QUANTIZATIONS ARISING FROM ABELIAN SUBGROUPS

Magnus B. Landstad

In recent years a lot of work has been done on quantum groups – although there is still no generally accepted definition of what this should be. However, there are many examples of quantum groups and the aim of this article is to show that many of these can be given a general description and is thus a step towards a unification of the theory.

We shall not try to give a general definition of a quantum group, all examples will be deformations of an "ordinary" group G. Here we take the view of e.g. [Rf1-3] that a deformation of a locally compact Hausdorff space X consists of taking a linear subspace A of $C_0(X)$ together with some continuous family $*_\omega$ of associative products on A such that in some norm $f *_\omega g \to fg$ (ordinary pointwise product) as $\omega \to 1$.

If $X = G$ is a group, it is natural to ask what remains of the group structure under the deformation. Many of the examples in the literature of deformations of $C_0(G)$ are obtained by doing some kind of Fourier transform of functions on G. So it seems natural to use $A(G) =$ the Fourier algebra as the subspace of $C_0(G)$ on which the product is changed – it is the canonical space on which to do Fourier transforms. Motivated by [Rf1] equivariant deformations of $C_0(G)$ were studied in [LR], i.e. products $*_\omega$ on $A(G)$ such that the action of $x \in G$ on $A(G)$ by left translations still is an automorphism. To a product $*_\omega$ on $A(G) = \mathcal{L}(G)_*$ there is a natural dual map $\delta_\omega : \mathcal{L}(G) \to \mathcal{L}(G) \otimes \mathcal{L}(G)$ given by

$$< \delta_\omega(T), f \otimes g >=< T, f *_\omega g >$$

and the deformation is called a **quantum group** if δ_ω is a comultiplication, i.e. a *-homomorphism of $\mathcal{L}(G)$. This is not true for the equivariant deformations in [LR] – there $\delta_\omega(T) = \delta_G(T)\omega^*$ with ω a unitary (a dual cocycle) in $\mathcal{L}(G) \otimes \mathcal{L}(G)$ and δ_G is the ordinary comultiplication on $\mathcal{L}(G)$ corresponding to the pointwise product in $A(G)$. If we instead look at the map

$$\delta_\omega : T \mapsto \omega\delta_G(T)\omega^*$$

this will be a *-homomorphism of $\mathcal{L}(G)$, and we get an associative product on $A(G)$ corresponding to a quantum group. This example of a quantum group is mentioned in [Dr, §10] and was shown by [L] and [Ws] to be the (still unsolved) obstruction to the classification of ergodic actions of compact groups on von Neumann algebras.

In §2 we shall study the deformations of $A(G)$ obtained by letting ω be a dual cocycle coming from an abelian subgroup H of G. On the outset this seems quite standard and it was some surprise to the author that not only many examples in the literature could be described this way – but also that this was not so straightforward to show. In fact, this part has taken most of the work to get in a satisfactory form.

In §3 and §4 we study deformations of $A(G)$ such that the corresponding comultiplication is of the form $T \mapsto U\delta_G(T)U^*$ with U a more general unitary operator on $L^2(G \times G)$. This construction depends on having a normal (in §3 central) subgroup H of G together with an action of $\widehat{H}$ on G satisfying certain properties.

The deformations of $A(G)$ studied in this article can be described two ways:

1) The corresponding comultiplication $\delta_U : \mathcal{L}(G) \to \mathcal{L}(G) \otimes \mathcal{L}(G)$ is of the form $\delta_U(T) = U\delta_G(T)U^*$ with U some unitary operator on $L^2(G \times G)$,
2) There is an abelian group H (which we for the moment assume is compact) and an action $(s, h)x = sxh$ of $\widehat{H} \times H$ on G. The product $f * g$ in $A(G)$ is then given by

$$f * g(x) = \sum_{s,t} f_{-s}(tx)g_t(sx) \quad \text{where} \quad f_s(x) = \int_H \overline{< h, s >} f(xh)dh.$$

Both descriptions are important, 1) shows that δ_U is a *-homomorphism, and 2) that we have an associative product on $A(G)$, i.e. the two important properties required for calling $(\mathcal{L}(G), \delta_U)$ or $(A(G), *)$ a quantum group. The above product looks like a crossed product and we shall in §2.1 give a definition of crossed product more suited to treat the algebras occurring in deformation theory. It turns out that it is not so new after all (Theorem 2.6), but still useful.

Furthermore 2) is important when one tries to extend the product to other classes of functions on G. If $f_s \neq 0$ only for finitely many $s \in \widehat{H}$, the above product is still well defined even when f is an unbounded function. This is useful in looking at non-compact matrix groups where functions like $x \mapsto x_{i,j}$ usually are unbounded, so such functions can be considered as unbounded elements affiliated the corresponding C*-algebra as defined in [Wo3]. We shall also see that at least in some cases the above product formula also works for H non-compact.

So far we have said nothing about the *-operation on $A(G)$, in most cases it is the usual complex conjugation, $f^*(x) = \overline{f(x)}$. If $\nu : \mathcal{L}(G) \to \mathcal{L}(G)$ is the anti-homomorphism such that $\nu(L_x) = L_{x^{-1}}$ for $x \in G$, this corresponds to that $(\mathcal{L}(G), \delta_U, \nu)$ is an involutive Hopf-von Neumann algebra. In some examples we also have to change the *-operation, the product is then instead given by the formula

$$f * g(x) = \sum_t f(tx)g_t(x)$$

(which is also seen to be an associative product) and the *-operation is given by the formula

$$f^*(x) = \overline{f(\nu(ux^*u^*))}$$

where u is some unitary operator on $L^2(G)$. Equivalently, if ν is replaced by $\tau : x \mapsto \nu(uxu^*)$ then $(\mathcal{L}(G), \delta_U, \tau)$ is an involutive Hopf-von Neumann algebra. In all the examples here $\tau^2 = i$, so we are inside the framework of Kac-algebras studied by Enock and Schwartz, c.f. [ES]. The algebras in question are quite often obtained by a crossed product of the form $G_1 \times_\alpha C_0(G_2)$ where G_1 and G_2 are both groups, this construction of Kac-algebras were first studied in generality in [C].

This article is written in the following way: Each § starts with showing how to define δ_U and τ such that $(\mathcal{L}(G), \delta_U, \tau)$ is an involutive Hopf-von Neumann algebra and what the corresponding product on $A(G)$ looks like. We then proceed to show that the given construction actually corresponds to cases already described elsewhere. This is usually the difficult part, furthermore it is many ways of describing the deformation of a group to a quantum group. In some cases it is described by changing the product on some space of functions on G ($C^\infty(G)$, Schwartz-functions etc.) The task is then to show that for the two descriptions there is 1) a common dense subalgebra and 2) there is a natural regular representation such that the corresponding C*-algebras are isomorphic.

Another point of view is to look at the Poisson bracket

$$\{f, g\} = \lim_{\lambda \to 0} \frac{1}{2i\lambda}(f *_\lambda g - g *_\lambda f)$$

when we have 1-parameter family $*_\lambda$ of products. A third point of view is to look at the Lie algebra $\mathfrak{g}$ and then look at deformations of the comultiplication $X \mapsto X \otimes I + I \otimes X$. The fourth point of view is the algebraic view, one looks at some natural space of algebraic functions on G and want certain algebraic relations to hold. All these views appear in the literature, and we shall try to compare with our point of view – although not in full detail.

The problem of quantization in our context is to find the "right" subalgebra of $C_0(G)$ to work with. Our candidate is $A(G)$, however we also have to work with some of its subalgebras. One of these is $A(G) \cap L^1(G)$ (note that this is an algebra under both pointwise product and convolution), however this is not always adequate either. If G is a Lie-group one could use some subspace of the smooth functions $C^\infty(G)$, like $\mathcal{S}(G)$ = the Schwartz-functions as done by Rieffel. The validity of some of the proofs here (changing the order of integration, Fourier inversion) would then be more obvious, however the author feels that although Lie groups are the most interesting examples, many aspects of deformations do not really depend on the differential structure. This – together with the fact that it is not at all obvious how to define $\mathcal{S}(G)$ for a general Lie group – is the reason for using the differential structure as little as possible.

Most groups here are of the form $G \cong \mathbb{R}^m \times \mathbb{T}^n$ (as topological spaces, not as groups), so $C_0(G)$ has a subalgebra $\cong L^1(\mathbb{R}^m \times \mathbb{Z}^n)$. Many deformations described in the literature are obtained by deforming the convolution product in $L^1(\mathbb{R}^m \times \mathbb{Z}^n)$, and then by taking the inverse transform one obtains a deformation of the product in $C_0(G)$, e.g. the classical non-commutative tori and [Rf1, Rf3-7]. The point of view taken here is the opposite, we deform the product in $A(G)$ directly (or equivalently the comultiplication in $\mathcal{L}(G)$), and we have to show that the constructions are essentially the same.

§§2-4 give examples of algebras in duality as studied by [Wl2]. If we start with a locally compact abelian group G the classical Pontryagin duality theorem can be phrased as follows: The algebras $L^1(G)$ and $L^1(\widehat{G})$ are in duality. §§2-4 show that it is possible to deform the product in both these algebras and still preserve duality between the group algebras. There has to be some relation between the two deformations, but it is still enough freedom such that it if we fix the product on $L^1(G)$ it is still usually a continuous family of products on $L^1(\widehat{G})$ satisfying this relation. So also between non-abelian groups there sometimes can be a duality in the sense that there is another group (usually several) such that there is a natural duality between the corresponding group algebras.

An important class of quantum groups **not** covered by this article is the quantization of the classical simple Lie groups (or Lie algebras) described in e.g. [Dr, ex. 6.2], [J] and [Wo1-2] – although we show that certain deformations of these described in [TV1-3] can be obtained in a similar way. The algebra being deformed is $C^\infty(G)$ – or in the algebraic case the polynomial functions on G – so the dual algebra is the enveloping algebra $\mathcal{U}(\mathfrak{g})$. This together with the fact that the involution τ does not satisfy $\tau^2 = i$ makes them different and interesting. The important difference is that in our examples the product on G (or $\mathcal{L}(G)$) is not changed – only the product in $A(G)$, in these other examples one also has to change the product in G at the same time. Sheu [S] has shown that also in this case the algebra in question can be considered as the deformation of some space of functions on the original space G. Although the basic ideas 1) and 2) described above also are involved here, we feel that the original approach by the above authors is the best way to present these quantum groups.

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§1. Preliminaries.

We use $< \cdot\, ,\ \cdot >$ for the pairing between a von Neumann algebra and its predual, and for that between an abelian group and its dual group. If $f : T \mapsto (T\xi|\eta)$ is a vector functional on a von Neumann algebra $M \subset B(\mathcal{H})$ (we sometimes write $f = \omega_{\xi,\eta}$), then S_f denotes the slice map $S_f : N \otimes M \to N$ (von Neumann algebra tensor product) characterized by

$$(S_f(T)h|k) = (T(h \otimes \xi)|k \otimes \eta) =< T, \omega_{\xi,\eta} \otimes f > .$$

For notations about $\mathcal{L}(G)$ = the von Neumann algebra of the left regular representation of G, the Fourier-algebra $A(G) = \mathcal{L}(G)_*$ etc., we refer to [L] or [Ws]. (Most of the definitions are meaningful also for non-compact groups.) The most used are: L = the left regular representation, and we shall often make the identification $L_x = x$ for $x \in G$. L^H is the restriction of L to a subgroup H, and W_G is the fundamental Takesaki-operator defined on $L^2(G \times G)$ by $W_G f(x, y) = f(x, x^{-1}y)$. Then $\delta_G : \mathcal{L}(G) \to \mathcal{L}(G) \otimes \mathcal{L}(G)$ and $\nu : \mathcal{L}(G) \to \mathcal{L}(G)$ are given by

$$\delta_G(x) = W_G(x \otimes I)W_G^* = x \otimes x \text{ and } \nu(x) = x^{-1} \text{ for } x \in G. \tag{1.1}$$

This defines the ordinary product and involution on $A(G)$:

$$< x, fg > \ = < \delta_G(x), f \otimes g > \tag{1.2}$$

$$< x, f^* > = \overline{< \nu(x^*), f >} \qquad \text{for } x \in \mathcal{L}(G). \tag{1.3}$$

A dual cocycle on G is a unitary element $\omega \in \mathcal{L}(G) \otimes \mathcal{L}(G)$ such that

$$(\omega \otimes I)\delta \otimes i(\omega) = (I \otimes \omega)i \otimes \delta(\omega). \tag{1.4}$$

Usually we require that ν is *normalized*, which means that

$$\sigma(\omega) = \omega^*, \quad \nu \otimes \nu(\omega) = \omega \quad \text{and} \quad i \otimes\ \nu(\omega) = \omega^* \tag{1.5}$$

where the last property should be interpreted as: $< \omega, f \otimes g \circ \nu >=< \omega^*, f \otimes g >$ and σ is the *flip* given by $\sigma(S \otimes T) = T \otimes S$. (In [L] there are some more properties, but they do not immediately make sense for non-compact G. We also have a slightly different definition in [LR], but they coincide in all known cases.) We can then define a new comultiplication by

$$\delta_\omega(x) = \omega \delta_G(x)\omega^*, \tag{1.6}$$

which satisfies

$$(\nu \otimes \nu)\delta_\omega(x) = \sigma \delta_\omega \nu. \tag{1.7}$$

This gives a new product on $A(G)$ by

$$< x, f * g > = < \delta_\omega(x), f \otimes g > . \tag{1.8}$$

Theorem 1.1. *$(\mathcal{L}(G), \delta_\omega, \nu)$ is an involutive Hopf-von Neumann algebra as in [T]. $A(G)$ with the product defined by (1.8) and the involution defined by (1.3) is a Banach *-algebra which we shall denote $A(G, \delta_\omega)$.*

Proof. That (1.7) holds and that δ_ω is a comultiplication – i.e. that $(\delta_\omega \otimes i)\delta_\omega = (i \otimes \delta_\omega)\delta_\omega$ – follows from (1.4) and (1.5). It is then standard that (1.8) defines an associative multiplication on $A(G)$. Since $f^*(T) = \overline{f(\nu(T^*))}$ for $T \in \mathcal{L}(G)$ it follows from (1.7) that this is an involution.

Note: This $A(G, \delta_\omega)$ is different from the algebra defined in [LR].

Both [L] and [Ws] realized that the maps δ_ω play an important role in the program of classifying the ergodic actions of compact groups on von Neumann algebras, and also the triangular QUE-algebras described in [Dr, §10] is a similar construction.

So far (c.f. [LR, Appendix]) the only known method of obtaining dual cocycles is the following: Let H be a closed abelian subgroup of G, and let ω be an ordinary 2-cocycle on $\widehat{H}$, i.e. $\omega : \widehat{H} \times \widehat{H} \to \mathbb{T}$ is measurable and satisfies the ordinary cocycle equation:

$$\omega(s, t)\omega(s + t, u) = \omega(t, u)\omega(s, t + u).$$

ω is then a unitary element of $L^\infty(\widehat{H} \times \widehat{H}) \cong \mathcal{L}(H) \otimes \mathcal{L}(H)$ and can then by Herz' result [H] be lifted to an element $\widetilde{\omega}$ of $\mathcal{L}(G) \otimes \mathcal{L}(G)$ which is then a dual cocycle on G, for details see [LR, Lemma 3.1]. Furthermore, we will only be interested in the case with $H \cong \mathbb{R}^m \times \mathbb{T}^n$, then Kleppner's characterization [K] of cocycles on $\widehat{H}$ applies, ω is equivalent to a cocycle of the form

$$\omega(s, t) = < \pi(s), t > \tag{1.9}$$

where $\pi : \widehat{H} \mapsto H$ is a continuous homomorphism which is *anti-symmetric*, i.e.

$$< \pi(s), s > = 1 \text{ for all } s \in \widehat{H}. \tag{1.10}$$

For such a π, $\pi(\widehat{H})^{\perp} = \ker \pi$, so by replacing H by $\overline{\pi(\widehat{H})}$ we may assume that π is injective ($\iff \pi(H)$ is dense in H), i.e. π is *totally skew* in the terminology of [BK]. As an element of $\mathcal{L}(H) \otimes \mathcal{L}(H)$ we have

$$\omega = \int_{\widehat{H}} \pi(s) \otimes dP_s = \int_{\widehat{H}} dP_s \otimes \pi(-s)$$

where $\{P_s\}$ is the projection valued measure associated with L^H, i.e.

$$L_h^H = \int_{\widehat{H}} < h, s > dP_s.$$

The lifting of ω to $\mathcal{L}(G) \otimes \mathcal{L}(G)$ is then given by

$$\widetilde{\omega} = \int_{\widehat{H}} L_{\pi(s)} \otimes d\widetilde{P}_s = \int_{\widehat{H}} d\widetilde{P}_s \otimes L_{\pi(-s)}$$

where $\{\widetilde{P}_s\}$ is the projection valued measure associated with L restricted to H, i.e.

$$L_h = \int_{\widehat{H}} < h, s > d\widetilde{P}_s \quad \text{for} \quad h \in H.$$

We will relax the notation somewhat and write

$$\omega = \int \pi(s) \otimes dP_s \quad \text{as an element of } \mathcal{L}(G) \otimes \mathcal{L}(G). \tag{1.11}$$

We have frequent use for this kind of construction, so we state it as:

Proposition 1.2. *Let $\{P_s\}_{s \in X}$ be a projection valued measure on $\mathcal{H}$, $\{u_s\}_{s \in X}$ a measurable family of unitary operators in $\mathcal{B}(\mathcal{H})$ such that u_t and $\int \phi(s) dP_s$ commute for all $t \in X$ and $\phi \in L^{\infty}(X)$. Then $u = \int u_s dP_s$ defines a unitary operator in $\mathcal{B}(\mathcal{H})$.*

Proof. $\{u_s \big| s \in X\}'' = L^{\infty}(Y)$ for some measure space Y. So there is a projection valued measure Q on $X \times Y$ such that $u_t = \int_Y \phi(s,t) dQ_{(s,t)}$ for some $\phi \in L^{\infty}(X \times Y)$ with $|\phi| = 1$, and also $\int_X f(s) dP_s = \int_{X \times Y} f(s) dQ_{(s,t)}$ i.e. $dP_s = \int_Y dQ_{(s,t)}$. So $u = \int_X u_s dP_s = \int_{X \times Y} \phi(s,t) dQ_{(s,t)}$ is a well defined unitary.

Corollary 1.3. *Suppose H is a locally compact abelian group and that we have commuting unitary representations S of H and U of $\widehat{H}$. Let $\{P_s\}_{s \in \widehat{H}}$ be the projection valued measure associated with S. Then $u = \int U_s dP_s$ defines a unitary operator.*

Note. In this case $U_t = \int < h, t > dQ_h$ for some projection valued measure Q on H. So $u = \int < h, t > dQ_h dP_t$.

The dual cocycles which will interest us most are the ones coming from $H = H_1 \times H_2$ and with π of the form

$$\pi(s,t) = (\pi_0(t), -\pi_0^t(s)) \text{ for } s \in \widehat{H}_1, t \in \widehat{H}_2 \tag{1.12}$$

where $\pi_0 : \widehat{H}_2 \to H_1$ is a continuous homomorphism. For instance when $\dim(H) = 2$ this is always the case. Then we will have for the corresponding ω in $\mathcal{L}(G) \otimes \mathcal{L}(G)$:

$$\omega = \int_{\widehat{H}_1} \int_{\widehat{H}_2} \big(\pi_0(t), -\pi_0^t(s)\big) \otimes dP_s^1 dP_t^2. \tag{1.13}$$

$\{P_s^i\}$ is here the projection valued measure associated with $L|H_i$, so

$$\begin{aligned}
\omega &= \int (\pi_0(t), -\pi_0^t(s)) dP_u^1 dP_v^2 \otimes dP_s^1 dP_t^2 \\
&= \int <\pi_0(t), u><-\pi_0^t(s), v> dP_u^1 dP_v^2 \otimes dP_s^1 dP_t^2 \\
&= \int <\pi_0(t), u> dP_u^1 \otimes dP_t^2 \int \overline{<\pi_0(v), s>} dP_v^2 \otimes dP_s^1 \\
&= \omega_0 \sigma(\omega_0)^* = \sigma(\omega_0)^* \omega_0 \quad \text{where} \\
\omega_0 &= \int <\pi_0(t), u> dP_u^1 \otimes dP_t^2 = \int_{\widehat{H}_2} (\pi_0(t), 0) \otimes dP_t^2 = \int_{\widehat{H}_1} dP_u^1 \otimes (0, \pi_0^t(u)).
\end{aligned}$$

So we have proved most of the following:

Proposition 1.4. *Suppose H is a closed abelian subgroup of G of the form $H = H_1 \times H_2$ and that $\pi_0 : \widehat{H}_2 \to H_1$ is a continuous homomorphism. Then $\pi : \widehat{H} \to H$ given by $\pi(s,t) = (\pi_0(t), -\pi_0^t(s))$ is an anti-symmetric homomorphism, and the corresponding dual cocycle in $\mathcal{L}(G) \otimes \mathcal{L}(G)$ satisfies*

(i) $\quad \omega = \int_{\widehat{H}} \pi(s) \otimes dP_s = \omega_0 \sigma(\omega_0)^*$ *where* $\omega_0 = \int_{\widehat{H}_2} \pi_0(t) \otimes dP_t^2,$

(ii) $\quad \nu \otimes \nu(\omega_0) = \omega_0$, *so* $\nu \otimes \nu(\omega) = \omega$ *and*

(iii) $\quad i \otimes \nu(\omega) = \omega^*$

Proof. (ii) follows from $\nu(x) = x^{-1}$ and $\nu(P_s) = P_{-s}$. (iii) has to be interpreted as in (1.5):

$$<\omega, \varphi \otimes \psi \circ \nu> = <\omega^*, \varphi \otimes \psi> \text{ for } \varphi, \psi \in A(G).$$

Corollary 1.5. *Suppose $H = H_1 \times \mathbb{T}$ with $H_1 = \mathbb{R}$ or $H_1 = \mathbb{T}$, $\lambda \in \mathbb{R}$. Replace π_0 in Proposition 1.4 by $\pi_\lambda : \mathbb{Z} \to H_1$ given by $\pi_\lambda(n) = \lambda n$ or $e^{i\lambda n}$, then*

$$\omega_0(\lambda) = \sum_n \pi_\lambda(n) \otimes P_n^2 \ , \quad \omega_\lambda = \sum_{m,n} \pi_\lambda(n) P_m^2 \otimes \pi_\lambda(-m) P_n^2 = \omega_0(\lambda) \sigma \omega_0(\lambda)^*$$

and $\sigma(\omega_\lambda) = \omega_\lambda^* = \omega_{-\lambda}$.

The same result holds if $H_2 = \mathbb{R}$ – one only has to replace the sums by direct integrals. We also are simplifying the notation: $\pi_\lambda(n) = (\lambda n, 0) \in G$.

Remark 1.6. If $x \in G$ and $\alpha_x(\varphi)(y) = \varphi(x^{-1}y)$ then α_x is an automorphism of $A(G, \delta_\omega) \iff \omega(x \otimes x)\omega^* = x \otimes x$. If $\omega = \int \pi(s) \otimes dP_s$ with π injective $\big(\iff \pi(\widehat{H})$ is dense in $H\big)$ this is the case $\iff xHx^{-1} = H$ and $x\pi(s)x^{-1} = \pi(xs)$ for all s, c.f. [LR, 4.7]. The action of x on $\widehat{H}$ is given by $< h, xs > = < x^{-1}hx, s >$.
If $H = \mathbb{R}^n$ then $xhx^{-1} = Ah$ for some $A \in GL(n, \mathbb{R})$, and $\pi : \widehat{H} \to H$ is given by a skew symmetric matrix. $x\pi(s)x^{-1} = \pi(xs)$ is then seen to be equivalent to $A\pi A^t = \pi$.
For $n = 2$ and $\pi_\lambda = \begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix}$, $A\pi A^t = \det(A)\pi$, so this is equivalent to $\det A = 1$.

§2. Structure of the algebra $A(G, \delta_\omega)$.

This part is devoted to a closer study of the algebra $A(G, \delta_\omega)$. We shall get explicit formulas for the product $f * g$ using that $A(G)$ also can be considered as functions on $\widehat{H} \times G$. Here also a "new" definition of crossed product is introduced. The representations of $A(G, \delta_\omega)$ are studied in §2.2, in particular it turns out that the regular representation is important and we define the C*-algebra $C_r^*(\widehat{G}, \delta_\omega)$ generated by this representation. The rest of §2 is devoted to examples, Poisson brackets in §2.3, then in §2.4 we show that certain deformations studied by Truini and Varadarajan can be described this way. We treat in §2.5 the semi-direct product $G = H \times_A K$ of two abelian groups – it turns out that on the C*-algebra level $C_r^*(\widehat{G}, \delta_\omega) \cong$ the crossed product $C_r^*(\widehat{H}, C_0(K), B)$ for some action of $\widehat{H}$ on K, (Theorem 2.27). This example seems new at least in this generality, although similar cases treated on the Lie algebra level in [Da2] and [Rf4] are related.

§2.1 Product formulas for $A(G,\delta_\omega)$ and a "new" crossed product.

We shall here first look closer at the structure of the algebra $A(G,\delta_\omega)$. For $f \in A(G)$ and $x \in G$ the functions $h \mapsto f(hx)$ and $h \mapsto f(xh)$ are both in $A(H) \cong L^1(\widehat{H})$, in fact we have as in [LR, Lemma 4.2]:

Proposition 2.1. *If $f \in A(G)$ there are functions $\widehat{f}_l,\ \widehat{f}_r : \widehat{H} \times G \to \mathbb{C}$ such that $x \mapsto \widehat{f}_l(\cdot,x)$ and $x \mapsto \widehat{f}_r(\cdot,x)$ are in $C_b(G, L^1(\widehat{H}))$ and*

$$\widehat{f}_l(s,hx) =< h,s > \widehat{f}_l(s,x) \tag{2.1}$$

$$\widehat{f}_r(s,xh) =< h,s > \widehat{f}_r(s,x) \tag{2.2}$$

$$\int_{\widehat{H}} \widehat{f}_l(s,x)ds = \int_{\widehat{H}} \widehat{f}_r(s,x)ds = f(x) \tag{2.3}$$

$$\sup_x \int_{\widehat{H}} |\widehat{f}_l(s,x)|ds \le \|f\|_{A(G)} \text{ and } \sup_x \int_{\widehat{H}} |\widehat{f}_r(s,x)|ds \le \|f\|_{A(G)}. \tag{2.4}$$

If the functions $h \mapsto f(hx)$ and $h \mapsto f(xh)$ are both in $L^1(H)$, then

$$\widehat{f}_l(s,x) = \int_H \overline{< h,s >} f(hx)\, dh \qquad \widehat{f}_r(s,x) = \int_H \overline{< h,s >} f(xh)\, dh.$$

This means that $\widehat{f}_l(s,x)ds =< dP_s x, f >$ and $\widehat{f}_r(s,xh)ds =< x dP_s, f >$, so we have:

Corollary 2.2. *The product on $A(G,\delta_\omega)$ defined by (1.8) is given by*

$$f * g(x) = \int \widehat{f}_r(s,\pi(t)x)\widehat{g}_l(t,x\pi(s))dsdt = \int \widehat{f}_l(s,x\pi(-t))\widehat{g}_r(t,\pi(-s)x)dsdt. \tag{2.5}$$

Proof. For $x \in G$ we have

$$\begin{aligned} < x, f * g > &=< \omega(x \otimes x)\omega^*, f \otimes g > \\ &= \int < \pi(t) \otimes dP_t(x \otimes x)dP_s \otimes \pi(s), f \otimes g > \\ &= \int \widehat{f}_r(s,\pi(t)x)\widehat{g}_l(t,x\pi(s))dsdt \end{aligned}$$

and similarly for the other formula.

This product is basically the same as in [Rf6, §1], just note that we also have at least formally (and if H is compact also in a rigorous way)

$$\begin{aligned} < x, f * g > &= \int < \pi(s)x\pi(-t), f >< dP_s x dP_t, g > \\ &= \int \overline{< h,s >< k,t >} f(\pi(s)x\pi(-t))g(hxk)\ dhdkdsdt \end{aligned}$$

and compare this with the formulas for $f \times_J g$ in [Rf6, §1].

We shall now look closer at ω coming from $H = H_1 \times H_2$ with H_2 *compact* as described in Corollary 1.5, it turns out that the product on $A(G, \delta_\omega)$ then can be extended to a larger subspace of $C(G)$. For $f, g \in A(G)$, $x \in G$ we now have

$$(2.6) \qquad < x, f * g >= \sum_{m,n,p,q} < \pi_0(p) P_m^2 x P_n^2 \pi_0(-q), f >< \pi_0(-m) P_p^2 x P_q^2 \pi_0(n), g > .$$

So if we for $f \in A(G)$ define the Fourier-coefficients $f_{m,n}(x) =< P_m^2 x P_n^2, f >$ the product is given by

$$(2.7) \qquad f * g(x) = \sum f_{m,n}\big(\pi_0(p) x \pi_0(-q)\big) g_{p,q}\big(\pi_0(-m) x \pi_0(n)\big).$$

Note that

$$(2.8) \qquad f_{m,n}(x) = \int_{H_2 \times H_2} \overline{< h, m >< k, n >} f(hxk) dh dk$$

and from this one sees that (2.8) makes sense for any $f \in C(G)$, hence we can extend (2.7) to define a product on

$$(2.9) \qquad CB^1(G) = \{ f \in CB(G) \big| \ \|\varphi\|_{\infty,1} < \infty \} \text{ where } \|f\|_{\infty,1} = \sum_{m,n} \sup_x \big| f_{m,n}(x) \big|.$$

With $\varphi^*(x) = \overline{\varphi(x)}$ this is a Banach *-algebra, and (if H_2 is a Lie-group) the old 2nd derivative trick tells us that the bounded C^∞-functions will form a dense *-subalgebra. In particular, $CB^1(G)$ is large enough to separate points in G. In fact, we have the following more general result:

Theorem 2.3. *Let X be a locally compact space on which a compact abelian group H acts continuously. For $f \in CB(X)$, $s \in \widehat{H}$ take*

$$(2.10) \qquad f_s(x) = \int_H \overline{< h, s >} f(hx) dh, \qquad \|f\|_{\infty,1} = \sum_{s \in \widehat{H}} \sup_x \big| f_s(x) \big| \qquad \text{and}$$

$$(2.11) \qquad CB^1(X) = \{ f \in CB(X) \ \big| \ \|f\|_{\infty,1} < \infty \}.$$

Suppose furthermore that there is an action of $\widehat{H}$ on X commuting with the action of H, written $(s, x) \to xs$. Then define a product and an involution by

$$(2.12) \qquad f * g(x) = \sum_{s,t} f_{-s}(xt) g_t(xs) \quad \text{and} \quad f^*(x) = \overline{f(x)}.$$

With these operations $CB^1(X)$ is a Banach *-algebra and $C_0^1(X) = CB^1(X) \cap C_0(X)$ is a closed 2-sided ideal.

Proof. (Compare also with [LR, Proposition 5.2].) $\|f\|_{\infty,1} < \infty$ means that $\sum_s f_s$ converges uniformly to f, hence the right side of (2.12) converges uniformly to a function in $CB(X)$. We leave it to the reader to prove that the product in (2.10) is associative and that $(f * g)^* = g^* * f^*$. If $f \in C_0^1(X)$, so is each f_s and $x \mapsto f_{-s}(xt)g_t(xs)$ is in $C_0(X)$. The uniform convergence in (2.12) then imply that $f * g \in C_0(X)$.

Remarks. 1. The product in (2.12) does **not** leave $C_{00}(X)$ invariant.
2. $C_0^1(X)$ is the cross-sectional algebra of a C^*-algebraic bundle over $\widehat{H}$ as defined by Fell. The fiber over $s = \{f_s \big| f \in C_0(X)\}$ c.f. [FD]. This can be utilized to determine the structure of the C^*-algebra generated by $C_0^1(X)$.
3. Since f_s makes sense for any continuous function on G – bounded or not – we could try to extend the product (2.12) also to unbounded functions. If $f_s \neq 0$ only for finitely many $s \in \widehat{H}$ this is possible, and this way such functions can be considered as unbounded elements affiliated the C*-algebra $C_r^*(\widehat{G}, \delta_\omega)$, c.f. [Wo3]. We shall make use of this later.

It would be useful to have a similar construction also when H is not compact, e.g. for $H = \mathbb{R}$. This would require assumptions on the nature of the action, but at least if $X = H \times Y$ it can be done, as the next example shows.

Example 2.4 Let $X = H \times Y$ with the obvious action $k(k', y) = (kk', y)$ of H and some action $(y, s) \to ys$ of $\widehat{H}$ on Y, lifted to X by $(k, y)s = (k, ys)$. Then it is easy to see that $CB^1(X) \cong L^1\big(\widehat{H}, CB(Y)\big)$, the isomorphism being the partial Fourier transform

$$f \in CB^1(X) \mapsto \widehat{f}(s, y) = \int_H \overline{< h, s >} f(h, y) dh. \tag{2.13}$$

Let us see how the operations in (2.12) translate to functions $\widehat{f}$ on $\widehat{H} \times Y$, note that $f_s(h, y) =< h, s > \widehat{f}(s, y)$ so

$$\begin{aligned}(f * g)\widehat{\ }(s, y) &= \int_H \overline{< h, s >} f * g(h, y) dh \\ &= \int \sum_{u,t} \overline{< h, s >} f_u(h, yt) g_t(h, y(-u)) dh \\ &= \int \sum_{u,t} \overline{< h, s >} < h, u > \widehat{f}(u, yt) < h, t > \widehat{g}(t, y(-u)) dh\end{aligned}$$

thus $t = s - u$ and

$$(2.14)\qquad (f * g)\widehat{\ }(s,y) = \sum_u \widehat{f}(u, y(s-u))\widehat{g}(s-u, y(-u)).$$

Since $f, g \in CB^1(X)$, the sum converges uniformly and this justifies the steps taken. It is straight forward to see that $\widehat{f^*}(s,y) = \overline{\widehat{f}(-s,y)}$.

The formula (2.14) looks very much like a crossed product, but not quite. Before showing that it actually corresponds to a crossed product, let us make an alternative definition of crossed products:

Definition 2.5. *Let H be an abelian locally compact group acting continuously by isometric *-automorphisms α_s on a Banach *-algebra A. Define a product and an involution on $L^1(H, A)$ by*

$$(2.15)\qquad f *_\alpha g(s) = \int \alpha_{s-u}(f(u))\alpha_{-u}(g(s-u))du \quad \text{and} \quad f^*(s) = f(-s)^*.$$

The proof that these operations make $L^1(H, A)$ a Banach *-algebra follows the same reasoning as for the usual crossed product, c.f. [P, 7.6] – and will in fact also follow from Theorem 2.6. The same kind of product also appears in [Rf7, 10.5]. Note that this product is more symmetric than the usual one in the following sense: $f *_\alpha g = g *_{-\alpha} f$ if A is abelian, here $-\alpha$ is the automorphism given by $(-\alpha)_s = \alpha_{-s}$.

Theorem 2.6. *The map $f \mapsto \widetilde{f}$ of $L^1(H, A)$ given by $\widetilde{f}(s) = \alpha_{-s}(f(s))$ identifies $L^1(H, A)$ with the operations in (2.15) with the ordinary crossed product $L^1(H, 2\alpha, A)$ where the operations are given by*

$$\varphi \star \psi(s) = \int \varphi(u)\alpha_{-2u}(\psi(s-u))du \qquad \varphi^\sharp(s) = \alpha_{-2s}(\varphi(-s)^*).$$

Proof. We have the following computations:

$$\begin{aligned}(f *_\alpha g)\widetilde{\ }(s) &= \int_H \alpha_{-u}(f(u))\alpha_{-s-u}(g(s-u))du = \int_H \widetilde{f}(u)\alpha_{-2u}(\widetilde{g}(s-u))du = \widetilde{f} \star \widetilde{g}(s)\\ (f^*)\widetilde{\ }(s) &= \alpha_{-s}(f(-s)^*) = \alpha_{-2s}(\widetilde{f}(-s)^*) = (\widetilde{f})^\sharp(s).\end{aligned}$$

Corollary 2.7. *If $X = H \times Y$ as in Example 2.4 the Banach *-algebras $C_0^1(X)$ and $L^1(H, C_0(Y))$ are isomorphic when the latter is given the crossed product structure rising from the action of $\widehat{H}$ on Y given by $(y, s) \to y(2s)$.*

§2.2 The regular representation.

Since we want to produce C*-algebras we shall proceed as in [BS] and [LR] to get *-representations of $A(G, \delta_\omega)$.

Definition 2.8. If ω is a dual cocycle on G, then a δ_ω-**corepresentation** of G on $\mathcal{H}$ is a unitary operator $V \in B(\mathcal{H}) \otimes L(G)$ such that

$$V_{12}V_{13} = i \otimes \delta_\omega(V). \tag{2.16}$$

(We use the now standard notation: $V_{12} = V \otimes I$ and $V_{13} = i \otimes \sigma(V \otimes I)$.) If V is a δ_ω-corepresentation of G on $\mathcal{H}$, then $V(f) = S_f(V)$ defines a norm-decreasing homomorphism of $A(G, \delta_\omega)$ into $B(\mathcal{H})$. This representation is non-degenerate, c.f. the proof of [LR, Theorem 1.14] or [BS, Appendice]. It is a *-representation $\iff$

$$\big(V^*(h \otimes \overline{\eta})\big|k \otimes \overline{\xi}\big) = \big(V(h \otimes \xi)\big|k \otimes \eta\big) \quad \text{for all} \quad h, k \in \mathcal{H}_V \text{ and } \xi, \eta \in L^2(G). \tag{2.17}$$

Let $J\xi(x) = \Delta(x)^{-1/2}\xi(x^{-1})$ for $\xi \in L^2(G)$, i.e. J is the equivalence between the left regular and right regular representations of G. Let $\widetilde{\omega} = (J \otimes I)\sigma(\omega^*)(J \otimes I)$.

Proposition 2.9. *$V = \omega W_G \widetilde{\omega}$ is a δ_ω-corepresentation of G over $L^2(G)$.*

Proof. We write $W = W_G$, $\delta = \delta_G$ and calculate

$$\begin{aligned}
i \otimes \delta_\omega(V) &= \omega_{23} i \otimes \delta(\omega) i \otimes \delta(W) i \otimes \delta(J \otimes I\sigma(\omega^*)J \otimes I)\omega_{23}^* \\
&= \omega_{12}\delta \otimes i(\omega)W_{12}W_{13}(J \otimes I \otimes I)(\sigma \otimes i)(i \otimes \sigma)[\delta \otimes i(\omega^*))\omega_{12}^*](J \otimes I \otimes I) \\
&= \omega_{12}W_{12}\omega_{13}W_{13}(J \otimes I \otimes I)(\sigma \otimes i)(i \otimes \sigma)[i \otimes \delta(\omega^*)\omega_{23}^*](J \otimes I \otimes I) \\
&= \omega_{12}W_{12}\omega_{13}(J \otimes I \otimes I)W_{13}^*(\sigma \otimes i)[W_{23}\omega_{12}^*W_{23}^*]\sigma(\omega^*)_{13}(J \otimes I \otimes I) \\
&= \omega_{12}W_{12}\omega_{13}(J \otimes I \otimes I)\sigma(\omega^*)_{12}W_{13}^*\sigma(\omega^*)_{13}(J \otimes I \otimes I) \\
&= \omega_{12}W_{12}\omega_{13}\widetilde{\omega}_{12}W_{13}\widetilde{\omega}_{13} = V_{12}V_{13}.
\end{aligned}$$

If ω is normalized as defined in [LR], we would guess that V then automatically would satisfy (2.17), however we can only prove it for the case where ω is inflated from an abelian subgroup. This however is good enough for our purposes here. We will need the following partial Fourier-transform for $f \in L^2(G)$:

$$\mathcal{F}f(s,x) = \int_H < h, s > f(hx)\, dh \tag{2.18}$$

so if $f \in A(G) \cap L^2(G)$ then $\mathcal{F}f(s,x) = \widehat{f}_l(-s,x)$.

We have $\mathcal{F}f(s,x)ds = dP_s f(x)$ so with $\omega = \sigma(\omega^*) = \int \pi(s)\otimes dP_s$ as before we have for $f \in L^2(G\times G)$ and with $\Delta = \Delta_G =$ the modular function on G that

$$I\otimes\mathcal{F}\omega f(x;s,y) = I\otimes\mathcal{F}f(\pi(-s)x;s,y) \tag{2.19}$$

$$I\otimes\mathcal{F}\widetilde{\omega} f(x;s,y) = \Delta\pi(s)^{1/2} I\otimes\mathcal{F}f(x\pi(s);s,y). \tag{2.20}$$

We shall also use frequently that

$$\mathcal{F}f(s,h^{-1}x) =< h,s > \mathcal{F}f(s,x) \text{ for } h\in H. \tag{2.21}$$

Proposition 2.10. *Let $\omega = \int \pi(s)\otimes dP_s$, then $V = \omega W_G \widetilde{\omega}$ satisfies (2.17).*

Proof. Suppose $h,k \in C_c(G)$ and that $\xi,\eta\in L^2(G)$ both have the property that $s\mapsto \mathcal{F}\xi(s,x)$ has compact support in s. Then the following substitutions and changes in the order of integration are valid:

$$\begin{aligned}
\big(V^*(h\otimes\overline{\eta})\big|k\otimes\overline{\xi}\big) &= \int \omega^*(h\otimes\overline{\eta})(x,xy)\overline{\widetilde{\omega}(k\otimes\overline{\xi})(x,y)}dxdy \\
&= \int I\otimes\mathcal{F}\omega^*(h\otimes\overline{\eta})(x;s,xy)\overline{I\otimes\mathcal{F}\widetilde{\omega}(k\otimes\overline{\xi})(x;t,y)}dsdtdxdy \\
&= \int h(\pi(s)x)\mathcal{F}\overline{\eta}(s,xy)\Delta\pi(t)^{1/2}\overline{k(x\pi(t))\mathcal{F}\overline{\xi}(t,y)}dsdtdxdy \\
&\quad (x\mapsto \pi(-s)x\pi(-t) \quad y\mapsto \pi(t)y) \\
&= \int h(x\pi(-t))\mathcal{F}\overline{\eta}(s,\pi(-s)xy)\Delta\pi(t)^{-1/2} \\
&\qquad \overline{k(\pi(-s)x)\mathcal{F}\overline{\xi}(t,\pi(t)y)}dsdtdxdy \\
&= \int \Delta\pi(t)^{-1/2}h(x\pi(-t))\mathcal{F}\xi(-t,y)\overline{k(\pi(-s)x)\mathcal{F}\eta(-s,xy)}dsdtdxdy \\
&= \int I\otimes\mathcal{F}\widetilde{\omega}(h\otimes\xi)(x;-t,y)\overline{I\otimes\mathcal{F}\omega^*(k\otimes\eta)(x;-s,xy)}dsdtdxdy \\
&= \big(\widetilde{\omega}(h\otimes\xi)\big|W_G^*\omega^*(k\otimes\eta)\big) = \big(V(h\otimes\xi)\big|k\otimes\eta\big).
\end{aligned}$$

This completes the proof, since such h,k,ξ,η form a dense subset of $L^2(G)$.

We can sum up the results so far in the following

Theorem 2.11. *Let $\omega = \int_{\widehat{H}} \pi(s)\otimes dP_s$, then $V = \omega W_G \widetilde{\omega}$ satisfies*

$$V_{12}V_{13} = i\otimes\delta_\omega(V)$$
$$\big(V^*(h\otimes\overline{\eta})\big|k\otimes\overline{\xi}\big) = \big(V(h\otimes\xi)\big|k\otimes\eta\big),$$

so $V(f) = S_f(V)$ defines a nondegenerate *-representation of $A(G, \delta_\omega)$.

Definition 2.12. *$V = \omega W_G \widetilde{\omega}$ is called the* **regular** *δ_ω-corepresentation and the corresponding C^*-algebra generated by $\{V(f)\}$ is denoted $C_r^*(\widehat{G}, \delta_\omega)$.*

As in [LR] we leave open the problems:
(i) whether every *-representation of $A(G, \delta_\omega)$ is given by slicing a δ_ω-corepresentation and
(ii) whether $C_r^*(\widehat{G}, \delta_\omega)$ is isomorphic to the C^*-enveloping algebra $C^*(A(G, \delta_\omega))$.

We shall need a more explicit expression for how the operators $V(f)$ act on $L^2(G)$, and in order to do this we need certain unbounded operators defined for $r \in \mathbb{R}$ by

$$\Theta^r = \int_{\widehat{H}} \Delta\pi(s)^r dP_s \quad \text{i.e.} \quad \mathcal{F}\Theta^r \xi(s, x) = \Delta\pi(s)^r \mathcal{F}\xi(s, x). \tag{2.22}$$

We leave it to the reader to figure out the exact domain of Θ^r, in what follows it will be enough to look at $\xi \in L^2(G)$ such that $\mathcal{F}\xi(s, x)$ has compact support in $s \in \widehat{H}$.

Lemma 2.13. *If $f = \omega_{\xi,\eta}$ then*

$$\widehat{f}_r(s, x) = \int_G \mathcal{F}\xi(s, y)\overline{\eta(xy)}dy \quad \text{and} \quad \widehat{f}_l(s, x) = \int_G \xi(y)\overline{\mathcal{F}\eta(s, xy)}dy.$$

Proof. For suitably nice functions we have

$$\begin{aligned}
\widehat{f}_r(s, x) &= \int_H < h, s > f(xh^{-1})dh \\
&= \int_{H \times G} < h, s > \xi(hy)\overline{\eta(xy)}dhdy = \int_G \mathcal{F}\xi(s, y)\overline{\eta(xy)}dy
\end{aligned}$$

and similarly for $\widehat{f}_l(s, x)$.

Proposition 2.14. *If $f = \omega_{\xi,\eta}$ is such that $\mathcal{F}\xi(s, x)$ has compact support in $s \in \widehat{H}$ and $g \in A(G) \cap L^2(G)$ then $V(f)g = f_{1/2} * g$. Here $f_{1/2} = \omega_{\Theta^{1/2}\xi,\eta}$, so $\widehat{(f_{1/2})}_r(s, x) = \Delta\pi(s)^{1/2}\widehat{f}_r(s, x)$.*

Proof. With the assumptions on f and g and with $h \in L^1(G) \cap L^2(G)$ the following substitutions and changes of order of integration should be justified:

$$
\begin{aligned}
\big(V(f)g\big|h\big) &= \big(Vg\otimes\xi\big)\big|h\otimes\eta\big) = \big(\widetilde{\omega}^*W_G^*\omega^*(g\otimes\overline{\eta})\big|h\otimes\overline{\xi}\big)\\
&= \int \mathcal{F}\otimes I\omega^*(g\otimes\overline{\eta})(s,x;xy)\overline{I\otimes\mathcal{F}\widetilde{\omega}(h\otimes\overline{\xi})(x;t,y)}dsdtdydx\\
&= \int \mathcal{F}g(s,x)\overline{\eta(\pi(-s)xy)\Delta\pi(t)^{1/2}h(x\pi(t))\mathcal{F}\overline{\xi}(t,y)}dsdtdydx\\
&= \int \mathcal{F}g(-s,x)\overline{\eta(\pi(s)xy)}\Delta\pi(t)^{-1/2}\overline{h(x\pi(-t))}\mathcal{F}\xi(t,y)dsdtdxdy
\end{aligned}
$$

$x\mapsto x\pi(t)$ and $y\mapsto\pi(-t)y$ give

$$
\begin{aligned}
&= \int \widehat{g}_l(s,x\pi(t))\overline{h(x)}\Delta\pi(t)^{1/2}\mathcal{F}\xi(t,\pi(-t)y)\overline{\eta(\pi(s)xy)}dydsdtdx\\
&= \int \widehat{g}_l(s,x\pi(t))\mathcal{F}\Theta^{1/2}\xi(t,y)\overline{\eta(\pi(s)xy)h(x)}dydsdtdx\\
&= \int \widehat{(\omega_{\Theta^{1/2}\xi,\eta})}_r(t,\pi(s)x)\widehat{g}_l(s,x\pi(t))\overline{h(x)}dsdtdx\\
&= \int f_{1/2}*g(x)\overline{h(x)}dx.
\end{aligned}
$$

Remark. If G is unimodular (in fact only $\Delta\pi = 1$ is needed) then $V(f)g = f*g$ and $\int f*g(x)dx = \int f(x)g(x)dx$. So the regular representation is in this case the same as the GNS-representation defined by the positive functional $f\mapsto\int f(x)dx$ on $A(G,\delta_\omega)\cap L^1(G)$.

§2.3 Poisson brackets.

If one has a 1-parameter family of products on an algebra A, the Poisson bracket is defined by

$$\{f,g\} = \lim_{\lambda\to 0} \frac{1}{2i\lambda}(f *_\lambda g - g *_\lambda f).$$

Let us try to see what this will be in the case where we have an action of $\mathbb{R}\times\mathbb{T}$ on a space X, written $(r,t)x = txr$. We then get a 1-parameter family of actions of $\mathbb{Z}\times\mathbb{T}$ on X by $(n,t)x = tx(\lambda n)$, so we can form the product $f *_\lambda g$ as in (2.12) and try to compute $\{f,g\}$. To be able to do this we require X to be a differential manifold and that the actions are differentiable. If, $f,g\in C^\infty(X)$ one sees that $g *_\lambda f = f *_{-\lambda} g$ and that

$$\begin{aligned}
\{f,g\}(x) &= \frac{1}{i}\frac{d}{d\lambda}\Big|_0 f *_\lambda g(x)\\
&= \frac{1}{i}\frac{d}{d\lambda}\Big|_0 \sum_{m,n} f_m(x(\lambda n))g_n(x(-\lambda m))\\
&= \frac{1}{i}\sum_{m,n} nD_2 f_m(x)g_n(x) - mf_m(x)D_2 g_n(x)\\
&= \frac{1}{i}D_2 f(x)\sum_n \int_{\mathbb{T}} nt^{-n}g(tx)dt - \frac{1}{i}D_2 g(x)\sum_m \int_{\mathbb{T}} mt^{-m}f(tx)dt\\
&= -D_2 f(x)D_1 g(x) + D_2 g(x)D_1 f(x)
\end{aligned}$$

i.e. $\{f,g\} = (D_1 f)(D_2 g) - (D_2 f)(D_1 g)$ where $D_1 f(x) = \frac{d}{d\lambda}\big|_0 f(e^{i\lambda}x)$ and $D_2 f(x) = \frac{d}{d\lambda}\big|_0 f(x\lambda)$.
Here we have taken fairly lightly the question whether these sums converge, but at least for sufficiently nice functions f,g the equations hold.

If we do the same computations for the product (2.14) on $A(G,\delta_\omega)$ we get

$$\begin{aligned}
&\{f,g\} = -(L_1 f)(L_2 g) + (L_2 f)(L_1 g) + (R_1 f)(R_2 g) - (R_2 f)(R_1 g),\\
\text{where}\quad & L_1 f(x) = \frac{d}{d\lambda}\Big|_0 f(\pi_\lambda(1)x),\quad R_1 f(x) = \frac{d}{d\lambda}\Big|_0 f(x\pi_\lambda(1))\\
& L_2 f(x) = \frac{d}{d\lambda}\Big|_0 f(e^{i\lambda}x),\quad R_2 f(x) = \frac{d}{d\lambda}\Big|_0 f(xe^{i\lambda})\ .
\end{aligned}$$

In many articles about quantization one starts with a given Poisson bracket and the problem is to find a deformation giving exactly this bracket. This is not the point of view taken here, we start with certain deformations and then we study the corresponding brackets.

§2.4 Quantum links.

Suppose $G = G_1 \times G_2$ and take $H = H_1 \times H_2$ with H_i a 1-parameter subgroup of G_i, and assume $H_2 \cong \mathbb{T}$. If φ is a function on G constant in the second variable we have that $\varphi_{m,n} = 0$ except for $m = n = 0$. So if ψ has the same property we get from (2.14) that $\varphi * \psi(x_1, x_2) = \varphi(x_1)\psi(x_1)$, i.e. the ordinary product. If φ and ψ are both constant in the first variable we have

$$\begin{aligned}\varphi * \psi(x_1, x_2) &= \sum < (\lambda p, 0)(x_1, x_2)(-\lambda q, 0), \varphi_{m,n} > < (-\lambda m, 0)(x_1, x_2)(\lambda n, 0), \psi_{p,q} > \\ &= \sum \varphi_{m,n}(x_2)\psi_{p,q}(x_2) = \varphi(x_2)\psi(x_2),\end{aligned}$$

so this is also the ordinary product.

This is what is called a **quantum link** in [TV1]. That the product (2.14) really is the same as in [TV1] is not so obvious, so we shall do some of the computations.

Take $G_1 = G_2 = SL(2, \mathbb{C})$ and for $z \in \mathbb{C} \setminus \{0\}$ let $D(z)$ be the diagonal matrix with entries z and z^{-1}. Then fix $q \in \mathbb{C} \setminus \{0\}$, let $H_1 = \{D(q^s) | s \in \mathbb{R}\}$ and $H_2 = \{D(t) | t \in \mathbb{T}\}$. Let the coordinate functions be defined by $a_{ij}(x, y) = x_{ij}$ and $\alpha_{ij}(x, y) = y_{ij}$. Then

$$(a_{ij})_{0,0} = a_{ij},\ (\alpha_{11})_{1,1} = \alpha_{11},\ (\alpha_{12})_{1,-1} = \alpha_{12},\ (\alpha_{21})_{-1,1} = \alpha_{21} \text{ and } (\alpha_{22})_{-1,-1} = \alpha_{22},$$

so it seems useful to change notation by replacing indices 2 by -1, so $(\alpha_{ij})_{m,n} = 0$ unless $m = i$ and $n = j$, with $i, j \in \{1, -1\}$. Then

$$\begin{aligned}a_{ij} * \alpha_{kl}(x, y) &= a_{ij}(D(q^k)xD(q^{-l}))\alpha_{kl}(y) = q^{ik-jl}a_{ij}(x)\alpha_{kl}(y) \\ \alpha_{kl} * a_{ij}(x, y) &= \alpha_{kl}(y)a_{ij}(D(q^{-k})xD(q^l)) = q^{jl-ik}\alpha_{kl}(y)a_{ij}(x),\end{aligned}$$

so we have shown that

$$a_{ij} * \alpha_{kl} = q^{2(ik-jl)}\alpha_{kl} * a_{ij}.$$

These functions are not bounded, but we have utilized Remark 3 after Theorem 2.3. To see that this coincides with §1 in [TV1] we have to check that under the map $1 \mapsto 1$, $2 \mapsto -1$ the symbol $\{ijkl\}$ in [TV1] corresponds to $\frac{1}{2}(jl - ik)$. This is done as follows: The map $\{1, -1\} \mapsto \{1, 2\}$ is given by $i \mapsto i' = \frac{1}{2}(3 - i)$ and then

$$2\{i'j'k'l'\} = (-1)^{j'+l'} - (-1)^{i'+k'} = (-1)^{\frac{i+k}{2}} - (-1)^{\frac{j+l}{2}} = jl - ik$$

since $(-1)^{\frac{i+k}{2}} = -ik$ for $i, k \in \{1, -1\}$. So our q correspond to q^{-4} in [TV1].

It is shown in [TV2-3] that this example has an obvious generalization to the case where G is a reductive Lie group: Let H be a Cartan subgroup and construct ω and δ_ω

from an anti-symmetric $\pi : \widehat{H} \to H$. Then we get a deformation of the ordinary product on $A(G)$, or equivalently of the comultiplication δ_G of $\mathcal{L}(G)$. Rieffel's examples in [Rf6] are also of this form.

Now after some thought about this construction one realizes that a similar deformation also is possible if one replaces δ_G with the comultiplication δ_q of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ defined in [Dr, Example 6.2], [J] or equivalently of the compact quantum groups studied in [Wo1-2]. It is shown in [TV2-3] that any deformation of a reductive Lie algebra satisfying certain natural conditions is in fact obtained via the Cartan subgroup from either δ_G or δ_q.

We claim that the comultiplication δ_ω in (1.6) is in essence the same as in [TV2-3]. We shall not do this in full detail, but the following special case should give an idea of what is involved. Suppose $X \in \mathfrak{g} = \mathfrak{sl}(4, \mathbb{C})$ and $(x, t) \in H \cong \mathbb{R} \times \mathbb{T}$ are given by

$$X = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad (x, t) = \begin{pmatrix} e^x & & & \\ & e^{-x} & & \\ & & t & \\ & & & t^{-1} \end{pmatrix}$$

and $\omega = \omega_0 \sigma(\omega_0^*)$ as in Corollary 1.5. Then $(x, 0)X(-x, 0) = e^{2x}X$ for $x \in \mathbb{R}$ and $(0, t)X(0, t^{-1}) = X$ for $t \in \mathbb{T}$. So $\omega_0(I \otimes X)\omega_0^* = I \otimes X$, and

$$\omega_0(X \otimes I)\omega_0^* = \sum (\lambda n, 0)X(-\lambda n, 0) \otimes P_n^2 = X \otimes \exp\,(2\lambda H_2)$$

where $H_2 \in \mathfrak{g}$ is the generator of $\{(0, t) \mid t \in \mathbb{T}\}$. Thus

$$\delta_\omega(X) = \omega_0 \sigma(\omega_0^*)(X \otimes I + I \otimes X)\omega_0^* \sigma(\omega_0) = X \otimes \exp\,(2\lambda H_2) + \exp(-2\lambda H_2) \otimes X.$$

This should now be compared with Proposition 3.1 in [TV2] or Proposition 4.1 in [TV3].
Remark. This construction can also be generalized by looking at spaces Y, Z on which H, respectively $\widehat{H}$ acts and take $X = Y \times Z$ with the obvious action of $H \times \widehat{H}$. This will give a product on $CB^1(X)$ and both $CB(Y)$ and $CB(Z)$ can be embedded in $M(CB^1(X))$. As above the product of two functions both constant in $y \in Y$ will be the pointwise product and the same is of coarse true if both functions are constant in $z \in Z$. This can also be considered as a particular case of Theorem 2.3. The C*-algebra we obtain this way can be considered a non-commutative tensor product of the spaces $C_0(Y)$ and $C_0(Z)$, a point of view due to [Wo4].

§2.5 Semi-direct products.

We shall look at the case of a semi-direct product $G = H \times_A K$ of two abelian groups. So we have a homomorphism $A : H \to \mathrm{Aut}K$, and we take $G = H \times K$ with the multiplication

$$(a, b)(a', b') = (a + a', A(-a')b + A(a)b') \tag{2.23}$$

i.e. we think of K as a vector group and $A(a)$ as a matrix. This is not the usual way of defining the semi-direct product, (c.f. [HR, 2.6]), but it is the one used when the "$ax + b$"-group is defined as the matrix group with elements of the form $\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$.
Furthermore, we have an anti-symmetric continuous homomorphism $\pi : \widehat{H} \to H$, ω the corresponding dual cocycle (1.11) and δ_ω defined by (1.6). It turns out that the computations needed to study the structure of $A(G, \delta_\omega)$ are simpler with this product on G.

The main result is that $C^*_r(\widehat{G}, \delta_\omega)$ in this case is isomorphic to the crossed product (as defined in Theorem 2.6) by $\widehat{H}$ and $C_0(K)$ with the action of $\widehat{H}$ on K given by $(s, k) \mapsto A\pi(2s)k$. The idea is similar to the one we shall later use in §4.1, we take the partial Fourier transform $\widehat{f}(s, k) = \int \overline{< h, s >} f(h, k)dh$ and then study what kind of operations we get when functions in $A(G, \delta_\omega)$ are looked at as functions on $\widehat{H} \times K$. So this also fits with Rieffel's philosophy: Look at the subspace of $C_0(G)$ as functions on $\widehat{H} \times K$, then change the product by introducing another product on these functions.

We shall need Fourier analysis not only on H, but also for the action of H on K. Therefore it is necessary with some assumption on the action.

Definition 2.15. For $k \in K$ let $H_k = \{h \in H | A(h)k = k\}$. A is said to be **regular** if the map $h' \mapsto A(h')k$ is a homeomorphism from H/H_k to K for each $k \in K$.

Examples. If H and K are connected Lie groups, then A is always regular. If $K = \mathbb{Q}_d$ (i.e. with discrete topology) and $H = \mathbb{R}$, then $A(q)x = e^q x$ is not regular.

Haar-measure. Left, respectively right Haar-measure on $G = H \times_A K$ is given by

$$\int f(x)d\mu_l(x) = \int f((h, 0)(0, k))dhdk = \int f(h, k)\Delta(h^{-1})dhdk$$
$$\int f(x)d\mu_r(x) = \int f((0, k)(h, 0))dhdk = \int f(h, k)\Delta(h)dhdk$$

where Δ is given by $\int f(A(h)k)dk = \Delta(h^{-1}) \int f(k)dk$. So $\Delta_G(h, k) = \Delta(h)^{-2}$.

Definition 2.16. For functions on G we shall need the following four partial Fourier transforms, where $s \in \widehat{H}$ and $p \in H_k^\perp$:

$$\widehat{f}(s, k) = \int \overline{< h, s >} f(h, k)dh \tag{2.24}$$

$$\widehat{f}_l(s,k) = \int \overline{<h,s>} f((h,0)(0,k))dh \tag{2.25}$$

$$\widehat{f}_r(s,k) = \int \overline{<h,s>} f((0,k)(h,0))dh \tag{2.26}$$

$$\widehat{\widehat{f}}(s,p,k) = \int_H \int_{H/H_k} \overline{<h,s><h',p>} f(h,A(h')k)dh'dh. \tag{2.27}$$

So with the notation in Proposition 2.1 we have:

$$\widehat{f}_l(s,(h,k)) = <h,s> \widehat{f}_l(s,A(-h)k) \quad \text{and} \quad \widehat{f}_r(s,(h,k)) = <h,s> \widehat{f}_r(s,A(h)k)$$

Also note that

$$\widehat{\widehat{f}}(s,p,A(h)k) = <h,p> \widehat{\widehat{f}}(s,p,k). \tag{2.28}$$

Definition 2.17. *Let $A^{1,1}(G)$ consist of all $f \in A(G)$ such that for all $k \in K$*

$$f(\cdot, A(\cdot)k) \text{ is in } L^1(H \times H/H_k) \text{ and} \tag{2.29}$$

$$\widehat{\widehat{f}}(\cdot,\cdot,k) \text{ is in } L^1(\widehat{H} \times H_k^\perp). \tag{2.30}$$

Lemma 2.18. *$A^{1,1}(G)$ is dense in $A(G)$.*

Proof. Let $g \in A(G)$ be such that $(h,h') \mapsto g(h, A(h')k)$ is integrable on $H \times H/H_k$ for all k, since the action of H on K is regular such functions are dense in $A(G)$. Next let $\varphi \in L^1(H \times H)$ be such that $\widehat{\varphi}$ is integrable over any closed subgroup of $\widehat{H} \times \widehat{H}$, and note that such functions are dense in $L^1(H \times H)$. Then

$$f(h,k) = \int \varphi(a,b) g((a,0)(h,k)(b,0))dadb = \int \varphi(a,b) g(a+h+b, A(a-b)k)dadb$$

is in $A(G)$, satisfies (2.29) and $\widehat{\widehat{f}}(p,s,k) = \widehat{\varphi}(s-p,-p-s)\widehat{\widehat{g}}(p,s,k)$ so $f \in A^{1,1}(G)$. It should be clear that functions of this form will be dense in $A(G)$.

We shall need the Poisson summation formula in the following form:

Lemma 2.19. *Let A be an abelian locally compact group and B a closed subgroup. If $\varphi \in L^1(A)$ is such that $t \mapsto \widehat{\varphi}(t+s)$ is in $L^1(B^\perp)$ for all $s \in \widehat{A}$, then*

$$\int_B \overline{<a+b,s>} \varphi(b)db = \int_{B^\perp} <a,t> \widehat{\varphi}(s+t)dt$$

for almost all $a \in A$. Equality holds for all a if the left hand side is continuous.

Proof. [HR, 31.46].

Lemma 2.20. *For $f \in A^{1,1}(G)$ we have*

$$\widehat{f}_l(s,k) = \int_{H_k^\perp} \widehat{\widehat{f}}(s-p,p,k)dp \tag{2.31}$$

$$\widehat{f}_r(s,k) = \int_{H_k^\perp} \widehat{\widehat{f}}(s+p,p,k)dp \tag{2.32}$$

$$\widehat{\widehat{f}}(s,q,k) = \int_{H_k} \overline{< h,q >}\widehat{f}_r(s-q,A(h)k)dh \tag{2.33}$$

$$\widehat{\widehat{f}}_{1/2}(s,q,k) = \Delta\pi(q-s)\widehat{\widehat{f}}(s,q,k). \tag{2.34}$$

Proof. Let $\varphi(h,h') = f(h,A(h')k)$, so $\widehat{f}_l(s,k) = \int_H \overline{< h,s >}\varphi(h,h)dh$. Use Lemma 2.19 with $A = H \times H/H_k$ and $B = \{(h,h+H_k)|h \in H\}$, so $B^\perp = \{(-p,p)|p \in H_k^\perp\}$.
With $a = (0,0)$, $s' = (s,0)$ we get:

$$\widehat{f}_l(s,k) = \int_{H_k^\perp} \widehat{\varphi}(s-p,p)dp = \int_{H_k^\perp} \widehat{\widehat{f}}(s-p,p,k)dp.$$

Note that the continuity property in Lemma 2.19 holds for $f \in A^{1,1}(G)$. The formula for $\widehat{f}_r(s,k)$ is obtained by taking $B = \{(h,-h+H_k)\big|h \in H\}$. If we make the substitution $h' \mapsto h' - h$ in the definition of $\widehat{\widehat{f}}$ we get (2.33), and if we use this together with the fact that $(f_{1/2})_r{}^\wedge(s,k) = \Delta_G\pi(s)^{1/2}\widehat{f}_r(s,k) = \Delta\pi(-s)\widehat{f}_r(s,k)$ we get (2.34).

We are now ready to look at the product δ_ω defines on $A(G)$, and as mentioned above we shall look at what the product corresponds to when we use the Fourier transform $f \mapsto \widehat{f}$.

Lemma 2.21. *For $f,g \in A^{1,1}(G)$*

$$f * g(h,k) = \int < h,s+t > \widehat{f}(s,A\pi(2t)k)\widehat{g}(t,A\pi(-2s)k)dsdt \tag{2.35}$$

$$\widehat{f * g}(t,k) = \int \widehat{f}(s,A\pi(2(t-s))k)\widehat{g}(t-s,A\pi(-2s)k)ds \tag{2.36}$$

$$\widehat{\widehat{f * g}}(t,q,k) = \int \widehat{\widehat{f}}(s,p,A\pi(2(t-s))k)\widehat{\widehat{g}}(t-s,q-p,A\pi(-2s)k)ds. \tag{2.37}$$

Proof. We have with $x = (h,k)$ and $\pi(t) = (\pi(t),0)$ in Corollary 2.2 that

$$\begin{aligned}
\widehat{f}_l(s,x\pi(-t)) &= \widehat{f}_l(s,(h-\pi(t),0)(0,A(\pi(2t)-h)k)) \\
&=< h-\pi(t),s > \widehat{f}_l(s,A(-h+2\pi(t))k) \\
\widehat{g}_r(t,\pi(-s)x) &= \widehat{g}_r(t,(0,A(h-\pi(2s))k)(h-\pi(s),0)) \\
&=< h-\pi(s),t > \widehat{g}_r(t,A(h-2\pi(s))k).
\end{aligned}$$

So by Corollary 2.2 we have that

$$
\begin{aligned}
f * g(h,k) &= \int < h - \pi(t), s > \widehat{f}_l(s, A(-h + 2\pi(t))k) \\
&\qquad < h - \pi(s), t > \widehat{g}_r(t, A(h - 2\pi(s))k) ds dt \\
&= \int < h, s+t > \widehat{f}_l(s, A(-h+2\pi(t))k)\widehat{g}_r(t, A(h-2\pi(s))k) ds dt \\
&= \int < h, s+t > \widehat{\widehat{f}}(s-p, p, A(-h+2\pi(t))k) \\
&\qquad \widehat{\widehat{g}}(t+q, q, A(h-2\pi(s))k) dp dq ds dt \\
&= \int < h, s-p+t+q > \widehat{\widehat{f}}(s-p, p, A(2\pi(t))k) \\
&\qquad \widehat{\widehat{g}}(t+q, q, A(2\pi(-s))k) ds dt dp dq
\end{aligned}
$$

By the assumptions on f and g the use of Fubini's Theorem above is valid and the substitutions $s \mapsto s+p$ and $t \mapsto t-q$ give

$$
\begin{aligned}
&= \int < h, s+t > \widehat{\widehat{f}}(s, p, A(2\pi(t-q))k)\widehat{\widehat{g}}(t, q, A(2\pi(-p-s))k) ds dt dp dq \\
&= \int < h, s+t >< 2\pi(-q), p >< 2\pi(-p), q > \widehat{\widehat{f}}(s, p, A(2\pi(t))k) \\
&\qquad \widehat{\widehat{g}}(t, q, A(2\pi(-s))k) dp dq ds dt.
\end{aligned}
$$

If we again use that π is anti-symmetric and do Fourier inversion with respect to p, q, we get (2.35). Combining this with (2.24) and (2.27) we also get (2.36) and (2.37).

Proposition 2.22. *$A^{1,1}(G)$ is a dense *-subalgebra of $A(G)$ and the map $\Phi(f) = \widehat{f}$ is a *-isomorphism between $A^{1,1}(G)$ and a dense *-subalgebra of $L^1(\widehat{H}, C_0(K))$ where the latter is given the operations*

$$
\varphi * \psi(t,k) = \int \varphi(s, A\pi(2(t-s))k)\psi(t-s, A\pi(-2s)k) ds \tag{2.38}
$$

$$
\varphi^*(t,k) = \overline{\varphi(-t,k)}, \tag{2.39}
$$

i.e. it is the crossed product of $C_0(K)$ by $\widehat{H}$ as defined in Theorem 2.6., the action on K being $(s,k) \mapsto A\pi(2s)k$.

Proof. For $f, g \in A^{1,1}(G)$ let $\varphi = \widehat{f}$ and $\psi = \widehat{g}$. Then we have from Lemma 2.21 that $\widehat{f * g} = \varphi * \psi$ and $\widehat{f^*} = \varphi^*$ so it remains to show that f^* and $f * g \in A^{1,1}(G)$. The map

$f \mapsto f^*$ obviously leaves $A^{1,1}(G)$ invariant, and it follows from (2.37) that

$$\begin{aligned} f * g(h, A(h')k) &= \int < h,t >< h',q > \widehat{\widehat{f}}(s,p,A\pi(2(t-s))k) \\ &\qquad \widehat{\widehat{g}}(t-s,q-p,A\pi(-2s)k)dsdtdpdq \\ &= \int < h,t+s >< h',q+p >< \pi(2t),p > \widehat{\widehat{f}}(s,p,k) \\ &\qquad \widehat{\widehat{g}}(t,q,A\pi(-2s)k)dsdtdpdq \\ &= \int < h,s >< h',p > \widehat{\widehat{f}}(s,p,A\pi(2t)k) \\ &\qquad g(h-\pi(2p),A(h'-\pi(2s))k)dsdtdpdq. \end{aligned}$$

From this it follows that

$$\int_{H \times H/H_k} |f * g(h, A(h')k)| dh dh' \leq \|\widehat{\widehat{f}}(\cdot,\cdot,k)\|_1 \|g(\cdot, A(\cdot)k)\|_1$$

so $f * g$ satisfies (2.29). The formula (2.37) is a "twisted" convolution on $\widehat{H} \times H_k^{\perp}$, so by the usual arguments $\widehat{\widehat{f * g}}\,(\cdot,\cdot,k)$ is integrable. Finally one observes that $\{\widehat{f}\,|f \in A^{1,1}(G)\}$ is dense in $L^1(\widehat{H}, C_0(K))$, c.f. also Lemma 2.18.

This proposition tells us that $A(G,\delta_\omega)$ and $L^1(\widehat{H}, C_0(K), A2\pi)$ have isomorphic dense *-subalgebras. One can however from this not in general conclude that the corresponding C*-algebras are isomorphic. For this we need a closer look at their regular representations. We have already defined the regular representation of $A(G,\delta_\omega)$ and for $L^1(\widehat{H}, C_0(K), A2\pi)$ it is done as follows:

Definition 2.23. *Let $\mathcal{H} = L^2(\widehat{H} \times K, A2\pi, d\mu_l)$ i.e. $d\mu_l(s,k) = \Delta\pi(2s)dsdk$. The left regular representation of $L^1(\widehat{H}, C_0(K), A2\pi)$ is then defined for $\varphi,\ \psi \in \mathcal{H}$ by*

$$\Pi(\varphi)\psi(t,k) = \int \varphi(s, A\pi(2(t-s))k)\psi(t-s, A\pi(-2s)k)ds. \tag{2.40}$$

Proposition 2.24. Π *is a nondegenerate *-representation of $L^1(\widehat{H}, C_0(K), A2\pi)$.*

Proof. That Π is well-defined – i.e. that $\Pi(\varphi)\psi \in \mathcal{H}$ – and that it is a *-representation is done exactly as with ordinary crossed products, see [P, 7.7].

If G is unimodular, i.e. $\Delta = 1$ it is not so difficult from Propositions 2.14 and 2.22 to conclude that Φ sets up an equivalence between the two regular representations. The main observation to make (and it took the author a while to figure this out) in the non-unimodular case is that one has to use a different Fourier transform than (2.24) between the corresponding L^2-spaces.

Definition 2.25. *Let $\mathcal{F}_A : L^2(H \times_A K) \to L^2(\widehat{H} \times_{A2\pi} K)$ be the operator defined by*

$$\mathcal{F}_A g(s,k) = \Delta\pi(s) \int_H \overline{< h, s >} \Delta(h)^{-1/2} g(h,k) dh. \tag{2.41}$$

This is a unitary operator when we use left Haar-measure for both groups, $\widehat{H} \times_{A2\pi} K$ is the group obtained by giving $\widehat{H} \times K$ the product

$$(s,k)(t,l) = (s+t, A\pi(2s)l + A\pi(-2t)k). \tag{2.42}$$

Proposition 2.26. *For $f, g \in A^{1,1}(G)$ we have $\mathcal{F}_A(f_{1/2} * g) = \Pi(\widehat{f})\mathcal{F}_A g$, so*

$$\mathcal{F}_A V(f) \mathcal{F}_A^{-1} = \Pi(\widehat{f}).$$

Proof.

$$\begin{aligned}
\mathcal{F}_A f_{1/2} * g(u,k) &= \Delta\pi(u) \int \overline{< h, u >} \Delta(h)^{-1/2} f_{1/2} * g(h,k) dh \\
&= \Delta\pi(u) \int \Delta(h)^{-1/2} < h, t-u > (f_{1/2} * g)\widehat{\ }(t,q,k) dt dq dh \\
&= \int \Delta\pi(u)\Delta(h)^{-1/2} < h, t-u > \widehat{\widehat{f}}_{1/2}(s,p,A\pi 2(t-s)k) \\
&\qquad \widehat{\widehat{g}}(t-s, q-p, A\pi(-2s)k) ds dp dt dq dh
\end{aligned}$$

so we get by (2.34) and the substitutions $t \mapsto t+s$ and $q \mapsto q+p$

$$\begin{aligned}
&= \int \Delta\pi(u+p-s)\Delta(h)^{-1/2} < h, t+s-u > \widehat{\widehat{f}}(s,p,A\pi(2t)k) \\
&\qquad \widehat{\widehat{g}}(t,q,A\pi(-2s)k) ds dp dt dq dh \\
&= \int \Delta\pi(u+p-s)\Delta(h)^{-1/2} < h, s-u > \widehat{\widehat{f}}(s,p,k) \\
&\qquad < h - 2\pi(p), t > \widehat{\widehat{g}}(t,q,A\pi(-2s)k) ds dp dt dq dh \\
&= \int \Delta\pi(u+p-s)\Delta(h)^{-1/2} < h, s-u > \widehat{\widehat{f}}(s,p,k) \\
&\qquad g(h-2\pi(p), A\pi(-2s)k) ds dp dh \\
&= \int < 2\pi(p), s-u > \widehat{\widehat{f}}(s,p,k)\Delta\pi(u-s)\Delta(h)^{-1/2} \\
&\qquad < h, s-u > g(h, A\pi(-2s)k) ds dp dh \\
&= \int \widehat{\widehat{f}}(s,p,A2\pi(u-s)k)\mathcal{F}_A g(u-s, A\pi(-2s)k) ds dp \\
&= \int \widehat{f}(s, A2\pi(u-s)k)\mathcal{F}_A g(u-s, A\pi(-2s)k) ds \\
&= \Pi(\widehat{f})\mathcal{F}_A g(u,k).
\end{aligned}$$

We can now show that our algebra is the claimed crossed product as follows:

Theorem 2.27. *Let $G = H \times_A K$ be a regular semi-direct product of abelian locally compact groups, $\pi : \widehat{H} \to H$ a skew homomorphism and ω the dual cocycle associated with π. Let $A2\pi$ be the action of $\widehat{H}$ on K defined by $(s,k) \mapsto A\pi(2s)k$ and let $L^1(\widehat{H}, C_0(K))$ have the convolution and adjoint defined in Theorem 2.6. Then $\mathrm{Ad}\mathcal{F}_A$ is a *-isomorphism between $C_r^*(\widehat{G}, \delta_\omega)$ and $C_r^*(\widehat{H}, C_0(K), A2\pi) = C^*(\Pi(L^1(\widehat{H}, C_0(K))))$.*

Proof. This now follows from the propositions 2.22, 2.24 and 2.26 since $\mathrm{Ad}\mathcal{F}_A$ is a *-isomorphism between dense subalgebras.

Notes. Since $C_0(K) = C^*(\widehat{K})$ and the action of $A2\pi$ on K corresponds to the homomorphism $A2\pi : \widehat{H} \mapsto \mathrm{Aut}(\widehat{K})$ given by $< k, A2\pi(t) >=< A2\pi(k), t >$ for $k \in K$, $t \in \widehat{K}$ we have that $C_r^*(\widehat{H}, C_0(K), A2\pi) \cong C_r^*(\widehat{H} \times_{A2\pi} \widehat{K})$ where $\widehat{H} \times_{A2\pi} \widehat{K}$ is the semi-direct product of $\widehat{H}$ and $\widehat{K}$ corresponding to (2.42). So we have $C^*((H \times_A K)\widehat{\ }, \delta_\omega) \cong C_r^*(\widehat{H} \times_{A2\pi} \widehat{K})$, i.e. the group $H \times_A K$ has $\widehat{H} \times_{A2\pi} \widehat{K}$ as a "dual group", c.f. [Wl2].

The Fourier transform $\widehat{f}$ is only defined over $A^{1,1}(G)$ and the proofs would be easier if it could be extended to the whole of $A(G)$. This will not be true, we have seen earlier that the maps $h \mapsto f((h,0)(0,k))$ and $h \mapsto f((0,k)(h,0))$ are in $A(H) \cong L^1(\widehat{H})$, but in general this will not be the case for the map $h \mapsto f(h,k)$.

Example 2.28. Let G be the "$az + b$"–group, i.e. if $(a,b) = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$ then $G = \{(a,b) \mid a, b \in \mathbb{C}, a \neq 0\}$. Let $H =$ the diagonal subgroup $\cong \mathbb{R} \times \mathbb{T}$ by $(x,t) \mapsto (e^x t, 0)$. We can take $\pi_0 : \mathbb{Z} \to G$ by $\pi_0(n) = (q^n, 0)$ where $q \in \mathbb{R} \setminus \{0\}$. Look at the following functions on G:

$$\phi(a,b) = \frac{a}{|a|} \qquad \psi(a,b) = |a| \qquad \beta(a,b) = b.$$

With the notation in (2.8) $\phi = \phi_{1,1}$, $\psi = \psi_{0,0}$ and $\beta = \beta_{1,-1}$, so we get by using (2.7) that

$$\begin{gathered} \phi^* * \phi = \phi * \phi^* = 1 \qquad \psi = \psi^* \qquad \beta * \beta^* = \beta^* * \beta = |\beta|^2 \\ \phi * \beta = q^{-2}\phi\beta \qquad \beta * \phi = q^2\phi\beta \qquad \psi * \beta = q^2\psi\beta \qquad \beta * \psi = q^{-2}\psi\beta. \end{gathered}$$

Hence the elements ϕ, β satisfies (3.3) in [Wo3] so the algebra they generate may be considered a quantization of the euclidian group $G_1 = \{(a,b) \mid |a| = 1\}$. Also compare with Example 2.5 in [Da2]. In the same way the elements ψ and β satisfy Example 2.4 in [Da2] so they generate an algebra related to the subgroup $G_2 = \{(a,b) \mid a \in \mathbb{R}\}$. These two examples do only partly fit into our general framework since δ_ω does not map $\mathcal{L}(G_i)$ into $\mathcal{L}(G_i) \otimes \mathcal{L}(G_i)$. However, it is an interesting exercise (which we leave to the reader) to compute the action of δ_ω on the generators of the Lie algebra $\mathfrak{g}$.

§3. Coactions from automorphisms.

Let us have another look at the comultiplication δ_ω in (1.6) with ω as in Proposition 1.4 and suppose now that the group H_2 is **central** in G. Then for $x \in G$

$$\delta_\omega(x) = \iint dP_s^2 \pi_0(t) x \pi_0(-t) \otimes dP_t^2 \pi_0(-s) x \pi_0(s) = \iint dP_s^2 \alpha_t(x) \otimes dP_t^2 \alpha_{-s}(x)$$

where $\alpha_t(x) = \pi_0(t) x \pi_0(-t)$. This suggests an obvious generalization:

Theorem 3.1. *Let H be a closed subgroup of the center of G and let $\alpha : \widehat{H} \to Aut(G)$ be a continuous homomorphism such that $\alpha_t(h) = h$ for $h \in H$. Then with $\{P_t\}$ the projection valued measure corresponding to $\lambda|H$*

$$\delta_\alpha(x) = \int dP_s \otimes dP_t (\alpha_t \otimes \alpha_{-s}) \circ \delta_G(x) \tag{3.1}$$

defines a comultiplication of $\mathcal{L}(G)$ satisfying

$$(\nu \otimes \nu)\delta_\alpha = \sigma \delta_\alpha \nu. \tag{3.2}$$

The corresponding operations on $A(G)$ are given for $x \in G$ by $f^*(x) = \overline{f(x)}$ and

$$f * g(x) = \int \widehat{f}_l(s, \beta_t(x)) \widehat{g}_l(t, \beta_{-s}(x)). \tag{3.3}$$

Proof. First we have to check that δ_α really is well defined. Define a unitary operator U_t on $L^2(G)$ by

$$U_t f(x) = \Delta_\alpha(t)^{-1/2} f(\alpha_{-t}(x)) \quad t \in \widehat{H},\ x \in G, \tag{3.4}$$

where $\Delta_\alpha(t)$ is the factor with which α_t scales the Haar-measure on G. Then

$$U_t L_x U_{-t} = L_{\alpha_t(x)} \tag{3.5}$$

and $U = \int U_t \otimes dP_t$ is by Proposition 1.2 a unitary operator on $L^2(G \times G)$ so

$$\delta_\alpha(x) = U \sigma(U^*) \delta_G(x) \sigma(U) U^*. \tag{3.6}$$

From this it follows that δ_α is a well defined *-homomorphism of $\mathcal{L}(G)$ into $\mathcal{L}(G) \otimes \mathcal{L}(G)$. $(\nu \otimes \nu)\delta_\alpha = \sigma \delta_\alpha \nu$ follows from the facts that $\nu(x) = Jx^*J$ and $(J \otimes J)U(J \otimes J) = U^*$

where $J\xi(x) = \overline{\xi(x)}$ for $\xi \in L^2(G)$. Using that $(I \otimes dP_q)\delta_G(dP_s) = dP_{s-q} \otimes dP_q$ we get for $x \in G$:

$$\begin{aligned}
(\delta_\alpha \otimes i)\delta_\alpha(x) &= \int \delta_\alpha(dP_s\alpha_t(x)) \otimes dP_t\alpha_{-s}(x) \\
&= \int dP_p \otimes dP_q(\alpha_q \otimes \alpha_{-p})\delta_G(dP_s\alpha_t(x)) \otimes dP_t\alpha_{-s}(x) \\
&= \int (dP_p \otimes dP_q)\delta_G(dP_s)\alpha_q\alpha_t(x) \otimes \alpha_{-p}\alpha_t(x) \otimes dP_t\alpha_{-s}(x) \\
&= \int dP_p\alpha_{q+t}(x) \otimes dP_q\alpha_{t-p}(x) \otimes dP_t\alpha_{-p-q}(x) \\
(i \otimes \delta_\alpha)\delta_\alpha(x) &= \int dP_s\alpha_t(x) \otimes \delta_\alpha(dP_t\alpha_{-s}(x)) \\
&= \int dP_s\alpha_t(x) \otimes (dP_p \otimes dP_q)(\alpha_q \otimes \alpha_{-p})\delta_G(dP_t\alpha_{-s}(x)) \\
&= \int dP_s\alpha_t(x) \otimes (dP_p \otimes dP_q)\delta_G(dP_t)(\alpha_q\alpha_{-s}(x) \otimes \alpha_{-p}\alpha_{-s}(x)) \\
&= \int dP_s\alpha_{p+q}(x) \otimes dP_p\alpha_{q-s}(x) \otimes dP_q\alpha_{-p-s}(x).
\end{aligned}$$

So δ_α is a comultiplication.

Example 3.2. Let G be the 3-dimensional Heisenberg group consisting of all 3×3 real matrices of the form

$$(x, y, z) = \begin{pmatrix} 1 & y & z \\ & 1 & x \\ & & 1 \end{pmatrix}$$

Then $H = \{(0,0,z) \mid z \in \mathbb{R}\}$ is central, and we can take $\alpha_t(x,y,z) = (e^{-t}x, e^t y, z)$. We shall look at what δ_α do to the elements of the Lie-algebra $\mathfrak{g}$ of G. $\mathfrak{g}$ is spanned by $p = -i\frac{d}{dx}\big|_0(x,0,0)$, $q = -i\frac{d}{dy}\big|_0(0,y,0)$ and $r = -i\frac{d}{dz}\big|_0(0,0,z)$, thus $\alpha_t(p) = e^{-t}p$, $\alpha_t(q) = e^t q$ and $\alpha_t(r) = r$, so

$$\begin{aligned}
\delta_\alpha(p) &= \int dP_s \otimes dP_t(\alpha_t(p) \otimes \alpha_{-s}(I) + \alpha_t(I) \otimes \alpha_{-s}(p)) \\
&= \int e^{-t}p \otimes dP_t + \int dP_s \otimes e^s p = p \otimes e^{-r} + e^r \otimes p.
\end{aligned}$$

Similarly $\delta_\alpha(q) = q \otimes e^r + e^{-r} \otimes q$ and $\delta_\alpha(r) = r \otimes I + I \otimes r$. This is the first example in [Da1].

We shall next see that there also is an "unsymmetric" version of Theorem 3.1, however here it is not so obvious what the *-operation on $A(G)$ should be, so we need a little preparation. With $u = \int dP_t U_t$ as in Theorem 3.1 the map $x \mapsto u^*xu = \int dP_t\alpha_{-t}(x)$ is well defined on $\mathcal{L}(G)$. Note also that in the terminology of Proposition 2.1 we have $\widehat{f_l} = \widehat{f_r}$.

Theorem 3.3. *Let the notation be as in Theorem 3.1 and take $u = \int U_t dP_t$. Then for $x \in \mathcal{L}(G)$*

$$\delta'_\alpha(x) = \int I \otimes dP_t(\alpha_t \otimes i) \circ \delta_G(x) \quad \text{and} \quad \tau(x) = u^*\nu(x)u = \nu(uxu^*) \tag{3.7}$$

makes $(\mathcal{L}(G), \delta'_\alpha, \tau)$ a coinvolutive Hopf-von Neumann algebra with $\tau^2 = i$. The product and *-operation on $A(G)$ are given for $x \in G$ by

$$f * g(x) = \int f(\alpha_s(x))\widehat{g}_l(s,x)ds \tag{3.8}$$

$$f^*(x) = \overline{f(\tau(x^*))} = \int \overline{\widehat{f}_l(s, \alpha_{-s}(x))}ds. \tag{3.9}$$

Proof. Here $\delta_\alpha(x) = U\delta_G(x)U^*$, the proof that δ_α is a comultiplication follows the same line of proof as Theorem 3.1, so we leave that and the formulas for the operations on $A(G)$ to the reader. The remaining property to check is then that for $x \in G$

$$\begin{aligned}
\sigma(\delta'_\alpha(\tau(x))) &= \int \sigma(U)\delta_G(dP_t\alpha_{-t}(x^{-1}))\sigma(U^*) \\
&= \int \sigma(U)\delta_G(dP_t)\alpha_{-t}(x^{-1}) \otimes \alpha_{-t}(x^{-1})\sigma(U^*) \\
&= \int \delta_G(dP_t)dP_s\alpha_{-t}(x^{-1}) \otimes \alpha_{s-t}(x^{-1}) \\
&= \int dP_s\alpha_{-t}(x^{-1}) \otimes dP_{t-s}\alpha_{s-t}(x^{-1}) \\
&= \int dP_s\alpha_{-s-t}(x^{-1}) \otimes dP_t\alpha_{-t}(x^{-1}) \\
&= \int u^*\alpha_{-t}(x^{-1})u \otimes u^*dP_t x^{-1}u \\
&= \int \tau\alpha_{-t}(x) \otimes \tau(xdP_{-t}) = (\tau \otimes \tau)\delta'_\alpha(x).
\end{aligned}$$

Example 3.4 Let $G = \mathbb{R} \times \mathbb{T}$, $\quad H = \mathbb{T}$. For $n \in \mathbb{Z} = \widehat{H}$ let $\alpha_n(x,t) = (e^{\lambda n}x, t)$. For $f \in A(G)$ let

$$f_n(x) =< (x,0)P_n, f >= \int_{\mathbb{T}} t^{-n}f(x,t)dt,$$

i.e. the partial Fourier transform. Then the product and involution is given by

$$\begin{aligned}
f * g(x,t) &= \sum_m f(\alpha_m(x,t)) < (x,t)P_m, g >= \sum_m t^m f(e^{\lambda m}x, t)g_m(x) \\
f^*(x,t) &= \sum_m \overline{< \alpha_{-m}(x,t)P_m, f >} = \sum_m t^{-m}\overline{f_m(e^{-\lambda m}x)}.
\end{aligned}$$

By remark 3 after Theorem 2.3 the product can be extended also to unbounded functions f, g such that f_n and $g_m \neq 0$ for only finitely many m, n. Take $f(x,t) = x$ and $g(x,t) = t$, so $f_n = 0$ for $n \neq 0$ and $g_m = 0$ for $m \neq 1$. We then have

$$f * g(x,t) = f(e^\lambda x, t) t g_1(x) = e^\lambda xt \quad \text{and} \quad g * f(x,t) = g(x,t) f_0(x) = tx$$

i.e. $f * g = e^\lambda g * f$. This could be called the quantum cylinder, note that this is different from the equivariant deformations of $A(\mathbb{R} \times \mathbb{T})$ studied in [LR] and [Rf7, 10.6]. This construction can of coarse be generalized to a product $G = H_1 \times H_2$ of abelian groups together with some homomorphism $\alpha : \widehat{H}_2 \to \mathrm{Aut}(H_1)$. So also abelian groups can be deformed as quantum groups.

§4. Comultiplications coming from homeomorphisms of G.

Our next step is to show that the formula (3.1) in some cases also defines a comultiplication when H is just a normal (but still abelian) subgroup of G. In particular we shall again look at the semi-direct product $G = K \times_A H$ of two abelian groups, but this time it is the normal subgroup H which is used to obtain the deformation. The construction in [Rf5] can be described this way, the method is again the one used by Rieffel: the Fourier transform maps $A(G)$ to functions on $\widehat{K} \times \widehat{H}$ and it turns out that the product in $A(G)$ corresponds to convolution in $L^1(\widehat{K} \times_\theta \widehat{H})$ where $\widehat{K} \times_\theta \widehat{H}$ is $\widehat{K} \times \widehat{H}$ with another product making it a nilpotent group. We end in §4.3 by comparing with the construction in §2, but this time we let the dual cocycle come from the normal subgroup H.

Definition. If H is an abelian, normal subgroup of G, we define an action of G on $\widehat{H}$ by

$$< h, xs > = < x^{-1} h x, s > . \tag{4.1}$$

We will then have $x dP_s x^{-1} = dP_{xs}$.

Theorem 4.1. *Suppose H is a closed, normal and abelian subgroup of G and that for each $s \in \widehat{H}$ there is a homeomorphism $\beta_s : G \to G$ such that $(s,x) \mapsto \beta_s(x)$ is continuous and*

$$\beta_s \circ \beta_t = \beta_{s+t} \tag{4.2}$$

$$\beta_s(kxh) = k \beta_s(x) h \quad \text{for} \quad h, k \in H, \tag{4.3}$$

$$\beta_s(xy) = \beta_s(x) \beta_{x^{-1}s}(y). \tag{4.4}$$

Furthermore, each β_t is assumed to scale the Haar-measure on G by a factor $\Delta_\beta(t)$ satisfying

$$\Delta_\beta(xt) = \Delta_\beta(t). \tag{4.5}$$

Then the formula

$$\delta_\beta(x) = \int dP_s \otimes dP_t(\beta_t \otimes \beta_{-s})\delta_G(x) \tag{4.6}$$

where P_t is the projection valued measure corresponding to $L^G|H$ defines a comultiplication on $\mathcal{L}(G)$ satisfying

$$(\nu \otimes \nu)\delta_\beta(x) = \sigma\delta_\beta\nu(x). \tag{4.7}$$

The corresponding operations on $A(G)$ are for $x \in G$ given by $f^*(x) = \overline{f(x)}$ and

$$f * g(x) = \int \widehat{f}_l(s, \beta_t(x))\widehat{g}_l(t, \beta_{-s}(x)). \tag{4.8}$$

Proof. Note first that from (4.4) it follows that

$$\beta_s(x)^{-1} = \beta_{x^{-1}s}(x^{-1}). \tag{4.9}$$

In the same way as before we define unitary operators U_t on $L^2(G)$ and U on $L^2(G \times G)$ by

$$U_t f(x) = \Delta_\beta(t)^{-\frac{1}{2}} f(\beta_{-t}(x)) \qquad U = \int U_t \otimes dP_t. \tag{4.10}$$

We then have for $x \in G$

$$\begin{aligned} Ux \otimes xU^* &= \int U_t x U_s^* \otimes dP_t x dP_s = \int U_t x U_s^* \otimes dP_t dP_{xs} x \\ &= \int U_t x U_{x^{-1}t}^* \otimes dP_t x = \int \beta_t(x) \otimes dP_t x. \end{aligned}$$

The last equality follows from the following computation, where we use (4.4), (4.5) and (4.9):

$$\begin{aligned} U_t x U_{x^{-1}t}^* f(y) &= \Delta_\beta(t)^{-\frac{1}{2}} U_{x^{-1}t}^* f(x^{-1}\beta_{-t}(y)) \\ &= \Delta_\beta(t)^{-\frac{1}{2}} \Delta_\beta(x^{-1}t)^{\frac{1}{2}} f(\beta_{x^{-1}t}(x^{-1}\beta_{-t}(y))) \\ &= f(\beta_{x^{-1}t}(x^{-1})y) = f(\beta_t(x)^{-1}y). \end{aligned} \tag{4.11}$$

Thus $\delta_\beta(x) = \int dP_s\beta_t(x) \otimes dP_t\beta_{-s}(x) = \sigma(U^*)U\delta_G(x)U^*\sigma(U)$. This proves that δ_β is a well defined *-isomorphism from $\mathcal{L}(G)$ to $\mathcal{L}(G) \otimes \mathcal{L}(G)$.

The computations showing that $(\delta_\beta \otimes i)\delta_\beta = (i \otimes \delta_\beta)\delta_\beta$ are exactly as in the proof of Theorem 3.1. It is only (4.2) and (4.3) which are needed for this, so we leave it to the reader to check that the steps are valid also in the present situation. We have

$$\begin{aligned}\nu \otimes \nu\delta_\beta(x) &= \int \beta_t(x)^{-1}dP_{-s} \otimes \beta_{-s}(x)^{-1}dP_{-t} = \int \beta_{-x^{-1}t}(x^{-1})dP_s \otimes \beta_{x^{-1}s}(x^{-1})dP_t \\ &= \int \beta_{-x^{-1}t}(x^{-1}dP_s) \otimes \beta_{x^{-1}s}(x^{-1}dP_t) = \int \beta_{-t}(dP_s x^{-1}) \otimes \beta_s(dP_t x^{-1}) \\ &= \int dP_s\beta_{-t}(x^{-1}) \otimes dP_t\beta_s(x^{-1}) = \sigma\delta_\beta\nu(x).\end{aligned}$$

Finally, the formulas for f^* and $f * g$ should be obvious.

§4.1 Semi-direct products of abelian groups.

We shall see in this part that Theorem 4.1 can be applied to get deformations of $A(G)$ for $G = K \times_A H$ – the semi-direct product of two abelian groups. Here we have reversed the roles of H and K from §2.5, H is always the subgroup used to deform $A(G)$. So let H and K be two abelian locally compact groups together with a continuous homomorphism $A : K \to \mathrm{Aut}(H)$. We shall look at $G = K \times_A H$ with the same multiplication as in (2.22):

$$(k,h)(k',h') = (k+k', A(-k')h + A(k)h').$$

Here we also might have used the ordinary definition of semi-direct product, but we have chosen to use the same definition throughout this article. We shall look at comultiplications on $\mathcal{L}(G)$ arising from the normal subgroup H this time, and we shall see in §4.3 that the construction in §2 taking a dual cocycle of the normal subgroup H quite often will just give the usual pointwise product on $A(G)$.

Lemma 4.2. *Suppose that $G = K \times_A H$ defined above satisfies*

$$A(2k)h = h \quad \text{for all} \quad h \in H \implies k = 0 \tag{4.12}$$

and that $\beta_s : G \to G$ is as in Theorem 4.1. Then (4.2–5) is satisfied if and only if β is of the form

$$\beta_s(k,0) = (0, \theta(s,k))(k,0) \tag{4.13}$$

where $\theta : \widehat{H} \times K \to H$ satisfies

$$s \mapsto \theta(s,k) \text{ is a continuous homomorphism for each } k \in K, \tag{4.14}$$

$$\theta(s, k+k') = \theta(s,k) + A(2k)\theta(k^{-1}s, k'). \tag{4.15}$$

Proof. (4.12) implies that H is maximal abelian. If β_s satisfies (4.2-4) then

$$h\beta_s(x)x^{-1} = \beta_s(xx^{-1}hx)x^{-1} = \beta_s(x)x^{-1}hxx^{-1} = \beta_s(x)x^{-1}h$$

for all $h \in H$, so $\beta_s(x)x^{-1} \in H$. Thus $\beta_s(k,0) = (0, \theta(s,k))(k,0)$ for some map $\theta : \widehat{H} \times K \to H$. Then (4.14) follows from (4.2) and (4.15) from (4.4) and (4.13). Here $\Delta_\beta(t) = 1$, and it is easy to check that if β satisfies (4.13-15), then Theorem 4.1 holds.

Lemma 4.3. *If θ satisfies (4.15), then $k \mapsto \theta(s,k)$ is a homomorphism if and only if*

$$\theta(ks,k') = A(2k)\theta(s,k') \quad \textit{for all } k \in K. \tag{4.16}$$

Proof. Obvious from (4.15), note that $ks = s \circ A(-2k)$.

Note. If $H = \mathbb{R}$ then (4.15) $\Rightarrow$ (4.16). But for e.g. $H = \mathbb{R}^2$ we will also have solutions of (4.15) which are not linear in K. Except for §4.3 we shall not pursue here what kind of comultiplications this will give, but from now on assume that θ is bilinear, i.e. that (4.16) holds.

Theorem 4.4. *Let H and K be two abelian locally compact groups, $A : K \to Aut(H)$ a continuous homomorphism, then form $G = K \times_A H$. Suppose $\theta : \widehat{H} \times K \to H$ satisfies $A(k)\theta(s \circ A(k), k') = \theta(s,k')$ for $k, k' \in K$, $s \in \widehat{H}$, is jointly continuous and a homomorphism in each variable separately. Then*

$$\delta_\theta(k,h) = \int dP_s(0,\theta(t,k))(k,h) \otimes dP_t(0,\theta(-s,k))(k,h) \tag{4.17}$$

defines a comultiplication on $\mathcal{L}(G)$ satisfying $(\nu \otimes \nu)\delta_\theta(x) = \sigma\delta_\theta\nu(x)$. The corresponding operations on $A(G)$ are for $x \in G$ given by $f^*(x) = \overline{f(x)}$ and

$$f * g(x) = \int \widehat{f}_l(s,(0,\theta(t,k))x)\widehat{g}_l(t,(0,-\theta(s,k))x)dsdt. \tag{4.18}$$

We shall look closer at the structure of the algebra $A(G,\delta_\theta)$, and will follow Rieffel's idea in [Rf5] and compare it with $L^1(\widehat{K} \times \widehat{H})$. It turns out that the Fourier transform is a homomorphism when $\widehat{K} \times \widehat{H}$ is given the structure of a nilpotent group, the new group law depending on θ.

The main difficulty in obtaining this is to find the right subspace to work on. Since we do not have that the norms in $A(G)$ and $L^1(\widehat{K} \times \widehat{H})$ coincide (c.f. [Wl1, Theorem 3]), we have to work on some dense subspace of $A(G)$.

Lemma 4.5. *Suppose $\theta : \widehat{H} \times K \to H$ is bilinear, then it corresponds to a bilinear map $\widetilde{\theta} : \widehat{H} \times \widehat{H} \to \widehat{K}$ by*

$$< k, \widetilde{\theta}(s,t) > = < \theta(s,k), t > . \tag{4.19}$$

(4.16) is then equivalent to

$$\widetilde{\theta}(s \circ A(k), t \circ A(k)) = \widetilde{\theta}(s,t). \tag{4.20}$$

Proof. Obvious.

Lemma 4.6. *With $\widetilde{\theta}$ as above, let $\widehat{\theta}(s,t) = \widetilde{\theta}(s,t) - \widetilde{\theta}(t,s)$. We can then define a group law on $\widehat{K} \times \widehat{H}$ by*

$$(p,s)(q,t) = (p + q + \widehat{\theta}(t,s), s+t). \tag{4.21}$$

Proof. We leave this to the reader. Note that $(p,s)^{-1} = (-p,-s)$ and $(p,s)^{-1}(q,t) = (q - p - \widehat{\theta}(t,s), t - s)$.

Definitions. We shall also need $\theta_0 : \widehat{H} \times K \to H$ given by

$$< \theta_0(s,k), t >=< k, \widehat{\theta}(s,t) > . \tag{4.22}$$

Next note that for $f \in A(G)$ the map $k \mapsto f((0,h)(k,0))$ is in $A(K) \cong L^1(\widehat{K})$ and the map $h \mapsto f((0,h)(k,0))$ is in $A(H) \cong L^1(\widehat{H})$. However, the map $(k,h) \mapsto f((0,h)(k,0))$ is generally not in $A(K \times H) \cong L^1(\widehat{K} \times \widehat{H})$.

Note also that as in §2.5 the right Haar-measure on G is given by

$$\int f d\mu_r = \int f((0,h)(k,0)) dh dk,$$

so for $f \in L^1(G, d\mu_r) = L^1_r(G)$ we can define the Fourier transform

$$\widehat{f}(p,s) = \int_H \int_K \overline{< k,p >< h,s >}\, f((0,h)(k,0)) dk dh. \tag{4.23}$$

Then define

$$A^{1,1}(G) = \{ f \in A(G) \cap L^1_r(G) \bigm| \widehat{f} \in L^1(\widehat{K} \times \widehat{H}) \}.$$

With $\widehat{f}_2(s,k) = \widehat{f}_l(s,(k,0))$ Proposition 2.1 then makes it possible to get a closer look at the product (4.18) defines on $A(G)$:

$$\begin{aligned}
f * g((0,h)(k,0)) &= \int \widehat{f}_l(s,(0,h+\theta(t,k))(k,0)) \widehat{g}_l(t,(0,h-\theta(s,k))(k,0)) ds dt \\
&= \int < h + \theta(t,k), s >< h - \theta(s,k), t > \widehat{f}_2(s,k) \widehat{g}_2(t,k) ds dt \\
&= \int < h, s+t >< k, \widehat{\theta}(t,s) > \widehat{f}_2(s,k) \widehat{g}_2(t,k) ds dt \\
&= \int < h, t >< k, \widehat{\theta}(t,s) > \widehat{f}_2(s,k) \widehat{g}_2(t-s,k) ds dt
\end{aligned}$$

(by $t \mapsto t - s$ and using $\widehat{\theta}(s,s) = 0$). From this we have

$$\widehat{(f * g)}_2(t,k) = \int < k, \widehat{\theta}(t,s) > \widehat{f}_2(s,k) \widehat{g}_2(t-s,k) ds, \tag{4.24}$$

$$(4.25)\quad f * g((0,h)(k,0)) = \int < h+\theta_0(t,k), s >< h,t > \widehat{f}_2(s,k)\widehat{g}_2(t,k)dsdt$$
$$= \int_{\widehat{H}} f((0,h+\theta_0(t,k))(k,0)) < h,t > \widehat{g}_2(t,k)dt \quad \text{and}$$
$$(4.26)\quad f * g((0,h)(k,0)) = \int < h,s >< h-\theta_0(s,k), t > \widehat{f}_2(s,k)\widehat{g}_2(t,k)dsdt$$
$$= \int < h,s > \widehat{f}_2(s,k)g((0,h-\theta_0(s,k))(k,0))ds.$$

Corollary 4.7. *$A(G) \cap L^1_r(G)$ is a two-sided ideal in $A(G,\delta_\theta)$ and right Haar measure μ_r is a positive linear functional on $A(G) \cap L^1_r(G)$:*

$$\int f * g(x)d\mu_r(x) = \int f(x)g(x)d\mu_r(x).$$

Proof. If $g \in A(G) \cap L^1_r(G)$ and $f \in A(G)$ we have from (4.26) that

$$\int \big|f * g((0,h)(k,0))\big|dkdh \le \int \big|\widehat{f}_2(s,k)g((0,h-\theta_0(s,k))(k,0))\big|dsdhdk$$
$$\le \int \big|\widehat{f}_2(s,k)g((0,h)(k,0))\big|dhdsdk \le \|f\|_{A(G)}\|g\|_1.$$

Similarly one shows using (4.25) that $A(G) \cap L^1_r(G)$ is also a right ideal. Integrating (4.25) now gives

$$\int f * g(x)d\mu_r(x) = \int f((0,h+\theta_0(t,k))(k,0)) < h,t > \widehat{g}_2(k,t)dtdhdk$$
$$= \int f((0,h)(k,0)) < h,t > \widehat{g}_2(t,k)dtdkdh$$
$$= \int f((0,h)(k,0))g((0,h)(k,0))dkdh = \int f(x)g(x)d\mu_r(x).$$

From this it is clear that $< f * f^*, \mu_r >= \int f(x)\overline{f(x)}d\mu_r(x) \ge 0$, however one does not have that $f * f^*(x) \ge 0$ in general.

Definition. The GNS-representation of $A(G) \cap L^1_r(G)$ and hence of $A(G,\delta_\theta)$ defined by μ_r is called *the regular representation* of $A(G,\delta_\theta)$, and the corresponding C^*-algebra is called $C^*_r(\widehat{G},\delta_\theta)$.

We are now ready for the main objective of this example, to compare the product δ_θ defines on $A(G)$ with Rieffel's convolution product [Rf5, §3] in $L^1(\widehat{K} \times \widehat{H})$ defined by the group law (4.21).

Theorem 4.8. *$A^{1,1}(G) = \{f \in A(G) \cap L^1_r(G) \mid \widehat{f} \in L^1(\widehat{H} \times \widehat{K})\}$ is a dense *-subalgebra of $A(G, \delta_\theta)$. The map $f \mapsto \widehat{f}$ is a *-homomorphism from $A^{1,1}(G, \delta_\theta)$ onto a dense *-subalgebra of $L^1(\widehat{K} \times \widehat{H}, \widehat{\theta})$, where $A^{1,1}(G, \delta_\theta)$ is given the operations from δ_θ and $L^1(\widehat{K} \times \widehat{H}, \widehat{\theta})$ is given the convolution and involution coming from the group law (4.21) on $\widehat{K} \times \widehat{H}$.*

Proof. We have from Corollary 4.7 that $f, g \in A^{1,1}(G) \Rightarrow f * g \in L^1_r(G)$, so it suffices to prove that $(f * g)\widehat{\ } \in L^1(\widehat{K} \times \widehat{H})$. This in turn will follow when we have proved that $(f * g)\widehat{\ }$ is the convolution of $\widehat{f}$ and $\widehat{g}$ in $L^1(\widehat{K} \times \widehat{H}, \widehat{\theta})$. With our earlier notation we have

$$\widehat{f}(p, s) = \int_K \overline{< k, p >} \widehat{f}_2(s, k) dk$$

so from (4.24) we get using Fourier inversion etc. that

$$\begin{aligned} (f * g)\widehat{\ }(q, t) &= \int \overline{< k, -\widehat{\theta}(t, s) + q >}\ \widehat{f}_2(s, k) \widehat{g}_2(t - s, k) ds dk \\ &= \int \widehat{f}(p, s) \widehat{g}(q - \widehat{\theta}(t, s) - p, t - s) dp ds \\ &= \int \widehat{f}(p, s) \widehat{g}((p, s)^{-1}(q, t)) dp ds = \widehat{f} * \widehat{g}(q, t). \end{aligned} \tag{4.27}$$

The assumptions on f, g make the above calculations valid. It is straight forward to see that $\widehat{f^*} = \widehat{f}^{\,*}$, and the map $f \mapsto \widehat{f}$ is obviously one-to-one.

To show that $A^{1,1}(G)$ is dense in $A(G)$, take $g(k, h) = (L_{(k,h)} \xi \mid \eta) \in A(G) \cap L^1_r(G)$ such that there is a compact set $A \subset K$ with $g(k, h) = 0$ for $k \notin A$. Also take $\varphi \in L^1(K)$ with $\widehat{\varphi} \in L^1(\widehat{K})$ and

$$f(k, h) = \int_K \varphi(a)\ g((a, 0)(k, h)) da = (L^K_\varphi L_{(k,h)} \xi \mid \eta).$$

So f is in $A(G)$, $\|f\|_1 \leq \|\varphi\|_1 \|g\|_1$ and f is in $L^1_r(G)$. Then $\widehat{f}(p, s) = \widehat{\varphi}(-p) \widehat{g}(p, s)$ so

$$\int \bigl|\widehat{f}(p, s)\bigr| dp ds = \int \bigl|\widehat{\varphi}(-p) \widehat{g}(p, s)\bigr| dp ds \leq \int \bigl|\widehat{\varphi}(-p) \widehat{g}_2(s, k)\bigr| dk ds dp \leq \|\widehat{\varphi}\|_1 \mu(A) \|g\|_{A(G)}$$

from Proposition 2.1, thus $\widehat{f} \in L^1(\widehat{K} \times \widehat{H})$. Since the functions of this form are dense in $A(G)$, we have proved that $A^{1,1}(G)$ is dense in $A(G)$.

Corollary 4.9. *The correspondence in Theorem 4.8 extends to a C^*-isomorphism between $C^*_r(\widehat{G}, \delta_\theta)$ and the C^*-algebra $C^*_r(\widehat{K} \times \widehat{H}, \widehat{\theta})$ of the left regular representation of the group $(\widehat{K} \times \widehat{H}, \widehat{\theta})$.*

Proof. The Fourier transform $\mathcal{F} f = \widehat{f}$ is unitary from $L^2_r(G)$ to $L^2(\widehat{K} \times \widehat{H})$. Let Π be the regular representation of $A(G, \delta_\theta)$ defined above and L the left regular representation of $(\widehat{K}\times\widehat{H},\widehat{\theta})$, then we have for $f,g,h\in A^{1,1}(G)$:

$$< \Pi(f)g,h> = \int_G f*g(x)\overline{h(x)}d\mu_r(x) = \int_G f*g*h^*(x)d\mu_r(x) = (f*g*h^*)^\wedge(0,0)$$
$$= \widehat{f}*\widehat{g}*\widehat{h^*}(0,0) = \int_{\widehat{K}\times\widehat{H}} \widehat{f}*\widehat{g}(z)\overline{\widehat{h}(z)}dz = < L_{\widehat{f}}\mathcal{F}g,\mathcal{F}h> .$$

So $\Pi(f)=\mathcal{F}^*L_{\widehat{f}}\mathcal{F}$. $A^{1,1}(G)$ is dense in $A(G)$ and so is its image in $L^1(\widehat{K}\times\widehat{H},\theta)$, this proves the claim, in fact we also have proved:

Corollary 4.10. *Under the correspondence $f\mapsto\widehat{f}$ in Theorem 4.8 we have $\mu_r(f)=\widehat{f}(0,0)$, so the representation of $A^{1,1}(G)$ constructed from μ_r corresponds to the left regular representation of $L^1(\widehat{K}\times\widehat{H},\widehat{\theta})$.*

We can now sum up the main results of this paragraph as follows :

Theorem 4.11. *Let $G=K\times_A H$ and $\theta:\widehat{H}\times K\to H$ be as in Theorem 4.4. Then (δ_θ,ν) gives a product and an involution on $A(G)$ making it a Banach *-algebra $A(G,\delta_\theta)$. If μ_r is right Haar measure $A^{1,1}(G)=\{f\in A(G)\cap L^1(G,\mu_r)\mid \widehat{f}\in L^1(\widehat{K}\times\widehat{H})\}$ is a dense *-subalgebra and $\mu_r(f)=\int f(x)d\mu_r(x)$ is a positive linear functional on $A^{1,1}(G)$. The GNS-construction on $(A^{1,1}(G),\mu_r)$ then gives a C^*-algebra $C_r^*(\widehat{G},\delta_\theta)$ which is isomorphic to the group C^*-algebra $C^*(\widehat{K}\times\widehat{H},\widehat{\theta})$ where $\widehat{K}\times\widehat{H}$ has the group structure given by Lemma 4.6.*

Proof. The only thing remaining is to observe that since $(\widehat{K}\times\widehat{H},\widehat{\theta})$ is amenable we have that $C_r^*(\widehat{K}\times\widehat{H},\widehat{\theta})\cong C^*(\widehat{K}\times\widehat{H},\widehat{\theta})$.

Remark. Since $\widehat{f}$ is the Fourier transform of the function $(k,h)\mapsto f((0,h)(k,0))$ Theorem 4.11 remains valid also with the usual definition of semi-direct product of K and H, contrary to the situation in §2.5.

Examples. From Theorem 4.11 it follows that the structure of $A(G,\delta_\theta)$ only depends on $\widehat{\theta}$, so if H is one of the "basic" abelian groups $\mathbb{R},\mathbb{T},\mathbb{Z}$ or $\mathbb{Z}_m$ then $\widehat{\theta}=0$ and $\delta_\theta=\delta_G$. If K is a product of such basic groups, $\widehat{\theta}$ is determined by its components $\widehat{\theta}_i$ which then all will be anti-symmetric forms on $\widehat{H}\times\widehat{H}$. We may therefore as well illustrate by having just one component. If $H=\mathbb{R}^2$ and $\widehat{\theta}(s,t)=s_1t_2-s_2t_1$ for $s,t\in\widehat{H}\cong\mathbb{R}^2$, then we have that

$$\widehat{\theta}(s\circ A(k),t\circ A(k))=\widehat{\theta}(s,t)\iff \det A(k)=1,$$

c.f. Remark 1.6. So A maps K into $SL(2,\mathbb{R})$, and $G=K\times_A\mathbb{R}^2$ is the corresponding semi-direct product. The structure of $A(G,\delta_\theta)$ is given by $(\widehat{K}\times\mathbb{R}^2,\widehat{\theta})$ and it is not difficult to see that if $K=\mathbb{R}$ this group is isomorphic to the Heisenberg group in Example 3.2.

Other possibilities include taking $H = \mathbb{T}^2$ and K a finite abelian subgroup of $SL(2,\mathbb{Z})$ acting on H.

§4.2 "Unsymmetric" comultiplications.

As in §3 we also have an "unsymmetric" version of Theorem 4.1. In order to get the involution here, we shall follow the procedure in Theorem 3.3 and define $u = \int dP_t U_t$. Then we have for $x \in G$ by (4.11) that

$$uxu^* = \int dP_t U_t x U_{-s} dP_s = \int dP_t U_t dP_{xs} x U_{-s} = \int dP_t U_t x U_{-x^{-1}t} = \int dP_t \beta_t(x)$$

and similarly $u^* x u = \int dP_t \beta_{-t}(x)$. So we have

$$\nu(uxu^*) = \int \beta_t(x)^{-1} dP_{-t} = \int \beta_{x^{-1}t}(x^{-1} dP_{-t}) = \int dP_t \beta_{-t}(x^{-1}) = u^* \nu(x) u.$$

Theorem 4.12. *With the same assumptions as in Theorem 4.1 and with $u = \int U_t dP_t$ the maps*

$$\delta'_\beta(x) = \int (I \otimes dP_s)(\beta_s \otimes i)\delta_G(x) \quad \text{and} \quad \tau(x) = u^* \nu(x) u = \nu(uxu^*) \tag{4.28}$$

make $(\mathcal{L}(G), \delta'_\beta, \tau)$ a coinvolutive Hopf-von Neumann algebra with $\tau^2 = i$. The product and *-operation on $A(G)$ are given for $x \in G$ by

$$f * g(x) = \int f(\beta_s(x)) \widehat{g}_l(s,x) ds \tag{4.29}$$

$$f^*(x) = \overline{f(\tau(x^*))} = \int \overline{\widehat{f}_l(s, \beta_{-s}(x))} ds. \tag{4.30}$$

Proof. With U as in Theorem 4.1 one sees that $\delta'_\beta(x) = U \delta_G(x) U^*$. The coassociativity is proved the same way as in Theorem 4.1, and as in Theorem 3.3 we leave the formulas for the operations on $A(G)$ for the reader to prove. To show that τ has the right properties note that we can repeat the first 6 equations in the proof of Theorem 3.3 and from there we have

$$\begin{aligned}\sigma(\delta'_\beta(\tau(x))) &= \int u^* \beta_{-t}(x^{-1}) u \otimes u^* dP_t x^{-1} u = \int u^* \beta_{-xt}(x)^{-1} u \otimes \tau(x dP_{-t}) \\ &= \int u^* \nu \beta_t(x) u \otimes \tau(dP_t x) = (\tau \otimes \tau)\delta'_\beta(x).\end{aligned}$$

If we use this on the situation in §4.1 we will get

Theorem 4.13. *The Fourier transform is an isomorphism between dense subspaces of $A(G,\delta'_\theta)$ and $L^1(\widehat{K}\times\widehat{H})$ such that the product and involution in $A(G,\delta'_\theta)$ correspond to the convolution product and the involution in the group algebra $L^1(\widehat{K}\times\widehat{H})$ obtained by giving $\widehat{K}\times\widehat{H}$ the group structure*

$$(p,s)(q,t)=(p+q+\widetilde{\theta}(t,s),s+t). \tag{4.31}$$

Proof. For a complete proof we will have to go through the same procedure as in §4.1, so we shall only show that the product and involution correspond. Using the same notation as in §4.1 and (4.29) one gets

$$\begin{aligned} f*g((0,h)(k,0)) &= \int f((0,h+\theta(s,k))(k,0))\widehat{g}_l(s,(0,h)(k,0))ds \\ &= \int <h+\theta(s,k),t><k,q>\widehat{f}(q,t)<h,s>\widehat{g}_2(s,k)dqdsdt \\ &= \int <h,s+t><k,p+q+\widetilde{\theta}(s,t)>\widehat{f}(q,t)\widehat{g}(p,s)dpdqdsdt. \end{aligned}$$

From this we will now have that

$$\begin{aligned} (f*g)\widehat{\ }(p,s) &= \int \widehat{f}(q,t)\widehat{g}(p-q-\widetilde{\theta}(s-t,t),s-t)dqdt \\ &= \int \widehat{f}(q,t)\widehat{g}((q,t)^{-1}(p,s))dqdt=\widehat{f}*\widehat{g}(p,s). \end{aligned}$$

As for the involution we have

$$\begin{aligned} \widehat{f^*}_2(s,k)ds &= f^*(dP_s(k,0))=\overline{f(u^*\nu((-k,0)dP_s)u)} \\ &= \overline{f(u^*dP_{-s}(k,0)u)}=\overline{f(dP_{-s}\beta_s(k,0))} \\ &= \overline{f(dP_{-s}(0,\theta(s,k))(k,0))}=<\theta(s,k),s>\overline{\widehat{f}_2(-s,k)}ds. \end{aligned}$$

We now get

$$\begin{aligned} \widehat{f^*}(p,s) &= \int \overline{<k,p>}<\theta(s,k),s>\overline{\widehat{f}_2(-s,k)}dk \\ &= \overline{\int <k,p-\widetilde{\theta}(s,s)>\widehat{f}_2(-s,k)dk} \\ &= \overline{\widehat{f}(\widetilde{\theta}(s,s)-p,-s)}=\overline{\widehat{f}((p,s)^{-1})}. \end{aligned}$$

Theorem 4.13 covers the examples in [Rf5], in particular we mention the example obtained by taking $H=\mathbb{Z}_2\times\mathbb{Z}_2$, and letting $K=\mathbb{Z}_2$ act on H by $A(1)(m,n)=(n,m)$ for $(m,n)\in H$, 1 being the generator of $\mathbb{Z}_2$. However, here we have to use the ordinary definition of semi-direct product given in [HR, 2.6], but as noted in the remark after

Theorem 4.11 this does not affect the conclutions in Theorem 4.13. So here $G = K \times_A H = \mathbb{Z}_2 \times \mathbb{Z}_2^2$ with product

$$(k,s)(l,t) = (k+l, A(k)t+s) \quad \text{for } k,l \in \mathbb{Z}_2, \quad s,t \in \mathbb{Z}_2^2.$$

(The product in (2.23) would in this case give an abelian group.) Note that this group is the 8-element dihedral group. With $\widetilde{\theta}(s,t) = s_1 t_2$ we have that $(\widehat{K} \times \widehat{H}, \widetilde{\theta})$ is the 8-element quaternionian group. As noted by Rieffel in [Rf5] this is the same as the famous example of Kac and Palyutkin in [KP].

§4.3 Comultiplications from dual cocycles of abelian, normal subgroups.

We shall finally return to where we started and look at the comultiplication δ_ω defined in §1, but this time coming from a dual cocycle ω of an abelian, normal subgroup H, given as before by $\omega(s,t) =< \pi(s), t >$. As noted in Remark 1.6, if $xhx^{-1} = A(x)h$ then $\delta_\omega(x) = x \otimes x \iff A(x)\pi A(x)^t = \pi$. For simplicity, let us concentrate on $H = \mathbb{R}^2$ with $\pi(s,t) = (\lambda t, -\lambda s)$, then this condition is equivalent to $\det A = 1$. So to get a nontrivial deformation we need actions which do not have determinant 1. We take $K = \mathbb{R}$ with $A(x)(y,z) = e^x(y,z)$, thus $G \cong \mathbb{R}^3$ with product

$$(x,y,z)(x',y',z') = (x+x', e^x y' + e^{-x'} y, e^x z' + e^{-x'} z)$$

isomorphic to the subgroup of $SL(2,\mathbb{C})$ consisting of matrices of the form $\begin{pmatrix} e^x & z \\ 0 & e^{-x} \end{pmatrix}$. Take $K = \{(x,0,0) | x \in \mathbb{R}\}$, $H_1 = \{(0,y,0) | y \in \mathbb{R}\}$, $H_2 = \{(0,0,z) | z \in \mathbb{R}\}$. We then have $\omega = \omega_0 \sigma(\omega_0)^*$ with $\omega_0 = \int_{\widehat{H}_2} \pi_0(t) \otimes dP_t^2$ where $\pi_0(t) = (0, \lambda t, 0)$. Then $\delta_\omega(h) = h \otimes h$ for $h \in H$, but for $x = (x,0,0) \in K$ we have

$$\begin{aligned} \omega_0 \delta(x) \omega_0^* &= \int \pi_0(s) x \pi_0(-t) \otimes dP_s^2 x dP_t^2 = \int \pi_0(s) x \pi_0(-x^{-1}s) \otimes dP_s^2 x \\ &= \int \theta_0(s,x) x \otimes dP_s^2 x \end{aligned}$$

where

$$\theta_0(s,x) = \pi_0(s) - x\pi_0(x^{-1}s)x^{-1} = (0, \lambda s, 0) - x(0, \lambda e^{2x} s, 0)x^{-1} = (0, \lambda s(1-e^{4x}), 0).$$

As in (4.17) we therefore have that

$$\delta_\omega(x) = \int dP_s \theta_0(t,x) x \otimes dP_t \theta_0(-s,x) x$$

and the corresponding product in $A(G)$ is as in (4.18)

$$f * g(x) = \int \widehat{f}_l(s, \theta_0(t,x)x)\widehat{g}_l(t, \theta(-s,x)x)dsdt.$$

In this example we shall look at the map δ_ω on the Lie-algebra $\mathfrak{g}$ which has basis

$$X = -i\frac{d}{dx}\Big|_0(x,0,0) \quad Y = -i\frac{d}{dy}\Big|_0(0,y,0) \quad Z = -i\frac{d}{dz}\Big|_0(0,0,z).$$

Since $-i\frac{d}{dx}\big|_0\theta_0(s,x) = -4\lambda sY$ and $Z = \int s dP_s^2$, we have

$$\begin{aligned}\delta_\omega(X) &= X \otimes I + I \otimes X - 4\lambda \int tY \otimes dP_t^2 + 4\lambda \int dP_s^2 \otimes sY \\ &= X \otimes I + I \otimes X + 4\lambda(Y \otimes Z - Z \otimes Y).\end{aligned}$$

We have $\delta_\omega(Y) = Y \otimes I + I \otimes Y$, $\delta_\omega(Z) = Z \otimes I + I \otimes Z$, $[Y,Z] = 0$, $[X,Y] = 2Y$ and $[X,Z] = 2Z$. This example is the same as the one in [Da1, 1.1], i.e. dual to example 3.2.

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The references only include articles directly needed here. For more guidance to the general subject of quantum deformations of groups, the reader should e.g. consult [Rf7] and [Dr].

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M. B. Landstad
Institute for Mathematics and Statistics, AVH
University of Trondheim
N-7055 Dragvoll
Norway

magnus.landstad@avh.unit.no