided ideal of $C_r^*(\widehat{G}/\Gamma, \rho)$ is invariant under β_k for $k \in K_\rho$.*

Proof. This follows because $\{\kappa(p)\}$ is dense in K_ρ.

Our goal is to study the ideal structure of $C_r^*(\widehat{G}/\Gamma, \rho)$, and Corollary 4.13 shows that any ideal will contain an ideal of $C_r^*(\widehat{G}/K_\rho\Gamma, \rho)$. Our next step corresponds to Proposition 3.6 and consists of using the functions $\{\phi_i\}$ from Corollary 4.6 on $C_1(G/K_\rho\Gamma)$.

Lemma 4.14. *Let $p \in \widehat{K}$, and choose $\{\phi_i\}$ as in Corollary 4.6. Then for $f \in C_1(G/K_\rho\Gamma)$ we have*

$$\sum_1^n \phi_i * f * \phi_i^*(x) = f(x\rho(-2p)\theta(p)).$$

Proof. *Since $f \in C_1(G/K_\rho\Gamma)$ we have $\widehat{f}(x,s) = 0$ for $s \notin K_\rho^\perp$ and $\widehat{\phi}_i(x,s) = 0$ for $s \neq p$. Thus*

$$\begin{aligned}
\sum_i \phi_i * f * \phi_i^*(x) &= \sum_{i,s} (\phi_i * f)\widehat{\ }(x\rho(-p), s)\overline{\phi_i(x\rho(-s))} \\
&= \sum_{i,s} \phi_i(x\rho(-p)\rho(s-p))\widehat{f}(x\rho(-2p), s-p)\overline{\phi_i(x\rho(-s))} \\
&= \sum_{i,s} \phi_i(x\rho(s-p))\overline{\phi_i(x\rho(-s-p))}\widehat{f}(x\rho(-2p), s).
\end{aligned}$$

Since the summands are only non-zero when $s \in K_\rho^\perp$, we can use (4.9) to continue:

$$
\begin{aligned}
&= \sum_{i,s} \phi_i(x\rho(-s-p)\widetilde{\theta}(s)\gamma(s))\overline{\phi_i(x\rho(-s-p))}\widehat{f}(x\rho(-2p), s) \\
&= \sum_{i,s} |\phi_i(x\rho(-s-p))|^2 \langle \widetilde{\theta}(s), p\rangle \widehat{f}(x\rho(-2p), s) \\
&= \sum_{s} \langle \theta(p), s\rangle \widehat{f}(x\rho(-2p), s) \\
&= \sum_{s} \widehat{f}(x\rho(-2p)\theta(p), s) = f(x\rho(-2p)\theta(p)),
\end{aligned}
$$

as required.

Theorem 4.15. *Suppose G, ρ, Γ satisfy (S1)-(S3), and let G_ρ be the closed subgroup of G generated by $\{\rho(-2p)\theta(p)K_\rho\Gamma \mid p \in \widehat{K}\}$. Then the center of $C_r^*(\widehat{G}/\Gamma, \rho)$ is isomorphic to $C(G/G_\rho)$, and $C_r^*(\widehat{G}/\Gamma, \rho)$ is simple if and only if $G = G_\rho$.*

Proof. (1) For $f \in C(G/G_\rho)$ we have $\widehat{f}(x, s) = 0$ for $s \notin K_\rho^\perp$, and so for any $g \in C_1(G/\Gamma)$ we have

$$
f * g(x) = \sum_{s,t} \widehat{f}(x\rho(t), s)\widehat{g}(x\rho(-s), t),
$$

where the sum is over $s \in K_\rho^\perp$, $t \in \widehat{K}$. For such s, $\rho(2s) \in K\Gamma$, so that by (4.9) we have

$$
\begin{aligned}
f * g(x) &= \sum_{s,t} \widehat{f}(x\rho(t), s)\widehat{g}(x\rho(s)\widetilde{\theta}(-s)\gamma(-s), t) \\
&= \sum_{s,t} \widehat{f}(x\rho(t), s)\langle \widetilde{\theta}(-s), t\rangle \widehat{g}(x\rho(s), t) \\
&= \sum_{s,t} \widehat{f}(x\rho(t), s)\langle \theta(-t), s\rangle \widehat{g}(x\rho(s), t) \\
&= \sum_{s,t} \widehat{g}(x\rho(s), t)\widehat{f}(x\rho(t)\theta(-t), s) \\
&= \sum_{s,t} \widehat{g}(x\rho(s), t)\widehat{f}(x\rho(-t), s) = g * f(x),
\end{aligned}
$$

where at the last step we used that $f(x\rho(2t)\theta(-t)) = f(x)$. Hence $C(G/G_\rho)$ is contained in the center.

(2) Suppose $a \in C_r^*(\widehat{G}/\Gamma, \rho)$ is central. Then for $t \in \widehat{K}$, $a(t) = \int_K \overline{\langle k, t\rangle}\beta_k(a)dk$ is also central, and by Corollary 4.6 we have $a(t) \in C_1(G/\Gamma)$. Lemmas 4.9 and 4.12 imply that $a(t) \in C_1(G/K_\rho\Gamma)$ so we can use Lemma 4.14 to conclude that $a(t) \in C(G/G_\rho)$. Finally, a little Fourier analysis shows that a can be approximated in norm by the $a(t)$'s, and we deduce that $a \in C(G/G_\rho)$. Thus the center is precisely $C(G/G_\rho)$.

(3) From the above it follows that if $G \neq G_\rho$ then $C_r^*(\widehat{G}/\Gamma, \rho)$ is not simple. Conversely, $G = G_\rho$ implies that $C_r^*(\widehat{G}/\Gamma, \rho)$ has trivial center. In general a C^*-algebra with trivial center need not be simple, but in this case we shall see that it is. Suppose $G = G_\rho$ and that $\mathcal{J}$ is a non-zero 2-sided ideal. By Lemma 4.12, $\mathcal{J}$ is invariant under β_k for $k \in K_\rho$, so

$$\mathcal{J}_0 := \{a \in \mathcal{J} \mid \beta_k(a) = a \quad \text{for all} \quad k \in K_\rho\} \neq 0.$$

Lemma 4.14 then implies that $\mathcal{J}_0$ is invariant under β_x for x of the form $\rho(-2t)\theta(t)$, so since $G = G_\rho$ this holds for all $x \in G$. In particular $\mathcal{J}_0$ is invariant under β_k with $k \in K$, and

$$\mathcal{J}_1 := \{a \in \mathcal{J}_0 \mid \beta_k(a) = a \quad \text{for all} \quad k \in K\}$$

is a non-zero ideal of the commutative C*-algebra $C(G/K\Gamma)$. Since $\mathcal{J}_1$ is invariant under β_x with $x \in G = G_\rho$ it must be all of $C(G/K\Gamma)$. Thus $1 \in \mathcal{J}_1 \subset \mathcal{J}$, and we deduce that $\mathcal{J} = C_r^*(\widehat{G}/\Gamma, \rho)$.

Remark 4.16. Theorem 4.15 tells us that the center of $C_r^*(\widehat{G}/\Gamma, \rho)$ is isomorphic to $C(G/G_\rho)$, but the product in (1.7) is *not* the usual pointwise product in $C(G/G_\rho)$. However, it turns out that the algebra $C_1(G/G_\rho)$ endowed with this product is isomorphic to $C(G/G_\rho)$ with the pointwise product. This will be treated in a later article: let us just mention here that if $\theta \equiv 1$ then the isomorphism Φ is given by $\Phi(f) : x \mapsto \sum_s \widehat{f}(x\rho(s), s)$.

Example 4.17. The obvious generalization of §3 is the $2n+1$–dimensional Heisenberg manifold. Let $G = \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{T}$ with the product

$$(x_1, y_1, z_1)(x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 z_2 \exp(2\pi i y_1 \cdot x_2)),$$

where $x_i, y_i \in \mathbf{R}^n$ and $x \cdot y$ denotes the Euclidean scalar product. Then take

$$\Gamma = \{(p, q, 1) \in G \mid p, q \in \mathbf{Z}^n\}, \qquad K = \{(0,0,t) | t \in \mathbf{T}\} = \text{the center of } G,$$

choose two vectors $\mu, \nu \in \mathbf{R}^n$, and define $\rho : \widehat{K} \mapsto G$ by

$$\rho(m) = (m\mu, m\nu, \exp(i\pi m^2 \mu \cdot \nu)).$$

We leave it to the reader to show that (S1)-(S3) hold, and that Theorem 4.15 says $C_r^*(\widehat{G}/\Gamma, \rho)$ is simple if and only if the entries of μ, ν and 1 are independent over $\mathbf{Q}$.

The deformations of homogeneous spaces described here will by Theorem 4.15 be simple C*algebras if and only if $G_\rho = G$, so we are interested in examples where this is the case. However, the next two propositions show that in many typical cases this only happens if $G = G_\rho$ is nilpotent, i.e. if G/Γ is a nilmanifold.

Proposition 4.18. *With the assumptions in Theorem 4.15, suppose $G = G_\rho$ and that $H = \overline{\rho(\widehat{K})K}$ is a maximal abelian subgroup. Then G is 3-step nilpotent.*

Proof. Since K commutes with Γ, $G = G_\rho$ implies K central. For $x \in \Gamma, h = \rho(s)$ we have by (S2) that $xhx^{-1} = hk$ for some $k \in K$, hence this holds for all $x \in G$. (k depends on x and h.) So for $y \in G$ with $yhy^{-1} = hk'$ we have

$$yxhx^{-1}y^{-1} = yhky^{-1} = yhy^{-1}k = hk'k = hkk' = xyhy^{-1}x^{-1}.$$

Hence $y^{-1}x^{-1}yx$ commutes with H, so it is itself in H, i.e. $[G,G] \subset H$. Since $xhx^{-1}h^{-1} \in K$ for $h \in H$, $x \in G$, we have $[G,[G,G]] \subset [G,H] \subset K$. Thus $[G,[G,[G,G]]] \subset [G,K] = \{e\}$, and G is a 3-step nilpotent group. $\square$

Many examples of homogenous manifolds are obtained by taking Γ as a discrete and cocompact subgroup of a Lie group G. The next result shows that if $G = G_\rho$, then it is likely to be nilpotent.

Proposition 4.19. *With the assumptions in Theorem 4.15, suppose that Γ is discrete, G is connected and $G = G_\rho$. Then G is a 2-step nilpotent group.*

Proof. As in Proposition 4.18, $G = G_\rho$ implies K central. Since $G_\rho \subset \overline{\rho(\widehat{K})}K\Gamma$, in order to determine $[G,G]$ it suffices to look at $xyx^{-1}y^{-1}$ with $x = hu$, $y = lv$ for some $h, l \in \rho(\widehat{K})$ and $u, v \in \Gamma$. As in Proposition 4.18 we then have $ul = luk$ and $vl = lvk'$ so

$$\begin{aligned} xyx^{-1}y^{-1} &= hulvu^{-1}h^{-1}v^{-1}l^{-1} = hlkuvu^{-1}h^{-1}l^{-1}v^{-1}k' \\ &= hlh^{-1}l^{-1}uvu^{-1}v^{-1}kk'k'' = uvu^{-1}v^{-1}kk'k'' \end{aligned}$$

for some $k, k', k'' \in K$. So $[G,G] \subset [\Gamma,\Gamma]K \cong [\Gamma,\Gamma] \times K$. Since $[G,G]$ is also connected, it must lie inside the connected component of $[\Gamma,\Gamma]\times K$. So $[G,G] \subset K$, and $[G,[G,G]] = \{e\}$. $\square$

For examples of nilpotent Lie groups with discrete, cocompact subgroups, we refer to [CG, Chapter 5].

Example 4.20. Let G be the group of 4×4 upper triangular matrices with 1 on the diagonal, Γ the subgroup with integer entries. The center K consists of the matrices of the form:

$$\begin{pmatrix} 1 & 0 & 0 & s \\ & 1 & 0 & 0 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix}$$

If we divide both G and Γ by $K \cap \Gamma \cong \mathbf{Z}$, the standard assumption (S2) holds. Proposition 4.19 then tells us that, regardless of our choice of homomorphism $\rho : \widehat{K} \mapsto G$ satisfying (S1)-(S3), we have $G \neq G_\rho$, so $C(G/\Gamma)$ cannot be deformed to a simple C*-algebra.

For examples of 2-step nilpotent groups, see [BP]. The connected ones seem to be very much like Heisenberg groups, c.f. [S]. We believe that in most 2-step nilpotent groups it should be possible to obtain simple C*-algebras by deforming the corresponding nilmanifolds.

§5. $C_r^*(\widehat{G}/\Gamma, \rho)$ as a generalised fixed point algebra of $C_r^*(\widehat{G})$.

Rieffel's proof of the simplicity of the algebras in §3 uses that they are generalised fixed point algebras under an action of Γ, and thus Morita equivalent to the corresponding crossed products by Γ. These crossed products are then shown to be simple by classical methods. As we shall see, it is quite easy to see directly that our deformations are the generalised fixed point algebras for the action β of Γ on $C_r^*(\widehat{G})$.

From [LR, Proposition 5.2] it follows that $C_{b,1}(G)$ multiplies $C_{0,1}(G)$, so the representation μ maps $C_{b,1}(G)$ into $M(C_r^*(\widehat{G}))$. By [LR, Corollary 4.7], $\beta_x(f)(y) = f(xy)$ is an automorphism of $A(G, \rho)$ for $x \in \Gamma$. Since β is unitarily implemented we have that $C_r^*(\widehat{G}/\Gamma, \rho) \subset M(C_r^*(\widehat{G}, \rho))^\Gamma$.

We shall now show that the action β of Γ on $C_r^*(\widehat{G}, \rho)$ is *proper* as defined in [R3], by producing a dense subalgebra $\mathcal{A}_0$ with the right properties; notice that our algebra is not the one used by Rieffel in [R1].

Proposition 5.1. *Let*

$$\mathcal{A}_0 = C_{c,0}(G) = \{f \in C_c(G) \mid \widehat{f}(\cdot, t) = 0 \text{ except for finitely many } t \in \widehat{K}\}.$$

*Then $\mathcal{A}_0$ is a dense *-subalgebra of $C_r^*(\widehat{G}, \rho)$ satisfying [R3, Definition 1.2].*

Proof. It should be obvious from (1.9) that $\mathcal{A}_0$ is a dense *-subalgebra. If $f, g \in \mathcal{A}_0$, then $\phi : \Gamma \mapsto \mathcal{A}_0$ given by

$$\begin{aligned} \phi(y)(x) :&= \Delta(y)^{-1/2} f * \beta_y(g)(x) \\ &= \Delta(y)^{-1/2} \sum_{s,t} \widehat{f}(x\rho(t), s)\widehat{g}(x\rho(-s)y, t) \end{aligned}$$

is in $L^1(\Gamma, \mathcal{A}_0)$, so [R3, 1.2.1] holds. Note that we have the natural map p of $\mathcal{A}_0$ into $C_{c,0}(G/\Gamma, \rho) \subset M(\mathcal{A}_0)$ given by

$$p(f)(x) = \int_\Gamma f(xy)dy.$$

Then $\int_\Gamma f * \beta_y(g^* * h)dy = f * p(g^* * h)$. The action is *saturated*, since the functions of the form

$$(y, x) \mapsto \phi(y)(x) = \Delta(y)^{-1/2} f * \beta_y(g)(x)$$

with $f, g \in \mathcal{A}_0$ span a dense subspace of $C_c(\Gamma \times G)$. □

By [R3, 1.7] we now have:

Theorem 5.2. *The crossed product by the right action β of Γ on $C_r^*(\widehat{G}, \rho)$ is Morita equivalent to $C_r^*(\widehat{G}/\Gamma, \rho)$.*

§6. Concluding remarks.

If a compact group Γ acts on a space X, we can form the quotient space X/Γ. There are obvious non-commutative versions of this, and we shall discuss what happens if an automorphism τ acts on our locally compact group G. If $\tau \in \mathrm{Aut}(G)$, then by [LR, Lemma 4.5] τ lifts to a *-automorphism of $C_r^*(\widehat{G}, \omega)$ if and only if $\tau \otimes \tau(\omega) = \omega$. The following generalisation of Lemma 4.1 is then useful:

Lemma 6.1. *Let ω be the dual cocycle defined in (1.2). If $\tau \in Aut(G)$ leaves K invariant, then $\tau \otimes \tau(\omega) = \omega$ if and only if*

$$B(s) := \tau(\rho(s \circ \tau))\rho(-s) \in K \text{ for all } s \in \widehat{K} \text{ and} \tag{6.1}$$

$$\langle B(s), t\rangle = \langle B(t), s\rangle \text{ for all } s, t \in \widehat{K}. \tag{6.2}$$

Proof. One notes that $\tau^{-1}(P_t) = P_{t\circ\tau}$, and then the proof follows that of Lemma 4.1.

Corollary 6.2. *If τ satisfies (6.1) and (6.2) then $\tau(f) = f \circ \tau$ defines a *-automorphism of $C_{b,1}(G, \rho)$. With $G^\tau = G/\tau$ we have that $C_{0,1}(G^\tau, \rho) := C_0(G^\tau) \cap C_{b,1}(G, \rho)$ is a closed *-subalgebra.*

Proof. If one notes that

$$\tau(f)\widehat{\ }(y, s) = \widehat{f}(\tau(y), s \circ \tau^{-1}) \text{ for all } f \in C_{b,1}(G), \tag{6.3}$$

then one can use the argument of Theorem 4.3.

This way we obtain deformations of $C_0(G^\tau)$. Examples of this kind of construction are the non-commutative 2-spheres in [BEEK]: there $G = \mathbf{T}^2$, the deformations $C_{b,1}(G, \rho)$ are the usual rotation algebras, and the automorphism τ is given by $\tau(s, t) = (s^{-1}, t^{-1})$. It is not hard to see that an automorphism of $G = \mathbf{T}^2$ satisfying Lemma 6.1 must have the form $\tau(s,t) = (s^a t^b, s^c t^d)$ for some $a,b,c,d \in \mathbf{Z}$ satisfying $ad - bc = 1$. Other such automorphisms of the rotation algebras have been studied by Farsi and Watling (e.g. [FW]).

We have in this article been mostly concerned with the structure of the algebras $C_r^*(\widehat{G}/\Gamma, \rho)$, and have not discussed what happens if we let ρ vary. One would like to investigate whether the algebras are *strict deformation quantizations* as defined by Rieffel in [R1, Definition 1.1]. It seems reasonable from (1.6)-(1.9) that if we fix K and let ρ vary in such a way that $\rho \mapsto \rho(s)$ is continuous for all $s \in \widehat{K}$, we will get a strict deformation quantization of $C_0(G)$. If we also fix a subgroup Γ satisfying (S2)-(S3) the same should hold for $C_0(G/\Gamma)$. Any reader who wants to change ρ, K and Γ simultaneously will have to do so without the help of the authors.

We have earlier noted that if G is a Lie group and G/Γ is compact, then $C^\infty(G/\Gamma)$ is a dense *-subalgebra of $C_1(G/\Gamma)$. If the maps $\rho \mapsto \rho(s)$ also are differentiable we can define *Poisson brackets* on $C^\infty(G/\Gamma)$ by

$$\{f, g\} = \lim_{\lambda \to 0} \frac{1}{2i\lambda}(f *_\lambda g - g *_\lambda f)$$

where $f *_\lambda g$ is the product we get from a 1-parameter differentiable family of maps ρ_λ with $\rho_0 = i$.

Since we are assuming that G/Γ is compact, there is a G-quasi-invariant probability measure $dr^\sharp$ on G/Γ, and it turns out that the map $f \in C(G/\Gamma) \mapsto \int f dr^\sharp$ extends to a G-quasi-invariant trace on $C_r^*(\widehat{G}/\Gamma, \rho)$. This, and a study of all traces on $C_r^*(\widehat{G}/\Gamma, \rho)$ like that of [A1] for the non-commutative Heisenberg manifolds, will be treated in a separate paper.

We close with a short discussion about relaxing the standard assumptions (S1)-(S3). First, is it necessary to have K compact? If K is an abelian locally compact group and $\rho : \widehat{K} \mapsto G$ is a continuous homomorphism satisfying (S1), then (1.2) will still define a dual cocycle if we replace sums with direct integrals, and (1.5) gives equivariant deformations of $A(G)$ as described in [LR]. Similar cases have been treated in [L] and [R6]. One would then like to make a definition of $C_{b,1}(G)$ as in (1.6). The corresponding space would then have to consist of functions such that $k \mapsto f(xk)$ is in $A(K) \cap L^1(K)$ and such that these maps behaves "nicely" when x varies. However, when we want to go on and define $C_{b,1}(G/\Gamma)$ it is not clear how to proceed. If G/Γ is compact, but K not, we might have $\rho(\widehat{K})\Gamma$ dense in G, and then there is no reason to believe that we have any non-zero functions $f \in C_b(G/\Gamma)$ for which we can do Fourier analysis on $k \mapsto f(xk)$.

One could also consider subgroups Γ of elements x for which (4.1) holds, but not (4.2) or (4.3). We can still define $C_r^*(\widehat{G}/\Gamma, \rho)$, but instead of (4.4) we get

$$\beta_x(f)\widehat{\ }(y, s) = \widehat{f}(yx, s \circ \mathrm{Ad}x),$$

and it is harder to determine the structure of $C_r^*(\widehat{G}/\Gamma, \rho)$. We challenge the reader to look at the possibilities when (4.1) holds and $K = \mathbf{T}$.

The assumption that $\Gamma \cap K = \{e\}$ in (S3) does not seem to be crucial: if for example K is central, we can reduce to that case by dividing both G and Γ by $\Gamma \cap K$. In fact this is what we did in §3 to get the group G from the usual Heisenberg group. It is also possible that the assumption that G/Γ is compact is not be really needed. In Lemma 4.5, Corollary 4.6 and Lemma 4.14 one could instead find a partition of unity $\{\phi_i\}$ satisfying (4.6) and (4.7) locally. However $C_r^*(\widehat{G}/\Gamma, \rho)$ would then be nonunital, which would make the proofs more complicated.

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