ERGODIC ACTIONS OF NON-ABELIAN COMPACT GROUPS

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Abstract. We show that an ergodic action of a compact group G on a von Neumann algebra $\mathcal{A}$ is completely characterized by a dual cocycle ω. Each ω defines a multiplication on $A(G)$ and a representation π into $\mathcal{L}(G)$ such that: (1) $\mathcal{A}$ is a factor $\iff$ (2) π is injective $\iff$ (3) π has dense range. The subalgebra $A^\circ(G)$ generated by G's characters is commutative and for $f \in A^\circ(G)$ $\pi(f) = \sum f(x_i(D))e^D_{ii}$ for some $x_i(D) \in G$. (1–3) are then also equivalent to (4) $x_i(D) \neq e$ for $D \not\sim 1$.

1. Introduction

Ergodic actions of compact groups may be thought of as generalizations of ordinary irreducible representations, the correspondence being as follows: If $D \in \widehat{G}$ acts on $\mathbf{C}^d$, $\operatorname{Ad} D_x$ gives an ergodic action of G on $\mathcal{M}(d, \mathbf{C})$. If we also allow D to be a projective representation, this correspondence is 1–1. So the ergodic actions of G on type I factors are the same as the projective representations of of G. The first indication that ergodic actions of compact groups are of a special kind is E. Størmer's result in [11]: If a compact abelian group G acts ergodically on a von Neumann algebra $\mathcal{A}$, then $\mathcal{A}$ has a unique invariant trace and is spanned by unitary eigenoperators, i.e. of elements $u(\alpha) \in \mathcal{A}$ where $\alpha \in \widehat{G}$ such that

$$\rho_x(u(\alpha)) = \alpha(x)u(\alpha) \quad \text{for all } x \in G. \tag{1.1}$$

The idea of thinking of ergodic actions of a group G as a dual object to G (following F. Klein's philosophy that an object should be studied by its homomorphisms) and the first classification of all the ergodic actions of a given group G started with the work [1] by S. Albeverio and R. Høegh-Krohn, and [6] by R. Høegh-Krohn and T. Skjelbred. This was later generalized to arbitrary compact abelian groups by D. Olesen, G.K. Pedersen and M. Takesaki in [10].

Abelian groups. The title of this article is chosen to emphasize that our main concern is non-abelian groups, although almost everything is true also for abelian groups. Therefore let us stop shortly to describe the abelian case. So suppose we have a W^*-dynamical system $(\mathcal{A}, \rho, G)$ over an abelian compact group G and let $u(\alpha)$ be the eigenoperators defined in (1.1). Then it is easy to see that there is a function $\omega : \widehat{G} \times \widehat{G} \to \mathbf{T}$ such that

$$u(\alpha)u(\beta) = \omega(\alpha, \beta)u(\alpha + \beta) \quad \text{for } \alpha, \beta \in \widehat{G}. \tag{1.2}$$

Ideas and methods in Mathematical Analysis, Stochastics, and Applications, ed. S. Albeverio *et al.*, Cambridge Univ. Press, 1992, vol. I, pp. 365–388.

ω must be a cocycle, i.e.

$$\omega(\alpha,\beta)\omega(\alpha+\beta,\gamma)=\omega(\alpha,\beta+\gamma)\omega(\beta,\gamma). \tag{1.3}$$

It turns out that ω completely determines $(\mathcal{A},\rho,G)$, so there is a 1–1 correspondence between ergodic W^*-dynamical systems $(\mathcal{A},\rho,G)$ and cocycles ω over $\widehat{G}$ such that $(\mathcal{A},\rho,G)\sim(\mathcal{A}',\rho',G) \iff \omega\sim\omega'$ as cocycles. So the classification of ergodic W^*-dynamical systems is the same as the classification of cocycles on $\widehat{G}$ up to equivalence.

This classification was in fact done by A. Kleppner in [8], where the classifying objects are the so called symplectic (or anti-symmetric) bicharacters of $\widehat{G}$. The function π (which is the same as $\omega^{(2)}$ in [8]) is defined by

$$\pi(\alpha,\beta)=\omega(\alpha,\beta)\overline{\omega(\beta,\alpha)} \tag{1.4}$$

and satisfies

$$\pi(\alpha,\beta)=\overline{\pi(\beta,\alpha)}\quad \pi(\alpha,\beta+\gamma)=\pi(\alpha,\beta)\pi(\alpha,\gamma)\quad \pi(\alpha,\alpha)=1. \tag{1.5}$$

So there is a homomorphism (also called π) from $\widehat{G}$ to G such that, $\pi(\alpha,\beta)=\langle\pi(\alpha),\beta\rangle$. Conversely, (and this is the hard part) to every homomorphism $\pi:\widehat{G}\to G$ which is symplectic, that is satisfies

$$\langle\pi(\alpha),\alpha\rangle=1 \tag{1.6}$$

there is a cocycle ω over $\widehat{G}$ such that $\langle\pi(\alpha),\beta\rangle=\omega(\alpha,\beta)\overline{\omega(\beta,\alpha)}$.

The classification of ergodic actions of a compact abelian group is therefore the same as the classification of all homomorphisms $\pi:\widehat{G}\to G$ which satisfy (1.6). So for instance if $G=\mathbf{T}^n$, then π is completely determined by its values on a set of generators of $\mathbf{Z}^n$.

Since $\pi(\alpha,\beta)=u(\alpha)u(\beta)u(\alpha)^*u(\beta)^*$ we may think of π as a measure on how non-commutative $\mathcal{A}$ is, for instance it should be obvious that $\mathcal{A}$ is commutative if and only if $\pi=1$. In the opposite direction one has

Theorem 1.1. *The following are equivalent:*

(1) *$\mathcal{A}$ is a factor*
(2) *π is injective*
(3) *$\pi(\widehat{G})$ is dense in G.*

Proof. See [1, 3.7],[10, 5.8].

So for abelian compact groups one really has a complete description of all ergodic actions.

Non-abelian groups. Let us now turn to the non-abelian compact groups. S. Albeverio and R. Høegh-Krohn made the following observation (although it was not included in the final version of [1]):

Theorem 1.2. *If $(\mathcal{A}, \rho, G)$ is an ergodic W^*-dynamical system then there is a closed subgroup H of G and an ergodic W^*-dynamical system $(\mathcal{B}, \beta, H)$ with $\mathcal{B}$ a factor such that*

$$\mathcal{A} \simeq \{f \in L^\infty(G, \mathcal{B}) \mid f(xh) = \beta_{h^{-1}}(f(x)) \quad \text{for } x \in G, h \in H\}$$

and ρ_x corresponds to the map $f \to \lambda(x)f$ where $\lambda(x)f(y) = f(x^{-1}y)$. We say that $(\mathcal{A}, \rho, G)$ is induced from $(\mathcal{B}, \beta, H)$.

So the classification of all ergodic actions of G is the same as classifying all ergodic actions of the closed subgroups of G on factors.

The first step toward a study of ergodic actions of non-abelian compact groups was taken in [5] where it was shown that $\mathcal{A}$ must be *finite*: $\mathcal{A}$ has a (unique) G-invariant trace and is an injective von Neumann algebra. So if $\mathcal{A}$ is a factor it is either an I_n-factor with $n < \infty$ or the hyperfinite factor R.

If N is a closed normal subgroup of G there is a natural ergodic action of G/N on the fixed–point algebra $\mathcal{A}^{\rho|N}$. As we noted earlier the ergodic actions of G on I_n-factors are the projective unitary representations of G, so the conjecture made in [5] seems quite natural:

Conjecture. *There are no ergodic actions of a simple compact group on the hyperfinite factor R.*

A. Wassermann has in [14] defined a *multiplicity map* which he has used together with a clever case by case approach to study the ergodic actions of the groups $SU(2)$, $SO(3)$, $SU(3)$, $SU(2) \times SU(2)$ and $SU(2) \times$any compact abelian group, see [14], [16] and [17]. His work confirms the conjecture so far.

The aim of this article is to show that one to a large extent can develop a theory of cocycles and symplectic bicharacters also in the non-abelian case, at least when we restrict to the case where ρ has what we shall call full multiplicity. The first step, from ergodic actions to so called dual cocycles is very similar to the abelian case when one uses that the non-abelian analogue of $L^\infty(G)$ is $\mathcal{L}(G)$= the von Neumann algebra generated by the left regular representation. A dual cocycle will then be a unitary $\omega \in \mathcal{L}(G) \otimes \mathcal{L}(G)$ satisfying an equation similar to (1.3), and we will also put some normalizing conditions on ω. The main result in part 3 is then that there is a 1–1 correspondence between ergodic W^*-dynamical systems and equivalence classes of normalized dual cocycles over G.

The next step, going from cocycles to bicharacters is less clear. By analogy with (1.4) π should be defined as $\pi = \omega\sigma(\omega^*)$ (σ is the usual flip), but it less clear which properties we should expect from π. In the abelian case, the group homomorhism $\pi : \widehat{G} \to G$ will implement an algebra homomorhism $\pi : l^1(\widehat{G}) \to l^\infty(\widehat{G})$ by

$$\pi(f)(\alpha) = \widehat{f}(\pi(\alpha)).$$

Since the non-abelian analogue of $l^1(\widehat{G})$ is the Fourier algebra $A(G)$, one would expect that π would give a homomorphism of $A(G)$ into $\mathcal{L}(G)$. This is not quite true however, π is only multiplicative when we give $A(G)$ another multiplication which can not be expected to be commutative. $A(G)$ with this multiplication is denoted by $A_\omega(G)$ since it depends on ω. But with these modifications we do get a homomorphism $\pi : A_\omega(G) \to \mathcal{L}(G)$ which satisfies the analogue of Theorem 1.1:

Theorem 4.14. *The following are equivalent:*

(1) $\mathcal{A}$ *is a factor*
(2) π *is injective*
(3) π *has dense image in* $\mathcal{L}(G)$

The classification problem now becomes the following: What are the possible multiplications on $A(G)$ and what are the possible (if any) representations into $\mathcal{L}(G)$ which are symplectic? (Precise definitions are given in part 4.) If G has a faithful irreducible unitary representation D, then π is uniquely determined by the restriction to $\mathcal{B}(D) \otimes \mathcal{B}(D)$. So the problem is reduced to a problem about finite matrices: Determine the matrices $\pi \in \mathcal{B}(D) \otimes \mathcal{B}(D)$ which can occur. The discovery by A. Wassermann that one of the equations π has to satisfy is the so called *Quantum Yang–Baxter equation* [17], gives a connection between ergodic actions of compact groups and the extensive theory of what often is called Quantum Groups. We refer to [3] for more on that. So far however, none of the many solutions of the Quantum Yang–Baxter equation have given any new ergodic actions.

$A_\omega(G)$ is not abelian, but it turns out that the subalgebra $A^\circ_\omega(G)$ generated by the characters of G is. So if we diagonalize the restriction of π to $A^\circ_\omega(G)$ we get that

$$\pi(f) = \sum_{i,D} f(x_i(D))e^D_{ii} \quad f \in A^\circ_\omega(G),$$

for some elements $x_i(D) \in G$ which are determined only up to conjugacy classes. These are the analogues of the group elements $\pi(\alpha) \in G$ in the abelian case, and we will get in the non-abelian case that (1–3) also are equivalent to

(4) $x_i(D) \neq 1 \quad \text{for } D \not\sim 1.$

For a group like $\mathbf{T}^n$ it was easy to see what the possible choices for the group elements $\pi(\alpha)$ are. But this is not at all clear in the general case.

We are here presenting results the author has slowly been gathering since the appearance of [5]. Many of these have been obtained independently by A. Wassermann and the author would like to acknowledge the various stimulating discussions we have had on this subject during the years. The reader should look in [14–17] for more detailed information on material not covered here.

2. Preliminaries

Notation. We shall first list some facts about compact groups and on the way also introduce the notation we are using.

$\lambda(x)$ and $r(x)$ are the left (respectively right) regular representations of G over $L^2(G)$, and $\mathcal{L}(G)$ is the von Neumann algebra generated by the left regular representation of G over $L^2(G)$. $A(G)$ is the predual of $\mathcal{L}(G)$. We have the maps $\delta : \mathcal{L}(G) \to \mathcal{L}(G) \otimes \mathcal{L}(G)$ and $\nu : \mathcal{L}(G) \to \mathcal{L}(G)$ defined by

$$\delta(x) = x \otimes x \quad \text{and} \quad \nu(x) = x^{-1} \quad \text{for } x \in G.$$

$A(G)$ is a commutative Banach $*$-algebra under pointwise multiplication and complex conjugation, and the spectrum of $A(G)$ is equal to G. We shall also need the subalgebra $A^\circ(G)$ of $A(G)$ generated by the characters of G, i.e.

$$f \in A^\circ(G) \iff f(xy - yx) = 0 \quad \text{for all } x, y \in \mathcal{L}(G).$$

The spectrum of $A^\circ(G)$ is the same as that of $C^\circ(G)$ and equals G° = the set of conjugacy classes of G.

We shall also need some facts from the theory of representations of compact groups. If $D \in \widehat{G}$ and $\{\xi_i\}$ is an orthonormal basis of the space $X(D)$ on which D acts, then let $(d = \dim D)$

$$\phi_{ij}^D(x) = \langle \xi_i, D_x \xi_j \rangle \qquad \chi_D = \frac{1}{d}\sum_{i=1}^{d} \phi_{ii}^D$$

$$e_{ij}^D = d\int_G \phi_{ij}^D(x)\lambda(x)\,dx \qquad e^D = \sum_{i=1}^{d} e_{ii}^D.$$

This gives an identification $\mathcal{L}(G) \cong \sum \mathcal{B}(D)$ with e_{ij}^D corresponding to matrix units in $\mathcal{B}(D)$. Here $\mathcal{B}(D) = \mathcal{B}(X(D)) = \mathcal{M}(d, \mathbf{C})$, and we shall often use this identification. One should then note that ν maps $\mathcal{B}(D)$ into $\mathcal{B}(\overline{D})$, and that the character of G as used in group representations is $d^2\chi_D$.

Let e^1 be the projection in $\mathcal{L}(G)$ corresponding to the trivial representation 1 of G, so $e^1 = \int_G \lambda(x)\,dx$. The following are then easily verified for all $x \in \mathcal{L}(G)$:

$$\delta(x) I \otimes e^1 = x \otimes e^1 \tag{2.1}$$

$$(I \otimes x)\delta(e^1) = (\nu(x) \otimes I)\delta(e^1) \tag{2.2}$$

We shall also need Takesaki's operator W defined over $L^2(G \times G)$ by

$$Wf(s,t) = f(s, st).$$

Then $\delta(x) = W^* x \otimes I\, W$ for all $x \in \mathcal{L}(G)$. Using standard results about representations of compact groups we also have

$$W = \sum_{i,j,D} \phi_{ji}^D \otimes e_{ij}^D,$$

where we have identified a function ϕ with the corresponding multiplication operator $\phi : f \to \phi f$ over $L^2(G)$. Therefore $i \otimes \nu(W)$ is well defined and

$$\nu \otimes \nu(W) = W^* = i \otimes \nu(W). \tag{2.3}$$

In working with non-abelian groups tensor products are an indispensible tool, and the following notation now seems to be standard. The flip σ is defined by $\sigma(a\otimes b) = b\otimes a$, and we then also get maps σ_{12} etc. of $\mathcal{L}(G) \otimes \mathcal{L}(G) \otimes \mathcal{L}(G)$ by

$$\sigma_{12}(a \otimes b \otimes c) = b \otimes a \otimes c \quad \sigma_{13}(a \otimes b \otimes c) = c \otimes b \otimes a.$$

If $U \in \mathcal{A} \otimes \mathcal{L}(G)$ we define $U_1, U_2 \in \mathcal{A} \otimes \mathcal{L}(G) \otimes \mathcal{L}(G) \otimes \mathcal{L}(G)$ by

$$U_1 = U \otimes I \otimes I \qquad U_2 = i \otimes \sigma_{12}(U_1) \quad \text{etc.}$$

If $\omega \in \mathcal{L}(G) \otimes \mathcal{L}(G)$ we shall write similarly

$$\omega_{12} = \omega \otimes I \quad \omega_{13} = \sigma_{23}(\omega \otimes I) \quad \omega_{23} = I \otimes \omega.$$

We shall also need slicemaps $\mathcal{L}(G) \otimes \mathcal{L}(G) \to \mathcal{L}(G)$ defined by

$$i \otimes f(a \otimes b) = f(b)a \quad \text{for } a, b \in \mathcal{L}(G)\ f \in A(G).$$

To increase readability we have tried to cut down on the number of parentheses and therefore certain formulas may at first sight seem ambiguous. For instance

$$i \otimes \nu(U)_2 = (i \otimes \sigma)(i \otimes \nu \otimes i)(U \otimes I).$$

For more about $A(G), \mathcal{L}(G)$ and Hopf–von Neumann algebras we refer to [12], and for more on compact groups we refer to [4].

Ergodic W^*-dynamical systems. If $(\mathcal{A}, \rho, G)$ is a W^*-dynamical system it is called *ergodic* if the fixed–point algebra

$$\mathcal{A}^\rho = \{a \in \mathcal{A} \mid \rho_x(a) = a \text{ for all } x \in G\} = \mathbf{C}I,$$

and these will be our main objects of study. One can also look at C^*-dynamical systems, but as noted in [9, 3B1] there is a 1–1 correspondence between ergodic C^*-dynamical systems and ergodic W^*-dynamical systems preserving all relevant structure. Some proofs are easier in the one case than the other, but we shall concentrate on the von Neumann algebra case.

Actions of compact groups on a locally convex space E are a well developed subject, we refer to [4, IX.7] for details. The basic fact we shall need is that one has a decomposition

$$E = \bigoplus_{D \in \widehat{G}} E(D)$$

where the action on each *spectral subspace* $E(D)$ is a multiple of D. In our situation with G acting on $\mathcal{A}$ we shall need not only the spectral subspaces $\mathcal{A}(D)$, but also the following: If D is an irreducible representation of G with $\dim(D)=d$, we call $U \in A \otimes \mathcal{M}(d, \mathbf{C})$ a *D-eigenoperator* for ρ if

$$\rho_x \otimes i(U) = UI \otimes D_x \quad \text{for all } x \in G. \tag{2.4}$$

It was shown in [5] that if $(\mathcal{A}, \rho, G)$ is ergodic the unique invariant state τ on $\mathcal{A}$ defined by $\tau(a)I = \int \rho_x(a)\, dx$ is in fact a trace on $\mathcal{A}$. It was also shown there that

$$\dim \mathcal{A}(D) \leq (\dim D)^2 \tag{2.5}$$

and in [9, Theorem 8] that equality holds in (2.5) if and only if there exists a *unitary* D-eigenoperator. If this is true for all $D \in \widehat{G}$ we shall say that ρ has *full* multiplicity, and for the rest of this article we shall always assume this to be the case. If we pick one unitary eigenoperator $U(D)$, any other D-eigenoperator is of the form $(I \otimes \xi)U(D)$ for some $\xi \in \mathcal{M}(d, \mathbf{C})$. If we write

$$U(D) = \sum_{i,j \leq d} u_{ij}^D \otimes e_{ji}^D, \tag{2.6}$$

then $\{u_{ij}^D \mid i, j \leq d\}$ is an orthonormal basis in $\mathcal{A}(D)$ under the inner product defined by τ. Using the theory of representations of G again, one has that

$$\mathcal{A}(D)\mathcal{A}(E) \subset \sum_{F \subset D \otimes E} \mathcal{A}(F) \quad \text{and} \quad \mathcal{A}(D)^* = \mathcal{A}(\overline{D}). \tag{2.7}$$

From this it now follows that $\{u_{ij}^D \mid i, j \leq d, D \in \widehat{G}\}$ spans a dense G-invariant $*$-subalgebra $\mathcal{A}_\circ$ of $\mathcal{A}$. One should note that these last statements are true also if we do not have full multiplicity.

Lemma 2.1. *If we have unitary eigenoperators for both D and E in $\widehat{G}$, then we have unitary eigenoperators for all $F \subset D \otimes E$ and also for $\overline{D}$.*

Proof. $F \subset D \otimes E$ means that there is a partial isometry

$$V : X(F) \to X(D) \otimes X(E) \quad \text{such that} \quad VF_x = D_x \otimes E_x V \quad \text{for } x \in G.$$

Suppose $U(D)$,$U(E)$ are the given unitary eigenoperators, and take

$$U'(F) = U(D)_1 U(E)_2 I \otimes V.$$

Then $\rho_x \otimes i(U'(F)) = U'(F) I \otimes D_x$, so $U'(F)U'(F)^* \in \mathcal{A}^\rho \otimes \mathcal{B}(D) \otimes \mathcal{B}(E)$, i.e. $U'(F)U'(F)^* = I \otimes p$ for some projection $p \in \mathcal{B}(D) \otimes \mathcal{B}(E)$. $U'(F)^* U'(F) = I \otimes e^F$, so there is a partial isometry $Q \in \mathcal{B}(X(D) \otimes X(E), X(F))$ such that $Q^*Q = e^F$ and $QQ^* = p$. Now take

$$U(F) = I \otimes Q^* U(D)_1 U(E)_2 I \otimes V \in \mathcal{A} \otimes \mathcal{B}(F),$$

then $U(F)$ is a unitary eigenoperator. Finally, observe that $i \otimes \nu(U(D)^*)$ is a unitary eigenoperator for $\overline{D}$. □

The author thanks E. Gootman and A. Lazar for discovering a mistake in an earlier version of Lemma 2.1.

Corollary 2.2. *Suppose D is a faithful irreducible representation of G. If ρ has a unitary D-eigenoperator, then it has unitary eigenoperators for all $E \in \widehat{G}$, so ρ has full multiplicity. Furthermore $\mathcal{A}(D)$ generates $\mathcal{A}$. In particular if G is a simple compact group then all $D \not\sim 1$ are faithful.*

Proof. This follows from the fact that if $D \in \widehat{G}$ is faithful, then by the Stone–Weierstrass theorem every $E \in \widehat{G}$ is contained in some tensor power of D and $\overline{D}$, (c.f. [4, IX,6.1]). So by Lemma 2.1 the $*$-algebra generated by $\mathcal{A}(D)$ equals $\mathcal{A}_\circ$, hence the von Neumann algebra generated by $\mathcal{A}(D)$ equals $\mathcal{A}$.

We are not going to use the theory of crossed products here, so we just mention the following result:

Theorem 2.3. *An ergodic system $(\mathcal{A}, \rho, G)$ has full multiplicity if and only if $\mathcal{A} \times_\rho G \simeq \mathcal{B}(L^2(G))$.*

Proof. See [9, Theorem 8] or [14, Theorem 15].

3. Ergodic actions and dual cocycles

As earlier G will always be a compact group, and we shall study ergodic W^*-dynamical systems $(\mathcal{A}, \rho, G)$ of full multiplicity. We pick one unitary eigenoperator $U(D)$ for each $D \in \widehat{G}$ and if we now define

$$U = \sum_{D \in \widehat{G}} U(D) \in \mathcal{A} \otimes \mathcal{L}(G), \tag{3.1}$$

then $\rho_x \otimes i(U) = U \; I \otimes \lambda(x)$ for $x \in G$. It is then easy to see that the operator

$$(U \otimes I) \; i \otimes \sigma(U \otimes I) \; i \otimes \delta(U^*)$$

must be invariant under $\rho_x \otimes i \otimes i$, hence it must be equal to $I \otimes \omega$ for some $\omega \in \mathcal{L}(G) \otimes \mathcal{L}(G)$.

Lemma 3.1. *ω satisfies*

$$(\omega \otimes I)\delta \otimes i(\omega) = (I \otimes \omega)i \otimes \delta(\omega) \tag{3.2}$$

Proof.

$$\begin{aligned}
&I \otimes (I \otimes \omega)(i \otimes \delta)(\omega) = (I \otimes \omega_{23})i \otimes i \otimes \delta[U_1 U_2 i \otimes \delta(U^*)] \\
&= U_1(I \otimes \omega_{23})(i \otimes i \otimes \delta)(i \otimes \sigma_{12})[U_1 i \otimes \delta(U^*)] \\
&= U_1(i \otimes \sigma_{12}\sigma_{23})(I \otimes \omega_{12})(i \otimes \sigma_{12}\sigma_{23})(i \otimes \delta \otimes i)[U_1]i \otimes (i \otimes \delta)\delta[U^*] \\
&= U_1(i \otimes \sigma_{12}\sigma_{23})[(I \otimes \omega)i \otimes \delta(U) \otimes I]i \otimes (\delta \otimes i)\delta[U^*] \\
&= U_1(i \otimes \sigma_{12}\sigma_{23})[U_1 U_2]i \otimes (\delta \otimes i)\delta[U^*] = U_1 U_2 U_3 i \otimes (\delta \otimes i)\delta[U^*] \\
&= (I \otimes \omega_{12})i \otimes \delta \otimes i[U_1 U_2 \delta(U^*)] = I \otimes (\omega \otimes I)\delta \otimes i(\omega). \quad \square
\end{aligned}$$

If G was abelian $\mathcal{L}(G) \otimes \mathcal{L}(G) \simeq L^\infty(\widehat{G} \times \widehat{G})$ and Lemma 3.1 would say that ω is a cocycle over the dual group $\widehat{G}$, so therefore we shall later call ω a dual cocycle. First we want to look a little closer at ω, and for this we are going to use the map $i \otimes \nu$. This should be done with caution: it is only defined on the algebraic tensor product and is neither an automorphism nor an anti-automorphism. But if we are careful we see that $Ui \otimes \nu(U)$ is a unitary operator fixed under $\rho_x \otimes i$, so there is a unitary $u \in \mathcal{L}(G)$ such that

$$i \otimes \nu(U) = U^* I \otimes u. \tag{3.3}$$

Lemma 3.2. *$u = \nu(u)$ and $\nu \otimes \nu(\omega) = \delta(u^*)\sigma(\omega^*)(u \otimes u)$.*

Proof. From (3.3) and $i \otimes \nu(U)^* = i \otimes \nu(U^*)$ it follows that

$$U = I \otimes \nu(u)i \otimes \nu(U^*) = I \otimes \nu(u)i \otimes \nu(U)^*.$$

Thus

$$I \otimes u = Ui \otimes \nu(U) = I \otimes \nu(u) \quad \text{i.e. } u = \nu(u).$$

For the second part we shall use $i \otimes \nu \otimes \nu$ on $U_1 U_2 = (I \otimes \omega)\, i \otimes \delta(U)$:

$$\begin{aligned}
&i \otimes \nu(U)_1 i \otimes \nu(U)_2 = i \otimes \nu \otimes \nu(i \otimes \delta)(U) I \otimes \nu \otimes \nu(\omega) \\
\iff &(U^* \otimes I)I \otimes u \otimes I(i \otimes \sigma)[(U^* \otimes I)I \otimes u \otimes I] \\
&\qquad = (i \otimes \delta)(i \otimes \nu)(U)I \otimes \nu \otimes \nu(\omega) \\
\iff &i \otimes \sigma[U \otimes I\, i \otimes \sigma(U \otimes I)]^* I \otimes u \otimes u = i \otimes \delta[U^* I \otimes u]I \otimes \nu \otimes \nu(\omega) \\
\iff &i \otimes \sigma[I \otimes \omega\, i \otimes \delta(U)]^* I \otimes u \otimes u = i \otimes \delta(U^*)I \otimes \delta(u)\nu \otimes \nu(\omega) \\
\iff &I \otimes \sigma(\omega^*)(u \otimes u) = I \otimes \delta(u)\nu \otimes \nu(\omega)
\end{aligned}$$

so

$$\nu \otimes \nu(\omega) = \delta(u^*)\sigma(\omega^*)(u \otimes u). \quad \square$$

So far we have seen that to an ergodic W^*-dynamical system with full multiplicity we can define a unitary eigenoperator $U \in \mathcal{A} \otimes \mathcal{L}(G)$, a unitary dual cocycle ω and an operator u. Next we shall see how ω and u vary with our choice of U:

Lemma 3.3. *Let U' be another unitary eigenoperator for ρ (so $U' = I \otimes \xi U$ for some $\xi \in \mathcal{L}(G)$). Then the corresponding elements ω' and u' are given by*

$$\omega' = \xi \otimes \xi \,\omega\, \delta(\xi^*) \text{ and } u' = \xi u \nu(\xi)$$

Proof. Straightforward.

The following definitions should therefore be natural and are consistent with the ordinary ones for G abelian:

Definition. *A unitary $\omega \in \mathcal{L}(G) \otimes \mathcal{L}(G)$ is called a dual cocycle if*

$$(\omega \otimes I)\delta \otimes i(\omega) = (I \otimes \omega) i \otimes \delta(\omega) \tag{3.2}$$

Two dual cocycles ω and ω' are said to be equivalent if

$$\omega' = \xi \otimes \xi \,\omega\, \delta(\xi^*) \quad \text{for some unitary } \xi \in \mathcal{L}(G).$$

The next lemma tells us that we can always pick U such that the corresponding $u = I$.

Lemma 3.4. *Let $(\mathcal{A}, \rho, G)$ be as before, then there is an eigenoperator U satisfying*

$$i \otimes \nu(U) = U^*.$$

Proof. If $D \not\sim \overline{D}$ we simply choose $U(D)$ and define $U(\overline{D})$ to be $i \otimes \nu(U(D)^*)$. If $D \sim \overline{D}$, first pick one eigenoperator $U'(D)$. Since the automorphism ν maps $B(D)$ into itself, Lemma 3.3 tells us that

$$U'(D) i \otimes \nu(U'(D)) = I \otimes u \quad \text{and} \quad u = \nu(u)$$

for some unitary $u \in B(D)$. Diagonalizing $u = \sum \lambda_i e_i$ one gets $e_i = \nu(e_i)$, so $v = \sum \lambda_i^{-\frac{1}{2}} e_i$ satisfies $v = \nu(v)$ and $v u \nu(v) = e^D$. Then $U(D) = I \otimes v\, U'(D)$ satisfies $i \otimes \nu(U(D)) = U(D)^*$. Finally, just take U as the sum of all $U(D)$'s as before. □

Example. *If $G = SU(2)$ then the only ergodic action of full multiplicity is on $L^\infty(G)$ with $\rho_x(f)(y) = f(x^{-1}y)$.*

Proof. Let D be the 2–dimensional representation of $SU(2)$, then the anti-automorphism ν of $\mathcal{B}(D) = \mathcal{M}(2, \mathbf{C})$ is given by

$$\nu \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

So for some $a, b, c, d \in \mathcal{A}$ we can write

$$U(D) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and thus } i \otimes \nu(U(D)) = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

$$\text{Thus} \quad i \otimes \nu(U(D)) = U(D)^* \iff \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}.$$

Hence $d = a^*, c = -b^*$ and the argument given in [16, Theorem 2] shows that $\{a, b\}$ generates a commutative W^*-algebra which by Corollary 2.2 must be $\mathcal{A}$. Full multiplicity leaves $L^\infty(G)$ as the only possibility. □

From now on we shall assume that U is chosen as in Lemma 3.4, so by Lemma 3.3 the corresponding ω will satisfy

$$\nu \otimes \nu(\omega) = \sigma(\omega^*). \tag{3.4}$$

Lemma 3.5.

$$\omega(I \otimes e^1) = I \otimes e^1 \tag{3.5}$$
$$\omega(e^1 \otimes I) = e^1 \otimes I \tag{3.6}$$
$$\omega\delta(e^1) = \delta(e^1) \tag{3.7}$$

Proof. If we use (2.1) we get

$$\begin{aligned} I \otimes \omega(I \otimes e^1) &= U_1U_2 i \otimes \delta(U^*) I \otimes I \otimes e^1 \\ &= U_1U_2(U^* \otimes e^1) = U_1U_2(I \otimes I \otimes e^1)U_1^* \\ &= U_1U_1^* I \otimes I \otimes e^1 = I \otimes I \otimes e^1, \end{aligned}$$

since $UI \otimes e^1 = I \otimes e^1$. So $\omega I \otimes e^1 = I \otimes e^1$ and similarly $\omega e^1 \otimes I = e^1 \otimes I$.

Next we use (2.2) to prove (3.7):

$$\begin{aligned} I \otimes \omega\delta(e^1) &= U_1U_2 i \otimes \delta(U^*) I \otimes \delta(e^1) = U_1U_2 I \otimes \delta(e^1) \\ &= U_1 i \otimes \nu(U)_1 I \otimes \delta(e^1) = I \otimes \delta(e^1). \quad \square \end{aligned}$$

We shall need two more properties of ω, and for this one should recall the properties of W given in (2.3).

Lemma 3.6.

$$i \otimes \nu(\omega\sigma(\omega^*)) = \sigma(\omega)\omega^* \tag{3.8}$$
$$i \otimes \nu(\omega W^*) = W\omega^* \tag{3.9}$$

Proof. Take $\pi = \omega\sigma(\omega^*)$, so $I \otimes \pi = U_1U_2U_1^*U_2^*$. Then (remember our earlier warnings about $i \otimes \nu$):

$$\begin{aligned} & I \otimes \pi U_2U_1 = U_1U_2 \\ \iff & i \otimes i \otimes \nu(U_2) I \otimes i \otimes \nu(\pi) U_1 = U_1 i \otimes i \otimes \nu(U_2) \\ \iff & i \otimes \nu(U)_2 I \otimes i \otimes \nu(\pi) U_1 = U_1 i \otimes \nu(U)_2 \\ \iff & U_2^* I \otimes i \otimes \nu(\pi) U_1 = U_1 U_2^* \\ \iff & I \otimes i \otimes \nu(\pi) = U_2U_1U_2^*U_1^* \\ \iff & I \otimes i \otimes \nu(\pi) = I \otimes \sigma(\pi) \end{aligned}$$

To prove (3.9) first note that $U_1U_2 = (I \otimes \omega)\, i \otimes \delta(U)$ and the properties of W give

$$U_1U_2 I \otimes W^* = (I \otimes \omega W^*)U_1,$$

so

$$\begin{aligned} I \otimes i \otimes \nu(\omega W^*)U_1 &= U_1 i \otimes i \otimes \nu(U_2 I \otimes W^*) \\ &= U_1 I \otimes i \otimes \nu(W^*) i \otimes \nu(U)_2 = U_1 I \otimes W\, U_2^* \\ &= I \otimes W\, i \otimes \delta(U)U_2^* = I \otimes W\omega_{23}^* U_1. \end{aligned}$$

Thus $i \otimes \nu(\omega W^*) = W\omega^*$. $\square$

Definition. *A normalized dual cocycle is a unitary $\omega \in \mathcal{L}(G)\otimes\mathcal{L}(G)$ which satisfies:*

$$(\omega \otimes I)\delta \otimes i(\omega) = (I \otimes \omega)i \otimes \delta(\omega) \tag{3.2}$$

$$\nu \otimes \nu(\omega) = \sigma(\omega^*) \tag{3.4}$$

$$\omega I \otimes e^1 = I \otimes e^1 \tag{3.5}$$

$$\omega e^1 \otimes I = e^1 \otimes I \tag{3.6}$$

$$\omega\delta(e^1) = \delta(e^1) \tag{3.7}$$

$$i \otimes \nu(\omega\sigma(\omega^*)) = \sigma(\omega)\omega^* \tag{3.8}$$

$$i \otimes \nu(\omega W^*) = W\omega^* \tag{3.9}$$

Two normalized dual cocycles ω and ω' are said to be equivalent if

$$\omega' = \xi \otimes \xi\, \omega\, \delta(\xi^*) \quad \textit{for some unitary } \xi \in \mathcal{L}(G) \quad \textit{with } \xi = \nu(\xi^*).$$

Theorem 3.8. *Suppose $(\mathcal{A}, \rho, G)$ and $(\mathcal{A}', \rho', G)$ are two ergodic W^*-dynamical systems which both have full multiplicity. Let ω and ω' be the corresponding normalized dual cocycles as constructed above. Then*

$$(\mathcal{A}, \rho, G) \sim (\mathcal{A}', \rho', G) \iff \omega \sim \omega' \quad \textit{with} \quad \xi = \nu(\xi^*).$$

Proof. $\Longrightarrow$: Suppose $\Phi : \mathcal{A} \to \mathcal{A}'$ is an equivariant isomorphism, $U \in \mathcal{A} \otimes \mathcal{L}(G)$, $U' \in \mathcal{A}' \otimes \mathcal{L}(G)$ such that

$$\rho_x \otimes i(U) = UI \otimes x, \quad \rho'_x \otimes i(U') = U'I \otimes x \quad \text{for } x \in G.$$

Then $V = \Phi \otimes i(U)$ also is a unitary eigenoperator for ρ' so $V = I \otimes \xi U'$ for some $\xi \in \mathcal{L}(G)$.

$$\begin{aligned} I \otimes \omega &= \Phi \otimes i \otimes i[U_1U_2 i \otimes \delta(U^*)] \\ &= [V_1V_2 i \otimes \delta(V^*)] = I \otimes \xi \otimes \xi\omega'\delta(\xi^*) \end{aligned}$$

So $\omega \sim \omega'$, and one checks that $\xi = \nu(\xi^*)$.

$\Longleftarrow$: Conversely, suppose $\omega = \xi \otimes \xi\, \omega'\, \delta(\xi^*)$ with $\xi = \nu(\xi^*)$. Let τ be the unique invariant trace on $\mathcal{A}$ and use the GNS-construction to obtain a faithful representation of $\mathcal{A}$ over a Hilbert space H, and define τ' and H' similarly. Now write as before

$$U(D) = \sum u_{ij}^D \otimes e_{ji}^D.$$

So $\{u_{ij}^D\}$ is a linearly independent set which spans a dense subalgebra $\mathcal{A}_\circ$ of $\mathcal{A}$ (c.f.(2.7)). $\mathcal{A}'_\circ$ is constructed the same way, so there is a linear map $\Phi : \mathcal{A}_\circ \to \mathcal{A}'_\circ$ such that

$$\Phi \otimes i(U(D)) = I \otimes \xi\, U'(D).$$

To show that Φ is multiplicative, note that

$$\begin{aligned} &\Phi \otimes i \otimes i(U(D)_1)\Phi \otimes i \otimes i(U(E)_2) \\ &= (I \otimes \xi \otimes I)U'(D)_1(I \otimes I \otimes \xi)U'(E)_2 \\ &= I \otimes (e^D \otimes e^E)(\xi \otimes \xi)\omega'\, i \otimes \delta(U') \\ &= I \otimes (e^D \otimes e^E)\omega\, (i \otimes \delta)[I \otimes \xi\, U'] \\ &= [I \otimes (e^D \otimes e^E)\omega](i \otimes \delta)[\Phi \otimes i(U)] \\ &= [I \otimes e^D \otimes e^E]\Phi \otimes i \otimes i[\omega_{23}(i \otimes \delta)(U)] \\ &= \Phi \otimes i \otimes i[U(D)_1U(E)_2], \end{aligned}$$

thus $\Phi(u_{ij}^D)\Phi(u_{kl}^E) = \Phi(u_{ij}^D u_{kl}^E)$, hence $\Phi(ab) = \Phi(a)\Phi(b)$ for all $a, b \in \mathcal{A}_\circ$.

$$\begin{aligned}\Phi \otimes i(U(D))^* &= U'(D)^* I \otimes \xi^* = i \otimes \nu(U'(\overline{D}))I \otimes \nu(\xi) \\ &= i \otimes \nu[I \otimes \xi U'(\overline{D})] = (i \otimes \nu)(\Phi \otimes i)[U(\overline{D})] \\ &= (\Phi \otimes i)(i \otimes \nu)[U(\overline{D})] = \Phi \otimes i(U(D)^*).\end{aligned}$$

So Φ is a $*$-isomorphism between $\mathcal{A}_\circ$ and $\mathcal{A}'_\circ$. Since $\tau' \circ \Phi = \tau$, Φ will extend to a unitary operator between H and H' which will implement a $*$-isomorphism between $\mathcal{A}$ and $\mathcal{A}'$. Finally $\Phi \circ \rho_x = \rho'_x \circ \Phi$, so Φ is equivariant. $\square$

The next result is the opposite of Theorem 3.8: To every normalized dual cocycle there is an ergodic W^*-dynamical system.

Theorem 3.9. *Let ω be a normalized dual cocycle. Then there is an ergodic W^*-dynamical system $(\mathcal{A}, \rho, G)$ with full multiplicity having ω as its normalized dual cocycle.*

Proof. Let $r(x)$ be the right regular representation of G over $L^2(G)$, and let W be as before. Take $U = \omega W^* \in \mathcal{B}(L^2(G)) \otimes \mathcal{L}(G)$. Then

$$r(x) \otimes I\, U = U r(x) \otimes \lambda(x)$$

and

$$\begin{aligned}U_1 U_2 &= (\omega \otimes I)(W^* \otimes I)\omega_{13} W_{13}^* \\ &= (\omega \otimes I)(\delta \otimes i)(\omega)(W^* \otimes I)W_{13}^* \\ &= (I \otimes \omega)(i \otimes \delta)(\omega)\, i \otimes \delta(W^*) \\ &= (I \otimes \omega)(i \otimes \delta)(U).\end{aligned}$$

By (3.9) $i \otimes \nu(U) = U^*$. From this all this it now follows that the "left coefficients" of U (= the images of U by all slicemaps $\mathcal{B}(L^2(G)) \otimes \mathcal{L}(G) \to \mathcal{B}(L^2(G))$) is a $*$-subalgebra $\mathcal{A}_\circ$ of $\mathcal{B}(L^2(G))$ invariant under $\rho_x = \mathrm{Ad}\nu(x)$. So the weak closure $\mathcal{A}$ of $\mathcal{A}_\circ$ is a W^*-dynamical system under ρ. It remains to show that it is ergodic, and since $\mathcal{A}_\circ^\rho$ is dense in $\mathcal{A}^\rho$, it suffices to show that $\mathcal{A}_\circ^\rho$ is trivial. Every $a \in \mathcal{A}_\circ^\rho$ is of the form $i \otimes \phi(\omega W^*)$ for some $\phi \in A(G)$, so

$$\begin{aligned}a &= \int \rho_x(a)\, dx = i \otimes \phi[\omega \int (r(x) \otimes I) W^* (r(x^{-1}) \otimes I)\, dx] \\ &= i \otimes \phi[\omega(I \otimes e^1)] = \phi(e^1) I.\end{aligned}$$

So $\mathcal{A}_\circ^\rho = \mathbf{C}I$, thus $\mathcal{A}^\rho = \mathbf{C}I$ and therefore ρ is ergodic. Since U is unitary ρ has full multiplicity, and ω is obviously the normalized dual cocycle corresponding to U. $\square$

The following corresponds to [15,Lemma 21]:

Proposition 3.10. *Let ω be a normalized dual cocycle. Then*

$$\omega \quad \textit{is trivial } (\sim I) \iff \omega \quad \textit{is symmetric } (\omega = \sigma(\omega)).$$

Proof. Only $\Longleftarrow$ needs to be proved. So suppose $\omega = \sigma(\omega)$ and as in Theorem 3.9 take $U = \omega W^*$ and let $\mathcal{B}$ be the C^*-algebra $\subset \mathcal{A}$ generated by the operators $U(\phi) = i \otimes \phi(U)$. Next note that (c.f. the proof of Lemma 3.6)

$$\omega = \sigma(\omega) \iff U_1 U_2 = U_2 U_1.$$

Since $U(\phi)U(\psi) = i \otimes \phi \otimes \psi(U_1 U_2)$ this means that $\mathcal{B}$ is an abelian C^*-algebra. Now let χ be a non-zero multiplicative linear functional on $\mathcal{B}$. Since $\|U(\phi)\| \leq \|\phi\|$ we must have

$$\chi(U(\phi)) = \langle \xi, \phi \rangle \quad \text{for some } \xi \neq 0 \quad \text{in } \mathcal{L}(G).$$

One also checks (we omit the details) that $U(\phi)U(\psi) = U(\phi \bigtriangleup \psi)$ where

$$\langle x, \phi \bigtriangleup \psi \rangle = \langle \omega\delta(x), \phi \otimes \psi \rangle \quad \text{for } x \in \mathcal{L}(G). \tag{3.10}$$

So since χ is multiplicative we have

$$\begin{aligned} &\langle \xi, \phi \bigtriangleup \psi \rangle = \langle \xi, \phi \rangle \langle \xi, \psi \rangle \\ \iff &\langle \omega\delta(\xi), \phi \otimes \psi \rangle = \langle \xi \otimes \xi, \phi \otimes \psi \rangle \quad \text{for all } \phi, \psi. \end{aligned}$$

So $\omega\delta(\xi) = \xi \otimes \xi \implies \delta(\xi^*\xi) = \xi^*\xi \otimes \xi^*\xi$. But it follows from [12, Lemma 10.4] that then $\xi^*\xi = I$. So ξ is unitary and $\omega = \xi \otimes \xi\delta(\xi)^* \sim I$. $\square$

Remark 3.11. This construction of a $*$-algebra from a dual cocycle is the dual version of A. Kleppner's construction in [7]. In fact (3.10) defines a product on $A(G)$ which makes it into a Banach algebra which generates Kleppner's left regular ω-representation, even when ω is not symmetric. The product (3.10) is also the one used by A. Wassermann in [15,§7].

The following should now be obvious:

Theorem 3.12. *There is a 1–1 correspondence between equivalence classes of ergodic W^*-dynamical systems $(\mathcal{A}, \rho, G)$ of full multiplicity and equivalence classes of normalized dual cocycles ω.*

Remark 3.13. From the observations in [9, 3B1] it follows that there also is a 1–1 correspondence between the equivalence classes of ergodic C^*-dynamical systems $(\mathcal{A}, \rho, G)$ of full multiplicity.

Remark 3.14. The problem of when a dual cocycle is equivalent to a normalized one has been treated in [15, Theorems 5 and 6], we shall not go into this here.

4. Dual cocycles and symplectic bicharacters

So far we have seen that the study of ergodic W^*-dynamical systems of full multiplicity is the same as that of normalized dual cocycles, exactly as for abelian groups. In [8] Kleppner showed that cocycles over $\widehat{G}$ can be classified by so called *symplectic bicharacters*, that is homomorphisms $\pi : \widehat{G} \to G$ which satisfy

$$\langle \pi(\alpha), \beta \rangle = \overline{\langle \pi(\beta), \alpha \rangle} \quad \text{and} \quad \langle \pi(\alpha), \alpha \rangle = 1$$

under the duality between G and $\widehat{G}$. (Observe that the second property implies the first one, but *not* the other way.) This is exactly what is needed to do the classification of ergodic W^*-dynamical systems over abelian compact groups as in [1] and [10]. We shall here show that we can form a symplectic bicharacter π also in the non-abelian case, and we shall see how properties of π correspond to properties of $(\mathcal{A}, \rho, G)$. The first result shows that dual cocycles (and hence also ergodic W^*-dynamical systems) give interesting examples of Hopf–von Neumann algebras.

Theorem 4.1. *Define $\delta_\omega : \mathcal{L}(G) \to \mathcal{L}(G) \otimes \mathcal{L}(G)$ by $\delta_\omega(x) = \omega\delta(x)\omega^*$, where ω is a normalized dual cocycle. Then $(\mathcal{L}(G), \delta_\omega, \nu, \mathrm{tr})$ is a Hopf–von Neumann algebra.*

Proof. For the definition of a Hopf–von Neumann algebra and for the definition of the trace tr on $\mathcal{L}(G)$, we refer to [12]. We omit the details of proof, note for instance that the property $(\delta_\omega \otimes i)\delta_\omega = (i \otimes \delta_\omega)\delta_\omega$ follows from (3.2), and that $\nu \otimes \nu\delta_\omega = \sigma\delta_\omega\nu$ follows from (3.4). □

A Hopf–von Neumann algebra may seem like a strange object, here we shall think of δ_ω as a structure giving $A(G)$ another multiplication:

Proposition 4.2. *For $f, g \in A(G)$ define $f \star g \in A(G)$ by*

$$\langle x, f \star g\rangle = \langle \delta_\omega(x), f \otimes g\rangle \quad \textit{for } x \in \mathcal{L}(G).$$

Together with the involution $^\#$ given by

$$\langle x, f^\#\rangle = \overline{\langle \nu(x^*), f\rangle}$$

and the usual norm as the predual of $\mathcal{L}(G)$, this makes $A(G)$ a Banach $*$-algebra which we shall denote $A_\omega(G)$.

Proof. This follows readily from [12], or directly from the properties of ω.

Remark 4.3. This is the first place we see an important difference between abelian and non-abelian groups: if G is abelian $A(G) = A_\omega(G)$ as Banach $*$-algebras.

Proposition 4.4. *Let $\pi = \omega\sigma(\omega^*)$, then*

$$(\pi \otimes I) i \otimes \sigma(\pi \otimes I) = i \otimes \delta_\omega(\pi) \tag{4.1}$$

$$(I \otimes \pi)\sigma \otimes i(\pi \otimes I) = \delta_\omega \otimes i(\pi) \tag{4.2}$$

$$\pi = \sigma(\pi^*) = \nu \otimes \nu(\pi) \tag{4.3}$$

$$\pi(e^1 \otimes I) = e^1 \otimes I \tag{4.4}$$

$$\pi(I \otimes e^1) = I \otimes e^1 \tag{4.5}$$

$$\pi\delta(e^1) = \delta(e^1) \tag{4.6}$$

$$i \otimes \nu(\pi) = \sigma(\pi) = \pi^* \tag{4.7}$$

Proof. We start with (4.1), and we will use only (3.2) and the properties of δ:

$$\begin{aligned} i \otimes \delta_\omega(\pi) &= \omega_{23} i \otimes \delta(\pi)\omega_{23}^* \\ &= \omega_{23} i \otimes \delta(\omega)(i \otimes \delta)\sigma(\omega^*)\omega_{23}^* \\ &= \omega_{12}\delta \otimes i(\omega)\sigma_{12}\sigma_{23}[(\delta \otimes i)(\omega^*)\omega_{12}^*] \\ &= \omega_{12}\delta \otimes i(\omega)\sigma_{12}\sigma_{23}[(i \otimes \delta)(\omega^*)\omega_{23}*] \\ &= \omega_{12}\delta \otimes i(\omega)\sigma_{12}[(i \otimes \delta)(\omega^*)\sigma_{23}(\omega_{23}^*)] \\ &= \omega_{12}\sigma_{12}[\omega_{12}^*\omega_{23}\sigma_{23}(\omega_{23})^*] \\ &= [\omega\sigma(\omega^*) \otimes I]\sigma_{12}[I \otimes \omega\sigma(\omega^*)] = \pi_{12}\pi_{23}. \end{aligned}$$

To prove (4.2), take the adjoint of (4.1) and act on both sides with $\sigma_{23}\sigma_{12}$:

$$\begin{aligned} &\sigma_{23}\sigma_{12}(\pi_{13}^*)\sigma_{23}(\pi_{12}) = \sigma_{23}\sigma_{12}[\omega_{23} i \otimes \delta(\pi^*)\omega_{23}^*] \\ \iff &[I \otimes \sigma(\pi^*)]\pi_{13} = \omega_{12}(\delta \otimes i)\sigma(\pi^*)\omega_{12}^* \\ \iff &\pi_{23}\pi_{13} = (\delta_\omega \otimes i)(\pi). \end{aligned}$$

(4.3–4.7) now follows from (3.4–3.9). $\square$

Remark 4.5. A.Wassermann has shown in [15, Lemma 26] that π also is a dual cocycle, and satisfies the important Quantum Yang–Baxter equation:

$$\pi_{12}\pi_{13}\pi_{23} = \pi_{23}\pi_{13}\pi_{12} \tag{QYBE}$$

which seems to be of importance in several parts of mathematics. He has also shown that any π satisfying this equation comes from a dual cocycle and hence from an ergodic action. The reader will see that there are many similarities here with ideas described by V.G. Drinfel'd in [3], and we refer to that article for references to the large literature on this subject. We shall comment more on this later.

Hopefully the statements of Proposition 4.4 becomes more meaningful when we phrase them with respect to the algebra $A_\omega(G)$. For $f \in A_\omega(G)$ let

$$\pi(f) = i \otimes f(\pi).$$

Then we have:

Theorem 4.6. *$\pi : f \to \pi(f)$ is a bounded $*$-representation of $A_\omega(G)$, i.e.*

$$\pi(f \star g) = \pi(f)\pi(g) \tag{4.8}$$
$$\pi(f^{\#}) = \pi(f)^* \tag{4.9}$$
$$\pi(1) = I \tag{4.10}$$
$$\nu(\pi(f)) = \pi(f \circ \nu) \tag{4.11}$$

and π is symplectic:

$$\langle \pi(f), g\rangle = \langle \pi(g \circ \nu), f\rangle = \overline{\langle \pi(g^{\#}), f^{\#}\rangle} \tag{4.12}$$

Proof. We start with (4.8):

$$\begin{aligned} \pi(f)\pi(g) &= i \otimes f \otimes g(\pi_{12}\pi_{13}) \\ &= i \otimes f \otimes g(i \otimes \delta_\omega(\pi)) = i \otimes f \star g(\pi). \end{aligned}$$

(4.11) follows from (4.3):

$$\begin{aligned} \langle \nu(\pi(f)), g\rangle &= \langle \pi, g \circ \nu \otimes f\rangle \\ &= \langle \pi, g \otimes f \circ \nu\rangle = \langle \pi(f \circ \nu), g\rangle \end{aligned}$$

(4.10) follows from (4.5):

$$\begin{aligned}&\langle\pi(1), f\rangle = \langle\pi, f\otimes 1\rangle = \langle\pi(I\otimes e^1), f\otimes 1\rangle\\ &= \langle I\otimes e^1, f\otimes 1\rangle = \langle I, f\rangle\end{aligned}$$

The first half of (4.12) follows from (4.7):

$$\begin{aligned}&\langle\pi(g\circ\nu), f\rangle = \langle i\otimes\nu(\pi), f\otimes g\rangle\\ &= \langle\sigma(\pi), f\otimes g\rangle = \langle\pi, g\otimes f\rangle = \langle\pi(f), g\rangle,\end{aligned}$$

and the last part from (4.3):

$$\begin{aligned}&\langle\pi(f), g\rangle = \langle a, g\otimes f\rangle = \overline{\langle\nu\otimes\nu(\pi^*), g^\#\otimes f^\#\rangle}\\ &= \overline{\langle\sigma(\pi), g^\#\otimes f^\#\rangle} = \overline{\langle(\pi(g^\#), f^\#\rangle}.\end{aligned}$$

We get (4.9) by using (4.12):

$$\begin{aligned}&\langle\pi(f)^*, g\rangle = \overline{\langle\pi(f), g^\#\circ\nu\rangle}\\ &= \langle\pi(g\circ\nu), f^\#\rangle = \langle\pi(f^\#), g\rangle.\end{aligned}$$

Finally, $\|\pi(f)\|\leq\|f\|$ should be obvious. □

$A_\omega(G)$ is a pre Hopf–von Neumann algebra if we define δ^t and ν^t by

$$\langle x\otimes y, \delta^t\rangle = \langle yx, f\rangle \quad\text{and } \nu^t(f) = f\circ\nu$$

Then

Corollary 4.7.

$$(\pi\otimes\pi)[\delta^t(f)] = \delta_\omega(\pi(f))$$

so $\pi : (A_\omega(G), \delta^t, \nu^t) \to (\mathcal{L}(G), \delta_\omega, \nu)$ is a Hopf algebra homomorphism.

We omit the proof (which follows from Theorem 4.1) since we shall not need it.

Corollary 4.8. *The map π defined in Theorem 4.6 is injective if and only if its image in $\mathcal{L}(G)$ is weakly dense.*

Proof. Follows from (4.12).

Remark. Since the main difference between abelian and non-abelian groups seems to be that in general $\delta_\omega\neq\delta$, it would be natural to investigate what happens if $\delta_\omega=\delta$, or more generally when $A_\omega(G)$ is commutative. By Proposition 4.2 this happens if and only if π commutes with $\delta(\mathcal{L}(G))$. We then have

Proposition 4.9. *If $A_\omega(G)$ is abelian then $(\mathcal{A}, \rho, G)$ is induced from an ergodic action of a closed abelian subgroup.*

Proof. This is essentially what was done in [9, 3B6].

Proposition 4.10. *If $D \in \widehat{G}$ is faithful, then $\pi(e^D \otimes e^D)$ uniquely determines π.*

Proof. For this first note that from Corollary 2.2 it follows that $\{\phi_{ij}^D\}$ generates $A_\omega(G)$. So from knowing $\pi(e^D \otimes e^D)$ one knows $\pi(\phi_{ij}^D)e^D$ and thus also $\pi(f)e^D$ for all f. From (4.12) we then also know $\pi(\phi_{ij}^D)$, hence $\pi(f)$ for all $f \in A_\omega(G)$ $\square$.

When G is abelian, the symplectic bicharacter π is a group homomorphism from $\widehat{G}$ into G. So far our non-abelian analogue is a homomorphism only on the group algebra level. We shall now see that π also can be thought of as a map on the set level from $\widehat{G}$ into G in the general case. We shall see that many of the abelian results still are true, but not so conclusive any longer.

Definition. *Let*

$$A_\omega^\circ(G) = \{f \in A_\omega(G) \mid f(xy - yx) = 0 \quad \textit{for all } x, y \in G\}.$$

So $A_\omega^\circ(G)$ is the subalgebra of $A_\omega(G)$ generated by the characters of G. Obviously $f \star g = fg$ for $f, g \in A_\omega^\circ(G)$, so $A_\omega^\circ(G) = A^\circ(G)$ as Banach $*$-algebras. The spectrum of $A^\circ(G)$ is the same as that of $C^\circ(G)$ which equals $G^\circ =$ the set of conjugacy classes of G. So π will be a $*$-representation of $A^\circ(G)$ into $\mathcal{L}(G)$ which can be diagonalized, i.e. there is a basis in the space of each $D \in \widehat{G}$ and elements $x_i(D) \in G^\circ$ such that

$$\pi(f) = \sum_{i,D} f(x_i(D))e_{ii}^D \quad \text{for } f \in A^\circ(G). \tag{4.13}$$

We shall call the set $\{x_i(D) \mid i \leq d, D \in \widehat{G}\}$ the *spectrum* of π, and note that $x_i(D)$ is only determined up to conjugacy. From (4.11) it follows that

$$f(x_i(\overline{D})) = f(x_i(D)^{-1}) \quad \text{for all } f \in A^\circ(G).$$

So $x_i(\overline{D})$ equals $x_i(D)^{-1}$ in G°, i.e. they are conjugate as elements of G. So if $D \not\sim \overline{D}$ we may assume they are equal also as elements of G. However if $D \sim \overline{D}$ we can *not* necessarily assume that we can pick an $x_i(D)$ with $x_i(D) = x_i(D)^{-1}$. For instance in the group $SU(2)$ x and x^{-1} are always conjugate, but usually not equal.

For $D \in \widehat{G}$ let

$$x(D) = \pi(\chi_D) \tag{4.14}$$

so $x(D) = \frac{1}{d}\sum_{i=1}^d x_i(D)$ as an element of $\mathcal{L}(G)$. From (4.12) we will now have:

$$\langle x(D), \chi_E \rangle = \overline{\langle x(E), \chi_D \rangle}. \tag{4.15}$$

Since $I \otimes \pi = U_1 U_2 U_1^* U_2^*$, one should expect that there is a relation between properties of π and the algebraic properties of $\mathcal{A}$. These relations shall be our next theme. First we shall observe that the elements $x_i(D)$ can be described more directly by $\mathcal{A}$. Let $U(D)$ be as before and look at the map $\Phi_D : \mathcal{A} \to \mathcal{A}$ given by

$$\begin{aligned}\Phi_D(a) &= i \otimes \chi_D[U(D)a \otimes IU(D)^*] \\ &= \frac{1}{d}\sum_{i,j} u_{ij}^D a u_{ij}^{D*}.\end{aligned}$$

Φ_D does not depend on which unitary $U(D)$ we have chosen, and

$$\Phi_D \circ \rho_x = \rho_x \circ \Phi_D \quad \Phi_D(a)^* = \Phi_D(a^*) \quad \tau(\Phi_D(a)b) = \tau(a\Phi_{\overline{D}}(b))$$

Furthermore

$$\begin{aligned}\Phi_D \circ \Phi_E(a) &= i \otimes \chi_D \otimes \chi_E[U(D)_1 U(E)_2 (a \otimes I \otimes I) U(E)_2^* U(D)_1^*] \\ &= i \otimes \chi_D \otimes \chi_E[\omega_{23} i \otimes \delta(Ua \otimes IU^*)\omega_{23}^*] \\ &= i \otimes \chi_D \otimes \chi_E[i \otimes \delta(Ua \otimes IU^*)] \\ &= i \otimes \chi_E \otimes \chi_D[i \otimes \delta(Ua \otimes IU^*)] = \Phi_E \circ \Phi_D(a).\end{aligned}$$

So $\{\Phi_D \mid D \in \widehat{G}\}$ is a commuting selfadjoint (with respect to the inner product defined by τ) family which leaves each spectral subspace $\mathcal{A}(E)$ invariant. Furthermore $\chi_D \to \Phi_{\overline{D}}$ is a representation of $A^\circ(G)$ over this inner product space. Hence the basis $\{u_{ij}^E\}$ of $\mathcal{A}(E)$ can be picked such that

$$\Phi_D(u_{ij}^E) = \overline{\chi_D(x_j(E))} u_{ij}^E \quad \text{for some } x_j(E) \in G^\circ.$$

That these really are the same $x_j(E)$ as above follows from the fact that

$$U_1^* I \otimes \pi = U_2 U_1^* U_2^* \implies I \otimes \pi(\chi_D) = U\Phi_D \otimes i(U^*).$$

Lemma 4.11. *$\Phi_D(a) = a \iff a$ commutes with $\mathcal{A}(D)$.*

Proof. $\Longleftarrow$ is obvious. So suppose $\Phi_D(a) = a$ and let $\mathcal{A} \otimes \mathcal{M}(d, \mathbf{C})$ have the norm from the inner product $\langle x, y\rangle = \tau \otimes \chi_D(y^* x)$. Then

$$\begin{aligned}&\|U(D)(a \otimes I)U(D)^* - a \otimes I\|^2 \\ &= \tau \otimes \chi_D[U(D)(a^*a \otimes I)U(D)^* - U(D)(a^* \otimes I)U(D)^*(a \otimes I) \\ &\quad - (a^* \otimes I)U(D)(a \otimes I)U(D)^* + a^*a \otimes I] \\ &= 2\tau \otimes \chi_D(a^*a \otimes I) - \tau(\Phi_D(a^*)a) - \tau(a^*\Phi_D(a)) = 0\end{aligned}$$

since $\Phi_D(a) = a \implies \Phi_D(a^*) = a^*$. So

$$U(D)(a \otimes I)U(D)^* = a \otimes I$$

which implies that a commutes with $\{u_{ij}^D\}$, hence with $\mathcal{A}(D)$. □

Lemma 4.12. *The center of $\mathcal{A}$, $\mathcal{Z}(\mathcal{A}) = \mathcal{A} \cap \mathcal{A}'$ is generated by*

$$\{u_{ij}^E \mid x_j(E) = e\}.$$

Proof. $\mathcal{Z}(\mathcal{A})$ is generated by $\mathcal{Z}(\mathcal{A})_\circ = \mathcal{A}_\circ \cap \mathcal{A}'$, so suppose we have a finite sum

$$a = \sum_{i,j,E} \lambda_{ij} u_{ij}^E \in \mathcal{A}_\circ \cap \mathcal{A}'.$$

Since $\Phi_D(a) = a$ for all D, we must have that

$$\lambda_{ij} \neq 0 \implies \chi_D(x_j(E)) = 1.$$

But if $\chi_D(x_j(E)) = 1$ for all $D \in \widehat{G}$ we must have $x_j(E) = e$. Conversely if $x_j(E) = e$ Lemma 4.11 applies to u_{ij}^E for all $D \in \widehat{G}$, so $u_{ij}^E \in \mathcal{Z}(\mathcal{A})$. □

Corollary 4.13. *$\mathcal{A}$ is a factor $\iff x_j(E) \neq e$ for all $E \neq 1$.*

Theorem 4.14. *Look at the following statements:*

(1) *$\mathcal{A}$ is a factor,*
(2) *$x_i(E) \neq e$ for $E \neq 1$,*
(3) *$\pi(f) = 0 \implies f = 0$,*
(4) *$\{\pi(f) \mid f \in A_\omega(G)\}$ is weakly dense in $\mathcal{L}(G)$,*
(5) *$\{x_j(D) \mid j \leq d, D \in \widehat{G}\}$ is dense in G°.*

Then (1) $\iff$ (2) $\iff$ (3) $\iff$ (4) $\implies$ (5)

Remark 4.15. As noted in [9, 3B2] (1) is also equivalent with the property that in the corresponding C^*-dynamical system the C^*-algebra is *simple* or *prime*. Theorem 4.14 corresponds to [15,Theorem 12].

Proof. (1) $\iff$ (2) is Corollary 4.13, (3) $\iff$ (4) is Corollary 4.8.

(3) $\implies$ (2): Suppose (2) is not true, so we have $x_i(E) = 1$ for some $E \neq 1$. Take $\phi = \phi^E_{ii}$. Then $\|\pi(\phi)\| \leq \|\phi^E_{ii}\| = 1$. From (4.12) we now have

$$\langle \pi(\phi), \chi_D \rangle = \langle \pi(\chi_{\overline{D}}), \phi^E_{ii} \rangle = \overline{\chi_D(x_i(E))} = 1.$$

So $\pi(\phi)e^D$ has norm ≤ 1 and trace $= 1$, thus $\pi(\phi)e^D = e^D$. This holds for all D, so $\pi(\phi) = I = \pi(1)$. So $\pi(1 - \phi) = 0$, but $1 \neq \phi$.

(2) $\implies$ (3): For this we will need the following elementary fact:

$$z \in \mathbf{C}, \quad |z| \leq 1, \quad z \neq 1 \implies \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} z^k = 0.$$

Now suppose (2) holds and let $\mathcal{M} = \{\pi(\phi) \mid \phi \in A_\omega(G)\}''$. So from (4.12)

$$\langle \mathcal{M}, f \rangle = 0 \iff \pi(f) = 0.$$

Now for $D \in \widehat{G}$, $n \in \mathbf{N}$ look at the following element in $\mathcal{M}$:

$$b(D, n) = \frac{1}{n} \sum_{k=1}^{n} \pi(\chi_D)^k = \frac{1}{n} \sum_{k=1}^{n} \sum_{i,E} \chi_D(x_i(E))^k e^E_{ii}.$$

So $\|b(D, n)\| \leq 1$ and

$$\lim_{n\to\infty} \langle b(D, n), f \rangle = \Big\langle \sum_{\chi_D(x_i(E))=1} e^E_{ii}, f \Big\rangle \quad \text{for all } f \in A_\omega(G).$$

So the following weak limit exists in $\mathcal{M}$:

$$b(D) = \lim_{n\to\infty} b(D, n) = \sum_{\chi_D(x_i(E))=1} e^E_{ii}.$$

But since $\chi_D(x_i(E)) = 1 \iff x_i(E) \in \ker D$, it follows from (2) that

$$e^1 = \inf_{D\in\widehat{G}} b(D) \in \mathcal{M}.$$

Thus

$$\begin{aligned}
&\pi(f)=0 \implies \pi(f^{\#}f)=0 \implies f^{\#}f\in\mathcal{M}^{\perp}\\
\implies\quad &0=\langle e^1, f^{\#}f\rangle=\langle\delta_\omega(e^1), f^{\#}\otimes f\rangle\\
&\quad=\langle\delta(e^1), f^{\#}\otimes f\rangle=\int|\langle\lambda(x),f\rangle|^2\,dx\\
\implies\quad &f=0.
\end{aligned}$$

(2) $\implies$ (5): Note that $b(D,n)$ and $b(D)$ as defined above in fact define states on $A^\circ_\omega(G)$:

$$\langle b(D,n), f^{\#}f\rangle\ \geq 0 \quad\text{and } \langle b(D,n),1\rangle=1$$

and hence extend to states on $C^\circ(G)$. This is true also for all finite products $\prod b(D_i,n_i)$ and a standard approximation argument shows that

$$\langle\prod_{D\in\widehat{G}} b(D), f\rangle=\langle e^1,f\rangle \quad\text{for all } f\in C^\circ(G).$$

Thus if $f\in C^\circ(G)^+$ vanishes on $\{x_i(D)\}$, then

$$\begin{aligned}
\langle\prod_{i=1}^m b(D_i,n_i), f\rangle=0 &\implies \langle\prod_{i=1}^m b(D_i),f\rangle=0\\
&\implies\langle e^1,f\rangle=0\implies f=0.
\end{aligned}$$

So (2) $\implies$ (5). □

Example 4.16. For abelian groups we also have (2) $\Longleftarrow$ (5), but that is not true in general:

Let G be a compact connected Lie group and H a maximal abelian subgroup. Let $\theta:\widehat{H}\to H$ be a symplectic bicharacter. Form the corresponding ergodic system $(\mathcal{B},\beta,H)$ as mentioned earlier, and let $(\mathcal{A},\rho,G)$ be the induced system. For each $D\in\widehat{G}$ pick a basis $\{\xi_i^D\}$ such that

$$D_h\xi_i^D=\alpha_i^D(h)\xi_i^D\quad\text{for some } \alpha_i^D\in\widehat{H}.$$

We leave it to the reader to show that the bicharacter corresponding to ρ is given by

$$\pi=\sum_{i,D} e_{ii}^D\otimes\lambda(\theta(\alpha_i^D)).$$

Hence $x_i(D)=\theta(\alpha_i^D)$. Since every element in G is conjugate to an element in H, the map $H\to G^\circ$ is onto. Thus if θ is 1–1 then $\{x_i(D)\}$ is dense in G°, but $\mathcal{A}$ is not simple.

Remark 4.17. The following should give an other indication that there must be some difference for non-abelian groups. If G is abelian then ρ_x is inner $\iff$ $x=\pi(\alpha)$, c.f. [10, 5.7]. So if $\mathcal{A}$ is a factor, the elements x for which ρ_x is inner is dense in G. But if G is a simple compact group it is also algebraically simple, so unless $\mathcal{A}=\mathcal{B}(D)$ it is only $x=e$ which will give ρ_x inner. (c.f. [2] and [13], for which we are indebted to P. de la Harpe.)

Final remarks. We have seen here that the program of classifying the ergodic actions of a non-abelian compact group can be performed along the same line as for abelian groups, we even get the existence of a map from $\widehat{G}$ into G having similar properties. The disappointing thing is that it is not so clear what kind of group elements one will get as solutions.

The encouraging fact is that the seemingly innocent question which started S. Albeverio and R. Høegh-Krohn has led to connections with other parts of mathematics, in particular the quantum groups in [3]. One might hope that some of the many solutions to (QYBE) would be helpful also in our situation, but so far none of these seem to correspond to any new ergodic W^*-dynamical systems. For instance, the solutions described in [3, 6.2] correspond to systems induced from a dual cocycle on a maximal abelian subgroup, the construction being as in Example 4.16. Proposition 4.4 tells us that the solutions of (QYBE) which we are interested in have to be rather special. If we are aiming at solving the conjecture in part 1, we will in addition be looking for solutions for which the "matrix-coeficients" π_{ij} act irreducibly on $\mathbf{C}^d$. So whether the conjecture is true or not, it should have interesting applications to the theory of (QYBE).

We have here only treated the case with full multiplicity. Some parts of the development also go through in the general case, the main difficulty is that ω and π then just will be partial isometries. One should note that the induction construction in Theorem 1.2 preserves full multiplicity, so also for ergodic actions which are not of full multiplicity the factor case is the generic one.

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