QUANTIZATIONS OF GROUPS AND HOMOGENEOUS SPACES

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1. Introduction

The basic idea of non-commutative geometry is to identify a geometric object X (e.g. a space, group) with an algebra $A(X)$ of functions on X and then one gets a "non-commutative X", a "deformation" or "quantization" of X by giving $A(X)$ a new product. When X has some extra structure, one wants as much as possible of that structure to be preserved: If X is an algebraic variety $A(X)$ should consist of algebraic functions over X; if X is a differential manifold $A(X)$ should consist of smooth functions; if X is a group $A(X)$ should be a group-like object (e.g. a quantum group) etc. This article will be a guided tour through parts of C^*-deformation theory with emphasis on some common features. The deformations we shall study are obtained from an action of an abelian group on the corresponding space X, and the structure we shall mainly be concerned with is *equivariance*, that is we also have a locally compact Hausdorff group G acting on the space X. We define

$$\alpha_y(f)(x) = f(y^{-1}x) \text{ for } f \in C_0(X), \tag{1.1}$$

we want $A(X)$ to be α_y-invariant and that all α_y should be homomorphisms of $A(X)$ for the new product. In Section 9 we shall also look at some quantum group deformations of G which are not equivariant, but they are also obtained from an abelian subgroup.

When we talk about deformations of a locally compact Hausdorff space X, we take the view of Rieffel: $A(X)$ should be a dense subalgebra of $C_0(X)$, a deformation quantization is then a family $*_\lambda$ of associative products on $A(X)$ such that in some norm $f *_\lambda g \to fg$ (ordinary pointwise product) as $\lambda \to 0$. (For a precise definition see [Rf1, Section 1].) The deformations we are concerned with can shortly be described as follows: The products $*_\lambda$ are obtained from an action of some abelian group H on X and the new product is defined by doing some Fourier analysis on H. Often the difficult part is to find the "right" space $A(X)$ of functions, we must be able to do Fourier analysis (with respect to H) for $f \in A(X)$, in particular we need the Fourier inversion formula.

Given a particular space X, it is not always clear what it means when we say that two quantizations are the same, since different authors may use different algebras $A(X)$. We shall say they are the same if the generating C^*-algebras are isomorphic and all other relevant information is preserved.

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2. Equivariant deformations of $C_0(G)$ and dual cocycles.

The first case we want to consider is $X = G$ with G acting by left multiplication. Since the deformations we are going to consider are obtained by doing Fourier analysis on G, it is natural to use $A(G)$ = the Fourier algebra as the dense subspace of $C_0(G)$, this is an algebra under the pointwise product. $A(G) \cong \mathcal{L}(G)_*$ – the predual of the left von Neumann algebra of G over $L^2(G)$, and with the corresponding norm $A(G)$ is a Banach *-algebra. (The involution is $f^*(x) = \overline{f(x)}$.) In fact, the map $L_x \mapsto L_x \otimes L_x$ for $x \in G$ extends to a *-homomorphism $\delta_G : \mathcal{L}(G) \mapsto \mathcal{L}(G) \otimes \mathcal{L}(G)$, and the product in $A(G)$ is given by

$$\langle a, fg\rangle = \langle \delta_G(a), f \otimes g\rangle \text{for } a \in \mathcal{L}(G). \tag{2.1}$$

$A(G)$ is invariant under the left actions α_y, and it is not so difficult to show that a product $*$ on $A(G)$ making it a Banach *-algebra and such that each α_y is a homomorphism must be of the form

$$\langle a, f * g\rangle = \langle \delta_G(a)\omega^*, f \otimes g\rangle \quad \text{for} \quad a \in \mathcal{L}(G) \tag{2.2}$$

where $\omega \in \mathcal{L}(G) \otimes \mathcal{L}(G)$ satisfies the *dual cocycle identity*

$$(\omega \otimes 1)\, \delta_G \otimes i(\omega) = (1 \otimes \omega)\, i \otimes \delta_G(\omega) \quad \text{in} \quad \mathcal{L}(G) \otimes \mathcal{L}(G) \otimes \mathcal{L}(G). \tag{2.3}$$

We shall here always assume that ω is *unitary*. In [LR1] it is shown that there is a natural notion of *normal* dual cocycles and there is a natural regular representation of $A(G)$ with the product (2.2). The C^*-closure thus gives a C^*-algebra $C_r^*(\widehat{G}, \omega)$ on which the left action α_y extends to an automorphism, this way $C_r^*(\widehat{G}, \omega)$ can be considered to be an equivariant quantization of $C_0(G)$.

The idea of considering a subalgebra of $C_0(G)$ with an equivariant product (different from the pointwise) as a quantization of the group G goes back to [AH], they showed that a classification of such products is possible for G compact and abelian – a complete solution for this case is described in [OPT]. The corresponding problem for G compact, non-abelian – i.e. to classify all unitaries $\omega \in \mathcal{L}(G) \otimes \mathcal{L}(G)$ satisfying (2.3) – has turned out to be a much more difficult task. Much of the theory and constructions can be carried over to compact non-abelian groups, this has been demonstrated in [HLS], [L1], [Wa1-4]. However, it is only for a few low-dimensional classical groups like SU(2) and SU(3) that Wassermann has obtained a complete description of the possible equivariant deformations.

3. Equivariant deformations of abelian groups

Let us first look at the general case with G abelian and locally compact. Then $A(G) \cong L^1(\widehat{G})$, $\mathcal{L}(G) \cong L^\infty(\widehat{G})$ and $\omega \in L^\infty(\widehat{G} \times \widehat{G})$ satisfies (2.3) if and only if

$$\omega(r, s)\omega(r + s, t) = \omega(s, t)\omega(r, s + t) \quad \text{for } r, s, t, \in \widehat{G} \tag{3.1}$$

i.e. ω is an ordinary Borel 2-cocycle on $\widehat{G}$. (This explains the name dual cocycle above.) If we assume that ω is normalized, then the product on $L^1(\widehat{G})$ corresponding to (2.2) is given by

$$f * g(t) = \int \overline{\omega(s, t-s)} f(s) g(t-s)\, ds, \tag{3.2}$$

the involution by $f^*(t) = \overline{f(-t)}$ and the left action of G correspond to

$$\alpha_x(f)(s) = \langle x, s\rangle f(s) \quad \text{for } x \in G,\ s \in \widehat{G}. \tag{3.3}$$

The formula (3.2), but now with $f \in L^1(\widehat{G})$ and $g \in L^2(\widehat{G})$ defines the regular representation. By taking the C^*-closure we obtain a C^*-algebra denoted $C_r^*(\widehat{G}, \omega)$ and since G is abelian we may drop the $_r$. The automorphism α_x extend to this algebra, and we have the desired equivariant deformation of $C_0(G)$. The structure of these algebras is determined by:

3.1 Theorem. *The C^*-algebra $C^*(\widehat{G}, \omega)$ is simple if and only if ω is totally skew, i.e. $\omega(s,t) = \omega(t,s)$ for all $t \in \widehat{G}$ only if $s = e$.*

The first proof of this in full generality seems to be [G, Proposition 32], but special cases has been treated earlier in [EH, Sl].

The classification of these equivariant deformations is therefore the same as finding all cocycles on $\widehat{G}$ up to equivalence, i.e. to determine $H^2(\widehat{G})$. For most locally compact abelian groups Kleppner showed in [K] that a cocycle on $\widehat{G}$ is equivalent to one of the form:

$$\omega(r, s) = \langle \pi(r), s\rangle \tag{3.4}$$

where π is a continuous homomorphism $\widehat{G} \mapsto G$ satisfying $\langle \pi(s), s\rangle \equiv 1$. The condition in Theorem 3.1 is then the same as the map $r \mapsto \pi(2r)$ being injective. In many cases one can even write

$$\omega(r, s) = \langle \rho(\widetilde{r}), s\rangle \langle \rho(-\widetilde{s}), r\rangle \tag{3.5}$$

where $\rho : \widehat{K} \mapsto G$ is a continuous homomorphism, K is a closed subgroup of G and $\widetilde{s}$ is the image of $s \in \widehat{G}$ in $\widehat{K} \cong \widehat{G}/K^{\perp}$. Then $\pi(s) = \rho(\widetilde{s}) - \rho^t(s)$, where $\rho^t : \widehat{G} \mapsto K$ is the transpose of ρ. The typical examples are $\mathbf{R}^2$, $\mathbf{R} \times \mathbf{T}$ or $\mathbf{T}^2$ with $K = \{(0, x)\}$ and $\rho(s) = (\lambda s, 0)$ for some number λ. If $G = \mathbf{T}^2$, the C^*-algebras obtained are denoted $C^*(\mathbf{Z}^2, \lambda)$, these are the famous rotation algebras which also here appear as important examples.

If ω is of the form (3.5), we write $C^*(\widehat{G}, \rho)$ instead of $C^*(\widehat{G}, \omega)$. We have the following expression for ω as an element of $\mathcal{L}(G) \otimes \mathcal{L}(G)$:

$$\omega = \int_{\widehat{K} \times \widehat{K}} L_{\rho(s)} dP_t \otimes L_{\rho(-t)} dP_s \tag{3.6}$$

where $\{P_s\}$ is the projection valued measure on $L^2(G)$ corresponding to the regular representation of G restricted to K, so over $L^2(G)$

$$L_k = \int_{\widehat{K}} \langle k, s\rangle dP_s \text{ for } \ k \in K. \tag{3.7}$$

In order to compute the product corresponding to (3.2) of two functions $f, g \in A(G)$, we have to take their Fourier transforms, convolve using (3.2) and then take the inverse Fourier transform to get $f * g$ as a function on G. If ω is of the form (3.6) it is not so difficult to see that only a partial Fourier transform is necessary: For $f \in A(G)$ take

$$\widehat{f}(x,s) = \int_K f(xk)\overline{\langle k, s\rangle}\, dk. \tag{3.8}$$

Then use (3.2) and (3.5) or (2.2) and (3.6) together with the Poisson formula [HR, 31.46(e)] to show that

$$f * g(x) = \int_{\widehat{K}} \int_{\widehat{K}} \widehat{f}(x\rho(t), s)\widehat{g}(x\rho(-s), t)\, ds\, dt. \tag{3.9}$$

This formula will turn out useful when we next want to look at a non-abelian group G. Note that without loss of generality we may assume that $\ker\rho = \{e\}$, if not – replace K by $K_1 = K \cap (\ker\rho)^{\perp}$ and ρ by the obvious map from $\widehat{K}_1 = K/\ker\rho$.

4. Equivariant deformations of non-abelian groups

We shall now look at how to obtain equivariant deformations of $C_0(G)$ in the general case, i.e. to find unitary solutions of (2.3). So far it is only one known method for this which is called *inflation*: If H is a closed subgroup of G and $\omega \in \mathcal{L}(H) \otimes \mathcal{L}(H)$ is a dual cocycle on H, then there is a natural embedding $\mathrm{Inf} : \mathcal{L}(H) \mapsto \mathcal{L}(G)$ such that $\mathrm{Inf}(L_h^H) = L_h^G$. Then $\mathrm{Inf}\,\omega = \mathrm{Inf} \otimes \mathrm{Inf}(\omega)$ is a dual cocycle on G, for more details we refer to [LR1]. So if H is a closed abelian subgroup of G and ω is a cocycle on $\widehat{H}$, we first consider ω as an element of $\mathcal{L}(H) \otimes \mathcal{L}(H)$ and then $\mathrm{Inf}\,\omega$ is a dual cocycle on G. There are in fact indications that this is the only way to obtain dual cocycles. The conjecture first asked in [HLS, Introduction] whether a compact, simple, non-abelian Lie group can act ergodically on the hyperfinite II_1-factor $\mathcal{R}$ is in fact equivalent to the conjecture that each unitary dual cocycle on the group is inflated from a closed abelian subgroup.

The structure of $C_r^*(\widehat{G}, \mathrm{Inf}\,\omega)$ follows from [LR1, Theorem 3.3]:

4.1 Theorem. *If H is a closed abelian subgroup of G and ω is a dual cocycle on H, define $\mathcal{B} = C_r^*(\widehat{H}, \omega)$ and $\alpha^H : H \mapsto Aut(\mathcal{B})$ as in (3.3). Then $C_r^*(\widehat{G}, Inf\omega)$ is isomorphic to the induced algebra*

$$\left\{ f \in C_b(G, \mathcal{B}) \,\middle|\, \begin{array}{l} f(xh) = \alpha^H_{h^{-1}}(f(x)) \text{ for } x \in G, h \in H, \\ \text{and } xH \mapsto \|f(x)\| \text{ is in } C_0(G/H) \end{array} \right\}.$$

The action of G on $C_r^*(\widehat{G}, Inf\omega)$ correspond to $\alpha_y(f)(x) = f(y^{-1}x)$.

Let us specialize to the case with a dual cocycle inflated from an ω of the form (3.6). If we identify ω and $\mathrm{Inf}\,\omega$, then ω is given by the same formula (3.6) only interpreted over $L^2(G)$, and the corresponding product on $A(G)$ is given by (3.8) and (3.9) – these formulas make perfectly sense also for G non-abelian. Theorem 4.1 can then be rephrased as follows:

4.2 Theorem. *Let K be a closed abelian subgroup of G and $\rho : \widehat{K} \mapsto G$ a continuous homomorphism such that $\rho(s)k = k\rho(s)$ for $s \in \widehat{K}$, $k \in K$. Then ω in (3.5) is a dual cocycle on $H = \overline{\rho(\widehat{K})K}$ and $C_r^*(\widehat{G}, Inf\omega)$ is an induced algebra as described in Theorem 4.1. The C^*-algebra $\mathcal{B} = C_r^*(\widehat{H}, \omega)$ is simple if and only if the map $s \mapsto \pi(2s) = \rho(2\widetilde{s}) - \rho^t(2s)$ in Section 3 is injective, and if so the center of $M(C_r^*(\widehat{G}, Inf\omega)) \cong C_b(G/H)$.*

Proof. The only new thing here is the statement about simplicity, this follows from Theorem 3.1 and the remarks thereafter.

It turns out that $A(G)$ may not be the best subalgebra of $C_0(G)$ in order to construct deformations. It is therefore very useful that the formula (3.9) extend to a larger space of functions on G, in particular we get a nice formulation if the subgroup K is compact.

4.3 Theorem. *Suppose K is a compact, abelian subgroup, and for $f \in C(G)$ and $s \in \widehat{K}$ define $\widehat{f}(x,s)$ as in (3.8). Then*

$$C_{b,1}(G) = \Big\{ f \in C_b(G) : \|f\|_{\infty,1} = \sum_s \sup_x |\widehat{f}(x,s)| < \infty \Big\} \tag{4.1}$$

*is a Banach *-algebra with norm $\|\cdot\|_{\infty,1}$ and operations*

$$f * g(x) = \sum_{s,t} \widehat{f}(x\rho(t), s)\widehat{g}(x\rho(-s), t), \tag{4.2}$$

$$f^*(x) = \overline{f(x)}. \tag{4.3}$$

This Banach *-algebra is very useful in what follows, for instance it is large enough to separate points in G. If G is a Lie group, then $C_b^\infty(G)$ will be a dense *-subalgebra, so it is easy to find the Poisson bracket associated with the deformation. Our use of $C_{b,1}(G)$ as the global algebra within which all deformations take place, shows that a differential structure is not needed for this, the existence of certain abelian subgroups is more important.

A similar method of deformation has been introduced by Rieffel (see [Rf6, 7, 9, 10]). He uses functions in $C^\infty(G)$ (or Schwartz-functions), the product is defined by a formula which formally is the same as (3.9), but now described as an *oscillatory integral*. The advantage of his method is that because the functions involved are smooth, one does not have to worry about the existence of these integrals (the Fourier inversions formula always holds), while with the method we describe one has to show that this actually is the case.

4.4 Remark. Theorem 4.3 is easily generalized to the situation where X is a locally compact space on which there are commuting actions of a compact abelian group K and its dual group $\widehat{K}$, see [L2, Theorem 2.3] for details. The same is true for Rieffel's method, [Rf7] considers deformations of $C_0(X)$ obtained from actions of $\mathbf{R}^d$.

Both methods are *analytic* in nature, however, many deformations appearing in the literature are *algebraic* in the sense that the space $A(X)$ is a space of polynomials on X with a non-commutative multiplication. The problem from a C^*-algebraic point of view is then to show the existence of a *-operation and a faithful representation over a Hilbert space. A very nice example of this is Woronowicz' work on compact quantum groups, see [Wo1-2]. This works well for compact groups like SU(2), but for non-compact groups there is yet no completely satisfactory theory, even for groups like SU(1,1) and SL(2,**R**). We shall come back to these quantizations later in Section 9, just note here that they are *not* equivariant over the underlying group. A word of warning is in its place, the results in [Ta] show that defining a C^*-algebra by generators and relations is a non-trivial task.

It is easy to describe the regular representation of $C_{b,1}(G)$, in fact we could use this as its definition:

4.5 Proposition. *For $f \in C_{b,1}(G,\rho)$ define*

$$\mu(f) = \sum_{s,t} L_{\rho(s)} M(\widehat{f}(\cdot,t)) L_{\rho(-s-t)} P_s = \sum_{s,t} P_t L_{\rho(s)} M(f) L_{\rho(-t)} P_s, \tag{4.5}$$

where M is the representation of $C_b(G)$ given by $M(f)\xi(x) = f(x^{-1})\xi(x)$ over $L^2(G)$. Then μ is a bounded, faithful *-representation of $C_{b,1}(G,\rho)$. It is α-covariant:

$$R_g \mu(f) R_{g^{-1}} = \mu(\alpha_g(f))$$

where R is the right regular representation of G over $L^2(G)$.

This proposition will be our main tool in obtaining C^*-deformations of many spaces, the first application being a third description of $C_r^*(\widehat{G},\rho)$ obtained by combining Propositions 1.17 and 5.2-3 in [LR1]:

4.6 Theorem. *$C_{0,1}(G,\rho) = C_b(G,\rho) \cap C_0(G)$ is a closed *-subalgebra of $C_{b,1}(G,\rho)$ and $\mu(C_{0,1}(G,\rho))$ is dense in $C_r^*(\widehat{G},\rho)$.*

5. Deformations of $C_0(\Gamma\backslash G)$

The left action of G is preserved under our deformations of $C_0(G)$, so it will be easy to obtain the corresponding deformations of $C_0(\Gamma\backslash G)$ when Γ is a closed subgroup of G.

5.1 Theorem. *If Γ is a closed subgroup of G, then $C_{0,1}(\Gamma\backslash G,\rho) = C_0(\Gamma\backslash G) \cap C_{b,1}(G,\rho)$ is a closed *-subalgebra of $C_{b,1}(G,\rho)$. By taking the C^*-closure of $\mu(C_{0,1}(\Gamma\backslash G,\rho))$ we get a deformation of $C_0(\Gamma\backslash G)$ denoted by $C_r^*(\Gamma\backslash\widehat{G},\rho)$.*

The structure of such an algebra is easy to describe by using Theorem 4.1, at least when Γ is compact:

5.2 Theorem. *If Γ is a compact subgroup of G, then with $(\mathcal{B},H,\alpha^H)$ as in Theorem 4.1*

$$C_r^*(\Gamma\backslash\widehat{G},\rho) \cong \left\{ f \in C_b(\Gamma\backslash G,\mathcal{B}) \,\middle|\, \begin{array}{l} f(xh) = \alpha^H_{h^{-1}}(f(x)) \text{ for } x \in G, h \in H, \\ \text{and } \Gamma x H \mapsto \|f(x)\| \text{ is in } C_0(\Gamma\backslash G/H) \end{array} \right\}.$$

An illustrating example is obtained by taking $G = U_2(\mathbf{C})$,

$$\Gamma = K = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} : |t| = 1 \right\} \cong \mathbf{T} \text{and } \rho(n) = \begin{pmatrix} q^n & 0 \\ 0 & 1 \end{pmatrix} \text{ for } n \in \widehat{K} = \mathbf{Z}.$$

Then $\Gamma\backslash G$ is isomorphic to the 3-sphere S^3 and $C_r^*(\Gamma\backslash\widehat{G},\rho)$ is the same as the non-commutative spheres studied in more detail in [M,NO].

6. Equivariant deformations of $C_0(G/\Gamma)$

Let us here first illustrate the problem of choosing an appropriate space of functions. We have earlier seen that $A(G)$ is a "good" space when we want to deform $C_0(G)$. A natural candidate for deforming $C_0(G/\Gamma)$ is therefore $B(G) \cap C_0(G/\Gamma)$ where $B(G) = C^*(G)^*$ is the Fourier-Stieltjes algebra (i.e. the linear span of continuous, positive definite functions on G). However, [LR1, Example 5.4] shows that this algebra is too small since it may not separate points in G/Γ.

Deformations of $C_0(G/\Gamma)$ are not the same as those of $C_0(\Gamma\backslash G)$, since we want to preserve the left action of G. We want to apply Theorem 4.3, but $C_{0,1}(G/\Gamma,\rho) = C_0(G/\Gamma) \cap C_{b,1}(G,\rho)$ is not in general a subalgebra of $C_{b,1}(G,\rho)$. If we define β by $\beta_x(f)(y) = f(yx)$, then a sufficient condition is that β_x is a *-automorphism of $C_{b,1}(G,\rho)$ and therefore also of $C_r^*(\widehat{G},\rho)$ for all $x \in \Gamma$. For many examples this condition can be described as follows:

6.1 Proposition. *Suppose $xk = kx$ for all $k \in K$, then β_x is a *-automorphism of $C_{b,1}(G,\rho)$ if and only if*

$$B_x(s) = x\rho(s)x^{-1}\rho(-s) \in K \text{ for all } s \in \widehat{K} \text{ and} \tag{6.1}$$

$$\langle B_x(s), t\rangle = \langle B_x(t), s\rangle \text{ for all } s,t \in \widehat{K}. \tag{6.2}$$

So if (6.1-2) hold for all elements of a subgroup Γ of G, $C_{0,1}(G/\Gamma,\rho) = C_0(G/\Gamma) \cap C_{b,1}(G,\rho)$ is an α-invariant subalgebra of $C_{b,1}(G,\rho)$. Via the regular representation in Proposition 4.5 we get a C^*-algebra $C_r^*(\widehat{G}/\Gamma,\rho)$ which is an equivariant deformation of $C_0(G/\Gamma)$. We summarize this in

6.2 Theorem. *Under the assumptions*

(S1) K is a compact abelian subgroup of G and $\rho : \widehat{K} \mapsto G$ is a homomorphism such that each $\rho(s)$ commutes with K and

(S2) Γ is a closed subgroup of G such that each $x \in \Gamma$ commutes with K and satisfies

$$B_x(s) = x\rho(s)x^{-1}\rho(-s) \in K \text{ for all } s \in \widehat{K} \text{ and}$$

$$\langle B_x(s), t\rangle = \langle B_x(t), s\rangle \text{ for all } s,t \in \widehat{K},$$

we have that $\beta_x(f)(y) = f(yx)$ defines a *-automorphism of $C_{b,1}(G,\rho)$ for all $x \in \Gamma$, and $C_{0,1}(G/\Gamma,\rho) = C_0(G/\Gamma) \cap C_{b,1}(G,\rho)$ is a closed *-subalgebra. By taking the C^*-closure of $\mu(C_{0,1}(G/\Gamma,\rho))$ we obtain a deformation of $C_0(G/\Gamma)$ denoted $C_r^*(\widehat{G}/\Gamma,\rho)$. The left translations α_y defined by $\alpha_y(f)(x) = f(y^{-1}x)$ extend to *-automorphisms of $C_r^*(\widehat{G}/\Gamma,\rho)$.

The C^*-algebras described in Theorem 6.2 can be thought of as generalized fixed point algebras as follows:

6.3 Theorem. *For $x \in \Gamma$, β_x extend to a *-automorphism of $C_r^*(\widehat{G},\rho)$. Then $C_r^*(\widehat{G}/\Gamma,\rho)$ is a generalized fixed point algebra as defined in [Rf4b] of $C_r^*(\widehat{G},\rho)$ under the action of $\{\beta_x | x \in \Gamma\}$. In particular $C_r^*(\widehat{G}/\Gamma,\rho)$ is Morita equivalent to the crossed product $C_r^*(\widehat{G},\rho) \times_\beta \Gamma$.*

This together with known results about crossed products can be used to determine when these C^*-algebras are simple, this is Rieffel's approach in [Rf1].

However, the structure of $C_r^*(\widehat{G}/\Gamma,\rho)$ can be described directly without using crossed products. One reason for this is that we can work with concrete functions in $C_{0,1}(G/\Gamma,\rho)$ and by carefully choosing these, the following is obtained in [LR2, Theorem 4.15]:

6.4 Theorem. *Suppose G,ρ,Γ satisfy (S1)-(S2) and*
(S3) G/Γ is compact and $\Gamma\cap K=\{e\}$.
Then there are closed subgroups K_ρ of K and G_ρ of G such that $K_\rho\Gamma$ is a normal subgroup of G_ρ and $G_\rho/K_\rho\Gamma$ is a compact abelian group. The center of $C_r^*(\widehat{G}/\Gamma,\rho)$ is isomorphic to $C(G/G_\rho)$ and $C_r^*(\widehat{G}/\Gamma,\rho)$ is simple if and only if $G=G_\rho$.

6.5 Remark The Heisenberg manifolds in [Rf1] are special cases of this theorem. If we take $\Gamma=\{e\}$, it is the same as Theorem 4.2 with G compact, so it also covers the higher dimensional non-commutative tori. The description of K_ρ and G_ρ is in general complicated, one always has

$$G_\rho\subset\{\rho(2t)K\Gamma\mid t\in\widehat{K}\}^{-}. \tag{6.3}$$

Let us just note the two special cases:

If $\rho(2t)\in K\Gamma\implies t=e$, then $K=K_\rho$ and equality hold in (6.3), this is used in the discussion of Heisenberg manifolds in Section 8.

If $\Gamma=\{e\}$ and $\rho(t)\in K$ for all t, then $K_\rho=\{e\}$ and $G_\rho=\{\pi(2t)\mid t\in\widehat{K}\}^{-}$ where π is as in Theorem 4.2. This covers the non-commutative tori, for instance by taking $G=K$.

Theorem 6.4 gives a new class of simple C^*-algebras which also are equivariant deformations of well-known homogeneous spaces. It is easy to see that $2n+1$–dimensional Heisenberg manifolds are examples of this. However, it seems like it is only for some nilmanifolds one will get simple C^*-algebras this way, as one has

6.6 Theorem. *If the assumptions in Theorem 6.4 hold and $G=G_\rho$, then*

(i) if $H=\overline{\rho(\widehat{K})K}$ is a maximal abelian subgroup, then G is a 3-step nilpotent group,

(ii) if Γ is discrete and G is connected, then G is a 2-step nilpotent group.

6.7 Berezin quantizations. One kind of deformations which we shall not discuss here are those of Berezin in [Bz], but we mention them since they also are equivariant, Rădulescu has in [Ra2] shown that Berezin deformations of the upper half plane can be made equivariant under the action of PSL(2,**R**).

7. Traces on $C_r^*(\widehat{G}/\Gamma,\rho)$

To determine the K-theory or other invariants of a C^*-algebra it is important to know its traces, and we shall see that for the C^*-algebras described by Theorem 6.4 there is a nice description. First it is natural to look at the map $E:C(G/\Gamma)\mapsto C(G/G_\rho)$ given by

$$E(f)(x)=\int_{G_\rho/\Gamma}f(xy)\,dy. \tag{7.1}$$

Here one should note that Γ may not be normal in G_ρ, but from Theorem 6.4 we have that G_ρ/Γ has a G_ρ-invariant measure given by

$$\int_{G_\rho/\Gamma} f(y)\,dy = \int_{G_\rho/K_\rho\Gamma}\int_{K_\rho} f(zk)\,dkdz. \tag{7.2}$$

Using Theorem 6.4 the following properties of E is proved in [L3]:

7.1 Theorem. *E has an extension to a conditional expectation $C_r^*(\widehat{G}/\Gamma,\rho) \mapsto C(G/G_\rho)$ given by*

$$E(a) = \int_{G_\rho/K_\rho\Gamma}\int_{K_\rho} \beta_{xk}(a)\,dkdx \tag{7.3}$$

satisfying $\alpha_x \circ E = E \circ \alpha_x$ and $E(ab) = E(ba)$. There is a 1-1 correspondence between normalized traces τ on $C_r^*(\widehat{G}/\Gamma,\rho)$ and probability measures ν on G/G_ρ given by

$$\tau(a) = \nu(E(a)). \tag{7.4}$$

τ is faithful if and only if ν is.

8. Non-commutative Heisenberg manifolds

We shall look at the Heisenberg manifolds somewhat closer for several reasons, [Rf1] is the first example of simple C^*-algebras which are equivariant deformations of homogeneous spaces G/Γ with Γ not normal. One should also note that the deformations are not of the form described in Remark 4.4 – the group K acts to the right on G/Γ, but $\widehat{K}$ does not. These non-commutative manifolds also illustrate all the difficulties one meets in the general case, and Theorem 6.6 at least partly explains why.

The group is the modified Heisenberg group $G = \mathbf{R}^2 \times \mathbf{T}$ with the product

$$(x_1, y_1, z_1)(x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 z_2 \exp(2\pi i y_1 x_2)). \tag{8.1}$$

(This is not the group of [Rf1], but we will end up with the same homogeneous space G/Γ as there.) We set $K = \{(0,0,t) | \, t \in \mathbf{T}\}$; we shall identify $t \in \mathbf{T}$ with $(0,0,t) \in K$. We define $\rho : \widehat{K} = \mathbf{Z} \mapsto G$ by

$$\rho(n) = (\mu n, \nu n, \exp(i\pi\mu\nu n^2)) \quad \text{for some fixed } \mu, \nu \in \mathbf{R}. \tag{8.2}$$

(S1) obviously holds for K, and a direct computation shows that all $x \in G$ satisfies (S2). So $C_r^*(\widehat{G}/\Gamma,\rho)$ can be defined for any closed subgroup Γ of G. The homogeneous manifolds studied in [Rf1] are then obtained by taking the subgroups

$$\Gamma_c = \{(m,n,z) \mid m,n \in \mathbf{Z},\ z^c = 1\}$$

where $c \in \mathbf{Z}$. Then $C_r^*(\widehat{G}/\Gamma_c,\rho) = D^c_{\mu,\nu}$ in the terminology of [Rf1, Ab?]. Note that in this case $C(G/K\Gamma_c) \cong C(\mathbf{T}^2)$ is a commutative C^*-subalgebra, and that the property mentioned in Remark 6.5 holds if and only if μ or ν is irrational. Then we have the following version of [Rf1, Theorem 6.2]:

8.1 Theorem. *$C^*_r(\widehat{G}/\Gamma_c, \rho)$ is simple $\iff$ $G = G_\rho$ $\iff$ $\{\mu, \nu, 1\}$ is independent over* $\mathbf{Q}$.

As just mentioned $G_\rho \supset K\Gamma_c$ if μ or ν is irrational, so $G/G_\rho \cong \mathbf{T}^2/H$ where H is the closed subgroup generated by $(\exp 4\pi i\mu, \exp 4\pi i\nu)$. Therefore Theorem 7.1 is the same as [Ab4, Proposition 3.1].

Abadie has determined much more of the structure of $D^c_{\mu,\nu}$, she has shown that many of the tools used for irrational rotation algebras also are available here. Both the existence of projections similar to Rieffel's projections and the analogue of the Pimsner-Voiculescu embedding into an AF-algebra are shown. Some of the main results of [Ab1-4] can be summarized as:

8.2 Theorem. *The K-theory of $D^c_{\mu,\nu}$ is described by $K_0 = \mathbf{Z}^3 \oplus \mathbf{Z}_c$ and $K_1 = \mathbf{Z}^3$. If μ or ν is irrational, all normalized traces on $D^c_{\mu,\nu}$ induce the same homomorphism on K_0 with image $\mathbf{Z} + 2\mu\mathbf{Z} + 2\nu\mathbf{Z}$. In particular, if at least one of μ, ν, μ' or ν' is irrational, then $D^c_{\mu,\nu} \cong D^c_{\mu',\nu'}$ if and only if (μ, ν) and (μ', ν') belong to the same orbit under the natural action of $GL(2,\mathbf{Z})$ on $\mathbf{T}^2$.*

9. Deformations of G as a quantum group

The equivariant deformations of $C_0(G)$ described in Sections 2-4 are not quantum groups, since the map $a \mapsto \delta_G(a)\omega^*$ in (2.2) dual to the multiplication in $A(G)$ is not a homomorphism. But actions of an abelian group on G can also be used to obtain quantum group deformations of G and we shall here describe some of these. The first observation is that the map $\delta_\omega : a \mapsto \omega\delta_G(a)\omega^*$ is a homomorphism, so if this map is used to define a product on $A(G)$, we will get a quantum group.

9.1 Theorem. *Let ω be a normalized dual cocycle on G. Then $A(G)$ with the product*

$$\langle a, f * g\rangle = \langle \omega\delta_G(a)\omega^*, f \otimes g\rangle \quad \text{for} \quad a \in \mathcal{L}(G) \tag{9.1}$$

and the *-operation $f^*(x) = \overline{f(x)}$ is a Banach *-algebra which we shall denote $A(G, \delta_\omega)$. This is a quantum group quantization of G, i.e. $(\mathcal{L}(G), \delta_\omega, \nu)$ is an involutive Hopf-von Neumann algebra as described in [T]. Here ν is the involution of $\mathcal{L}(G)$ such that $\nu(L_x) = L_{x^{-1}}$ for $x \in G$.

If ω is of the form (3.6) for some abelian subgroup K and a homomorphism $\rho : \widehat{K} \mapsto G$, we write δ_ρ instead of δ_ω. We then want to describe the C^*-algebra generated by $A(G, \delta_\rho)$ in its regular representation (which exists). We state the result in a special case:

9.2 Theorem. *If we take*

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a, b \in \mathbf{C}, a \neq 0 \right\}, \ K = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : |a| = 1 \right\} \cong \mathbf{T} \textit{and}$$

$$\rho(n) = \begin{pmatrix} q^n & 0 \\ 0 & q^{-n} \end{pmatrix},$$

the C^*-algebra generated by $A(G, \delta_\rho)$ is isomorphic to the C^*-algebra of the group

$$G_q = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a, b \in \mathbf{C}, |a| \in q^{\mathbf{Z}} \right\}.$$

The examples in [Rf6] and the *quantum links* of [TV1] can be obtained in a similar way, see [L2, Section 2.4].

Theorem 9.2 is a special case of [L2, Theorem 2.27] where G is a semidirect product $H_1 \times_A H_2$ of two abelian groups, and ω is a cocycle on $\widehat{H}_1$ of the form (3.5). Then the C^*-algebra generated by $A(G, \delta_\omega)$ is isomorphic to the C^*-algebra of a semidirect product $\widehat{H}_1 \times_B \widehat{H}_2$ where B depends on both A and ω. Some particular deformations treated on the Lie algebra level in [Da2] and [Rf4] are special cases of this result for the corresponding groups.

Rieffel's idea is as follows: As a space (not as a group) $G \cong H_1 \times H_2$, so $C_0(G) \cong C^*(\widehat{H}_1 \times \widehat{H}_2)$ which has $L^1(\widehat{H}_1 \times \widehat{H}_2)$ as a dense subalgebra. The product here is now changed by giving $\widehat{H}_1 \times \widehat{H}_2$ a new (non-commutative) group structure. In this way one can say that there is a duality between the two non-abelian groups $H_1 \times_A H_2$ and $\widehat{H}_1 \times_B \widehat{H}_2$. There are other examples of this idea in [Rf5], for instance a Heisenberg group can be dual to a solvable group as in the classical example of Kac and V. G. Palyutkin in [KP]. This construction and some others (e.g. [Da1]) can also be described as in Theorem 9.1, but then ω no longer is in $\mathcal{L}(G) \otimes \mathcal{L}(G)$, it is enough that δ_ω is a comultiplication and maps into $\mathcal{L}(G) \otimes \mathcal{L}(G)$ as in [L2, Theorems 3.1 and 4.4]. The deformations of Kac-algebras by Enock and Vaĭnerman in [EV] uses the same idea, and the earlier work of De Cannière in [C] can also be described in this framework.

9.3 DJW-quantum groups. The methods described above do not cover the important quantum groups defined by Drinfel'd, Jimbo and Woronowicz, in lack of a better name we call them DJW-quantum groups. The quantum groups we have described so far are obtained by deforming the product in $C_0(G)$, while all other structures (such as the product and inverse in G) are unchanged. For the DJW-quantum groups one also have to change the product on G (or rather on the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of the Lie algebra) and the involution is no longer bounded or periodic. However, Sheu has shown in [S1] that $\mathrm{SU}_q(2)$ (and also Podleś' quantum spheres described below) are quantum deformations in Rieffel's sense, it is further shown in [Rf7, Chapter 12] that they can be obtained by an action of $\mathbf{R}^2$ on the corresponding space as described in Remark 4.4. However, the coproduct and the involution are not so easily described this way, and this presentation is therefore not so useful. [S3] explains why these difficulties arise.

Deformations as described in Theorem 9.2 also play a role in the DJW-theory: Now the comultiplication is a homomorphism $\delta_q : \mathcal{U}(\mathfrak{g}) \mapsto \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$ and δ_q can be deformed using a Cartan subgroup exactly as δ_G was in Theorem 9.1. It is shown in [TV3] that any deformation of a reductive Lie algebra satisfying certain natural conditions can be obtained this way via the Cartan subgroup from either δ_G or δ_q, so one obtains triangular QUE-algebras as described in [Dr, Section 10].

There are many aspects of the theory of quantum groups which we have not discussed here, one is the existence of Haar measure in the non-compact case.

9.4 Homogeneous spaces. For the DJW-groups there also is a theory of homogeneous spaces. We describe it for $\mathrm{SU}_q(2)$ as worked out in [P1-2]. The algebra of functions is $\mathcal{P}(G)$ = all polynomials in the matrix functions on SU(2). Although the deformations are not SU(2)-equivariant, they are so under the diagonal subgroup $\cong \mathbf{T}$, so $\mathcal{P}(G)^{\mathbf{T}}$ is a subalgebra which give deformations of $\mathrm{SU}(2)/\mathbf{T} \cong S^2$, i.e. the 2-sphere S^2.

This illustrates a point made in the introduction, given a space X we may get different deformations of $C_0(X)$ depending on what additional structure we want to preserve. One also has $S^2 \cong$ quotient space of $\mathbf{T}^2$ under the action $\alpha : (z, w) \mapsto (z^{-1}, w^{-1})$. Now α is also an automorphism of $C^*(\mathbf{Z}^2, \lambda)$ as defined earlier, so the fixed point algebra $C^*(\mathbf{Z}^2, \lambda)^\alpha$ is a different deformation of $C(S^2)$ – these are the non-commutative spheres of [BEEK]. Neither of these are equivariant, in fact it follows from [Wa3] that there are no nontrivial deformations of $C(\mathrm{SU}(2)/\mathbf{T})$ which are equivariant under the action of SU(2). One also has that SU(2)$\cong S^3$ as topological spaces, but the $\mathrm{SU}_q(2)$ mentioned in 9.3 is different from the non-commutative S^3 described after Theorem 5.2.

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