# TWISTED DUAL-GROUP ALGEBRAS: EQUIVARIANT DEFORMATIONS OF $C_0(G)$

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It is standard practice to deform the multiplication on a group algebra using a 2-cocycle $\sigma$ on the group: the group algebra is spanned by a copy $\{\delta_x : x \in G\}$ of the group $G$, with the multiplication defined by $\delta_x \delta_y = \delta_{xy}$, and the *twisted group algebra* is spanned by a family $\{\delta_x\}$ satisfying $\delta_x\delta_y = \sigma(x,y)\delta_{xy}$. This works for discrete groups, purely algebraically, and for locally compact groups, where one uses either the $L^1$-group algebra $L^1(G)$ or the group $C^*$-algebra $C^*(G)$. Many of the recent constructions of non-commutative spaces have been obtained by recognising the space as the dual of a locally compact group $\widehat{G}$, and then deforming $C_0(G) \cong C^*(\widehat{G})$ into a twisted group algebra $C^*(\widehat{G},\sigma)$: thus, for example, the non-commutative tori are by definition the twisted group algebras $C^*(\mathbf{Z}^n,\sigma)$. Of course, this only works if the underlying group $G$ is abelian. Here we shall describe a method of directly deforming $C_0(G)$ for arbitrary $G$, using the dual cocycles introduced by Wassermann and Landstad in their classification of ergodic actions of compact groups on von Neumann algebras [23, 12]. Our deformations are equivariant, in the sense that $G$ acts by left translation on each of the deformed algebras; they are not quantum groups, although similar techniques can be used to deform $C_0(G)$ into quantum groups [13].

Just as deformations of $C^*(\widehat{G})$ are defined first on $L^1(\widehat{G})$, we begin with the Fourier algebra $A(G) \subset C_0(G)$. If we view $A(G)$ as the predual of the von Neumann algebra $\mathcal{L}(G)$, the usual pointwise multiplication on $A(G)$ can be described in terms of the comultiplication $\delta_G : \mathcal{L}(G) \to \mathcal{L}(G) \otimes \mathcal{L}(G)$ by $\langle T, fg\rangle = \langle \delta_G(T), f \otimes g\rangle$ for $T \in \mathcal{L}(G)$. A *dual cocycle* is a unitary element $\omega$ of the von Neumann algebra tensor product $\mathcal{L}(G) \otimes \mathcal{L}(G)$ such that

$$(\omega \otimes 1)\, \delta_G \otimes i(\omega) = (1 \otimes \omega)\, i \otimes \delta_G(\omega) \quad \text{in } \mathcal{L}(G) \otimes \mathcal{L}(G) \otimes \mathcal{L}(G),$$

and the *twisted Fourier algebra* $A(G,\omega)$ is then the usual Banach space $A(G)$ with multiplication defined by

$$\langle T, f *_\omega g\rangle = \langle \delta_G(T)\omega^*, f \otimes g\rangle \quad \text{for } T \in \mathcal{L}(G).$$

In §1, we explain carefully how, for an abelian group $G$, dual cocycles are the Fourier transforms of ordinary cocycles on $\widehat{G}$. We then discuss the extra normalisation condition we need to ensure that $A(G,\omega)$ is a $^*$-algebra with the usual involution: this is one of the technical problems which arises in extending the basic theory of [23, 12] from compact to locally compact groups. The algebra $A(G,\omega)$ has a regular representation on $L^2(G)$, and the *twisted dual-group algebra* $C^*_r(\widehat{G},\omega)$ is by definition the closure of $A(G,\omega)$ in $B(L^2(G))$. (For technical reasons, we have chosen to work with this algebra rather than the enveloping algebra $C^*(A(G,\omega))$, but we conjecture that they coincide.) The action of $G$ by left translation on $A(G)$ extends to an action $\alpha : G \to \operatorname{Aut} C^*_r(\widehat{G},\omega)$, so that we have constructed equivariant deformations of $C_0(G) = C^*_r(\widehat{G}, 1\otimes 1)$.

The main examples of dual cocycles are the Fourier transforms of ordinary cocycles on abelian groups, and those *inflated* from these dual cocycles on abelian subgroups: if $H$ is a subgroup of $G$, $\mathcal{L}(H)$ embeds in $\mathcal{L}(G)$, and a dual cocycle $\omega$ on $H$ gives a dual cocycle $\operatorname{Inf}_H^G \omega$ on $G$ (Proposition 3.2). (There is some evidence that all dual cocycles arise this way; see the comments at the start of §4 and our Appendix.) Our main theorem says that the twisted dual-group algebra $C^*_r(\widehat{G}, \operatorname{Inf}\omega)$ is then determined by $C^*_r(\widehat{H},\omega)$ and the action $\alpha$ of $H$ by left translation, as an induced $C^*$-algebra:

$$C^*_r(\widehat{G}, \operatorname{Inf}_H^G \omega) \cong \operatorname{Ind}_H^G \left(C^*_r(\widehat{H},\omega),\alpha\right);$$

curiously, this appears to be an interesting result even for abelian groups (see Remark 3.5). The proof of this theorem in §3 requires some basic properties of induced $C^*$-algebras and their representations, which we present in §2. In particular, we show there that if $(\pi, S)$ is a covariant representation of $(B,\beta)$, then there is a natural representation $\widetilde{\pi}$ of $\operatorname{Ind}(B,\beta)$ on the Hilbert space of the induced representation $\operatorname{Ind} S$.

Our twisted dual-group algebras, then, provide a family of deformations of $C_0(G)$. We would like to extend our construction to give deformations of homogeneous spaces $C_0(G/\Gamma)$, at least in examples like those considered by Rieffel [21]. In order to preserve the left action of $G$ on such a deformation, we want $\Gamma$ to act by right translation on the algebras $C^*_r(\widehat{G},\omega)$, and then the fixed-point algebras $C^*_r(\widehat{G},\omega)^\Gamma$ should be deformations of $C_0(G/\Gamma)$. Now there certainly are circumstances – including, encouragingly, those studied in [21] – in which such an action of $\Gamma$ exists (see the end of §4), but in general the fixed-point algebra will be trivial. Rieffel has found one way round this which involves taking a generalised fixed-point subalgebra in the multiplier algebra. We propose an alternative approach, in which we directly enlarge the subalgebra of $C_0(G)$ to include bounded functions on $G$ which are invariant under $\Gamma$. To this end, we work out integral formulas for the multiplication on $A(G, \operatorname{Inf}\omega)$, in the case where $\omega$ is inflated from an abelian subgroup $H$ and the original cocycle on $\widehat{H}$ has an explicit, but common, form. We do this in §4, and then in §5 discuss a particular dense subalgebra $C_{0,1}(G)$ of $C_0(G)$, which can be readily extended to a much larger space of bounded continuous functions on which our formulas still make sense. We plan to decribe deformations of $C_0(G/\Gamma)$ and their properties in a subsequent article.

**Notation.** We denote by $L^G$ and $R^G$, respectively, the left and right regular representations of a locally compact group $G$ on $L^2(G)$; we shall use left Haar measure throughout, so that $R^G$ is defined by $R^G_x(\xi)(y) = \Delta(x)^{1/2}\xi(yx)$, where $\Delta$ is the modular function of $G$. We write $\mathcal{L}(G)$ for the von Neumann algebra $L^G(G)''$, and $\otimes$ for the von Neumann algebra tensor product. Then $\delta_G : \mathcal{L}(G) \to \mathcal{L}(G) \otimes \mathcal{L}(G)$ is the comultiplication on $\mathcal{L}(G)$: on the one hand, if $W_G$ is the unitary operator on $L^2(G) \otimes L^2(G) = L^2(G \times G)$ defined by $W_G\xi(x,y) = \xi(x, x^{-1}y)$, then $\delta_G(T) = W_G(T \otimes 1)W_G^*$, and on the other hand, we can view $\delta_G$ as the unique normal homomorphism such that $\delta_G(L^G_x) = L^G_x \otimes L^G_x$. We write $\Sigma$ for the flip automorphism of $\mathcal{L}(G) \otimes \mathcal{L}(G)$, 1 for the identity of an algebra, and $i$ for the identity mapping between algebras. We shall freely use subscript notation for operators acting on tensor product spaces: thus, for example, if $T \in \mathcal{L}(G) \otimes \mathcal{L}(G)$, then $T_{23}$ stands for $1 \otimes T$, or $T_{13}$ for $\Sigma \otimes i(1 \otimes T) = i \otimes \Sigma(T \otimes 1)$. We use $\langle \cdot, \cdot \rangle$ for the pairing between a von Neumann algebra and its predual, and for that between an abelian group and its dual. If $f : T \mapsto (T\xi|\eta)$ is a vector functional on a von Neumann algebra $M \subset B(\mathcal{H})$ (we sometimes write $f = \omega_{\xi,\eta}$), then $S_f$ denotes the slice map $S_f : N \otimes M \to N$ characterised by

$$(S_f(T)h|k) = (T(h \otimes \xi)|k \otimes \eta) = (T|\omega_{\xi,\eta} \otimes f).$$

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## §1. Dual cocycles and twisted dual-group algebras.

By a cocycle on a locally compact group $G$, we shall mean a Borel 2-cocycle with values in $\mathbf{T}$; that is, a Borel function $\sigma : G \times G \to \mathbf{T}$ satisfying

$$\sigma(r,s)\sigma(r+s,t) = \sigma(s,t)\sigma(r,s+t) \quad \text{for } r,s,t, \in G. \tag{1.1}$$

(We use additive notation because all our cocycles will be on abelian groups or their duals.) The *twisted group algebra* $L^1(G,\sigma)$ is the Banach space $L^1(G)$ equipped with the multiplication and involution

$$\begin{gathered} f *_\sigma g(t) = \int f(s)g(t-s)\sigma(s,t-s)\,ds \\ f^*(t) = \Delta(t)^{-1}\overline{\sigma(t,-t)}\,\overline{f(-t)}. \end{gathered} \tag{1.2}$$

The *twisted group $C^*$-algebra* $C^*(G,\sigma)$ is the enveloping $C^*$-algebra of $L^1(G,\sigma)$. (See [7, p.417] for a discussion of enveloping algebras which does not assume bounded approximate identities, or [16] for an alternative approach.)

**Definition 1.1.** A *dual cocycle* on a locally compact group $G$ is a unitary $\omega \in \mathcal{L}(G) \otimes \mathcal{L}(G)$ satisfying

$$\omega_{12}\, \delta_G \otimes i(\omega) = \omega_{23}\, i \otimes \delta_G(\omega) \text{ in } \mathcal{L}(G) \otimes \mathcal{L}(G) \otimes \mathcal{L}(G). \tag{1.3}$$

To justify this definition, note that if $G$ is abelian, we can view a cocycle $\sigma$ on $\widehat{G}$ as a unitary element of $L^\infty(\widehat{G}) \otimes L^\infty(\widehat{G}) = L^\infty(\widehat{G} \times \widehat{G})$. Since the Fourier transform of $\mathcal{L}(G)$ onto $L^\infty(\widehat{G})$ converts $\delta_G$ into the comultiplication $\alpha_G : L^\infty(\widehat{G}) \to L^\infty(\widehat{G} \times \widehat{G})$ satisfying $\alpha_G(f)(s,t) = f(s+t)$, and the cocycle identity (1.1) for $\sigma$ can be written

$$\sigma_{12}\, \alpha_G \otimes i(\sigma) = \sigma_{23}\, i \otimes \alpha_G(\sigma),$$

the Fourier transform $\omega \in \mathcal{L}(G) \otimes \mathcal{L}(G)$ of $\sigma \in L^\infty(\widehat{G}) \otimes L^\infty(\widehat{G})$ is a unitary operator satisfying (1.3).

The twisted dual-group algebra will be based on a twisted version of the Fourier algebra $A(G)$ [6]. Given a dual cocycle $\omega$ on $G$ and $f, g \in A(G)$, we define $f *_\omega g \in A(G) = \mathcal{L}(G)_*$ by

$$\langle T, f *_\omega g\rangle = \langle \delta_G(T)\omega^*, f \otimes g\rangle \quad \text{for } T \in \mathcal{L}(G); \tag{1.4}$$

since $T \to \delta_G(T)\omega^* = W_G(T \otimes 1)W_G^*\omega^*$ is weakly continuous, the right-hand side is weakly continuous in $T \in \mathcal{L}(G)$, and there is a unique functional $f *_\omega g \in \mathcal{L}(G)_*$ satisfying (1.4). To motivate (1.4), suppose $G$ is abelian, $\omega$ is the Fourier transform of a cocycle $\sigma$ on $\widehat{G}$, and $f, g$ are the (inverse) Fourier transforms of $h, k \in L^1(\widehat{G})$ – so, for example, $f(x) = \mathcal{F}h(x) = \int \langle x, s\rangle h(s)\, ds$. Then

$$f *_\omega g(x) = \langle L_x, f *_\omega g\rangle = \big\langle \delta_G(L_x)\omega^*, \widehat{h} \otimes \widehat{k}\big\rangle.$$

Under the Fourier transform $L_x$ goes to the function $f_x : t \mapsto \overline{\langle x, t\rangle}$, and $\delta_G(L_x)\omega^*$ to the function $(s,t) \mapsto \overline{\langle x, s+t\rangle\, \sigma(s,t)}$. Thus

$$\begin{aligned}
\langle L_x, f *_\omega g\rangle &= \iint \overline{\langle x, s+t\rangle\, \sigma(s,t)}\, h(s)k(t)\, ds\, dt \\
&= \int \overline{\langle x, t\rangle}\, \Big(\int \overline{\sigma(s, t-s)}\, h(s)k(t-s)\, ds\Big)\, dt \\
&= \langle f_x, h *_{\overline{\sigma}} k\rangle.
\end{aligned}$$

Since we know that $\mathcal{F}$ is an isometric Banach space isomorphism of $L^1(\widehat{G})$ onto $A(G)$, this proves that the Fourier transform induces an isometric Banach algebra isomorphism of $L^1(\widehat{G}, \overline{\sigma})$ onto $(A(G), \omega)$.

**Lemma 1.2.** *For an arbitrary locally compact group $G$, the Banach space $A(G)$ is a Banach algebra with respect to the multiplication $*_\omega$ defined by (1.4); we denote this Banach algebra by $A(G,\omega)$.*

**Proof.** The various distributive laws seem to be clear, so we have to check associativity and submultiplicativity of the norm. But the first follows from the coassociativity of $\delta_G$ and the cocycle identity (1.3):

$$\begin{aligned}(f *_\omega g) *_\omega h(x) &= \big\langle \delta_G \otimes i(\delta_G(L_x)\omega^*)\omega_{12}^*, f \otimes g \otimes h\big\rangle \\ &= \big\langle \delta_G \otimes i(\delta_G(L_x))(\omega_{12}\, \delta_G \otimes i(\omega))^*, f \otimes g \otimes h\big\rangle \\ &= \big\langle i \otimes \delta_G(\delta_G(L_x))(\omega_{23}\, i \otimes \delta_G(\omega))^*, f \otimes g \otimes h\big\rangle \\ &= \big\langle i \otimes \delta_G(\delta_G(L_x)\omega^*)\omega_{23}^*, f \otimes g \otimes h\big\rangle \\ &= f *_\omega (g *_\omega h)(x),\end{aligned}$$

and the second is easy. (The comultiplication $\delta_G$ is isometric, $\|\omega\| = 1$ and $\|f \otimes g\| = \|f\|\,\|g\|$. )

To define the involution on $A(G,\omega)$ analogous to that in (1.2), we would need a noncommutative (variable-free) analogue of the factor $\sigma(t,-t)$. The function $t \mapsto \sigma(t,-t)$ can be recovered from a $\sigma$-representation $V : \widehat{G} \to \mathcal{H}$ by comparing $V_t$ with $V_{-t}$:

$$V_t V_{-t} = \sigma(t,-t)V_{t-t} = \sigma(t,-t)1.$$

If we view $V$ as the element $t \mapsto V_t$ of $B(\mathcal{H}) \otimes L^\infty(\widehat{G}) \cong L^\infty(\widehat{G}, B(\mathcal{H}))$, then $t \mapsto V_{-t}$ is $i \otimes \mu(V)$, where $\mu$ is the homomorphism on $L^\infty(\widehat{G})$ given by $\mu(f)(t) = f(-t)$, and then $u : t \mapsto \sigma(t,-t)$ is characterised by $i \otimes \mu(V)V = 1 \otimes u$. The Fourier transform of $\mu$ is the linear map $\nu : \mathcal{L}(G) \to \mathcal{L}(G)$ such that $\nu(L_x) = L_{x^{-1}}$; this always makes sense – it can be defined spatially by $\big(\nu(T)h|k\big) = \big(T\overline{k}|\overline{h}\big)$ for $h, k \in L^2(G)$ – but is anti-multiplicative, and hence is not a homomorphism unless $G$ is abelian. Thus it is not obvious – and in general false – that the linear map $i \otimes \nu : B(\mathcal{H}) \odot \mathcal{L}(G) \to B(\mathcal{H}) \otimes \mathcal{L}(G)$ extends to the von Neumann algebra $B(\mathcal{H}) \otimes \mathcal{L}(G)$; even if it does, it is not clear what algebraic properties the map $i \otimes \nu$ will have. (In both [12] and [23], some more detailed arguments might have been helpful.)

We can dodge the factor $\sigma(t,-t)$ by normalising the cocycle $\sigma$ so that $\sigma(t,-t) = 1$: every cocycle is equivalent to a normalised one. Equivalently, $\sigma$ is normalised if any $\sigma$-representation $V$ satisfies $i \otimes \mu(V) = V^*$. The corresponding property of a dual cocycle would be $i \otimes \nu(V) = V^*$, which we can interpret, without direct reference to $i \otimes \nu$, as

$$\big(V(h \otimes \overline{\eta})\big|k \otimes \overline{\xi}\big) = \big(V^*(h \otimes \xi)\big|k \otimes \eta\big) \quad \text{for } h, k \in \mathcal{H},\ \xi, \eta \in L^2(G). \tag{1.5}$$

We shall now define $\omega$-corepresentations, and declare $\omega$ to be normalised if there is an $\omega$-representation satisfying (1.5). We do not know if every dual cocycle is equivalent to a normalised one, although [23, Theorem 6] says this is true when $G$ is compact.

**Definition 1.3.** If $\omega$ is a dual cocycle on $G$, then an *$\omega$-corepresentation* of $G$ on $\mathcal{H}$ is a unitary operator $V \in B(\mathcal{H}) \otimes \mathcal{L}(G)$ such that

$$V_{12}V_{13} = i \otimes \delta_G(V)\omega_{23}^* \quad \text{in } B(\mathcal{H}) \otimes \mathcal{L}(G) \otimes \mathcal{L}(G); \tag{1.6}$$

when $\omega = 1 \otimes 1$, we say $V$ is a corepresentation.

**Remark 1.4.** Since $\Sigma \circ \delta_G = \delta_G$, a unitary $V$ in $B(\mathcal{H}) \otimes \mathcal{L}(G)$ is an $\omega$-corepresentation if and only if

$$V_{12}^* V_{13}^* = \Sigma(\omega)_{23}\, i \otimes \delta_G(V^*). \tag{1.7}$$

Again, we look at the case of abelian $G$ for motivation. A $\overline{\sigma}$-representation of $\widehat{G}$ is a Borel homomorphism $U : G \to U(\mathcal{H})$ such that

$$U_s U_t = \overline{\sigma(s,t)}\, U_{s+t} \quad \text{for } s, t \in \widehat{G}. \tag{1.8}$$

If we view $U$ as the unitary element $t \mapsto U_t$ of $L^\infty(\widehat{G}, B(\mathcal{H})) \cong B(\mathcal{H}) \otimes L^\infty(\widehat{G})$, then $B(\mathcal{H}) \otimes L^\infty(\widehat{G}) \otimes L^\infty(\widehat{G}) \cong L^\infty\big(\widehat{G} \times \widehat{G}, B(\mathcal{H})\big)$, and (1.8) is equivalent to

$$U_{12}U_{13} = \overline{\sigma}_{23}\, i \otimes \alpha_G(U).$$

Since $\omega$ is the Fourier transform of $\sigma$, the Fourier transform $V$ of $U$ satisfies (1.6). In the abelian case, $\sigma_{23}$ is central, and it does not matter which side $\sigma_{23}$ goes on; in general we have to be more careful. We have chosen to put the $\omega^*$ on the right of $i \otimes \delta_G(V)$ because we want $\operatorname{Ad} V : T \mapsto V(T \otimes 1)V^*$ to be a coaction of $G$ on $B(\mathcal{H})$:

$$\begin{aligned}(\operatorname{Ad} V \otimes i) \circ \operatorname{Ad} V(T) &= (\operatorname{Ad} V_{12}V_{13})(T \otimes 1 \otimes 1) = \operatorname{Ad}\,(i \otimes \delta_G(V)\omega_{23}^*)(T \otimes 1 \otimes 1) \\ &= \operatorname{Ad}\,(i \otimes \delta_G(V))(T \otimes \delta_G(1)) = i \otimes \delta_G(\operatorname{Ad} V(T)).\end{aligned}$$

**Lemma 1.5.** *If $V$ is an $\omega$-corepresentation of $G$ on $\mathcal{H}$, then $\mu(f) = S_f(V)$ defines a norm-decreasing homomorphism of $A(G,\omega)$ into $B(\mathcal{H})$.*

**Proof.** We have $\|\mu(f)\| \leq \|f\|$ because $\|V\| = 1$ and $\|S_f\| = \|f\|$, and a calculation shows that $\mu$ is multiplicative: if $\phi \in B(\mathcal{H})_*$, then

$$\begin{aligned}\langle \mu(f *_\omega g), \phi \rangle &= \langle V, \phi \otimes (f *_\omega g) \rangle \\ &= \langle i \otimes \delta_G(V)\omega_{23}^*, \phi \otimes f \otimes g \rangle \\ &= \langle V_{12}V_{13}, \phi \otimes f \otimes g \rangle \\ &= \langle V(\mu(g) \otimes 1), \phi \otimes f \rangle \\ &= \langle \mu(f)\mu(g), \phi \rangle.\end{aligned}$$

**Lemma 1.6.** *If $V = W_G^* \Sigma(\omega^*)$, then $V$ is an $\omega$-corepresentation of $G$ on $L^2(G)$, which we call the regular $\omega$-corepresentation of $G$.*

**Proof.** We calculate

$$\begin{aligned}
V_{12}V_{13} &= (W_G^*)_{12}\Sigma(\omega^*)_{12}(W_G^*)_{13}\Sigma(\omega^*)_{13} \\
&= (W_G^*)_{12}(W_G^*)_{13}\, \Sigma \otimes i\big((W_G)_{23}\omega_{12}^*(W_G^*)_{23}\big)\Sigma(\omega^*)_{13} \\
&= (W_G^*)_{12}(W_G^*)_{13}\, \Sigma \otimes i\big(i \otimes \delta_G(\omega^*)\big)\Sigma(\omega^*)_{13} \\
&= i \otimes \delta_G(W_G^*)(\Sigma \otimes i) \circ (i \otimes \Sigma)\big(i \otimes \delta_G(\omega^*)\omega_{23}^*\big) \\
&= i \otimes \delta_G(W_G^*)(\Sigma \otimes i) \circ (i \otimes \Sigma)\big(\delta_G \otimes i(\omega^*)\omega_{12}^*\big).
\end{aligned}$$

Since $(\Sigma \otimes i) \circ (i \otimes \Sigma) \circ (\delta_G \otimes i) = (i \otimes \delta_G) \circ \Sigma$, this implies

$$V_{12}V_{13} = i \otimes \delta_G(W_G^*)\, i \otimes \delta_G(\Sigma(\omega^*))\omega_{23}^* = i \otimes \delta_G(V)\omega_{23}^*,$$

as required.

To see why we call this the regular $\omega$-corepresentation, suppose $G$ is abelian and $\omega$ is the Fourier transform of $\sigma \in Z^2(G, \mathbf{T})$. A quick calculation shows that $\mathrm{Ad}\,(\mathcal{F} \otimes \mathcal{F})(W_G)$ is the operator $V_{\widehat{G}}$ on $L^2(\widehat{G} \times \widehat{G})$ defined by $V_{\widehat{G}}\xi(s,t) = \xi(s+t,t)$, and hence $\mathrm{Ad}\,(\mathcal{F} \otimes \mathcal{F})(W_G^*\sigma(\omega^*))$ is the operator $U$ defined by

$$U(\xi)(s,t) = \overline{\sigma(s, s-t)}\,\xi(s-t,t).$$

This is multiplication by the operator-valued function $t \mapsto U_t$, where

$$(U_t\xi)(s) = \overline{\sigma(s, s-t)}\,\xi(s-t)$$

is one version of the regular $\overline{\sigma}$-representation of $\widehat{G}$ on $L^2(\widehat{G})$.

**Definition 1.7.** A dual cocycle $\omega$ on $G$ is *normalised* if there is an $\omega$-corepresentation $U$ of $G$ on a Hilbert space $\mathcal{H}$ satisfying

$$\big(U(h \otimes \overline{\eta})\big|k \otimes \overline{\xi}\big) = \big(U^*(h \otimes \xi)\big|k \otimes \eta\big) \quad \text{for } h,k \in \mathcal{H},\ \xi,\eta \in L^2(G). \tag{1.9}$$

**Proposition 1.8.** *Suppose $\omega$ is a normalised dual cocycle on $G$. Then for any $\omega$-corepresentation $V$ of $G$ on a Hilbert space $\mathcal{H}_V$, we have*

$$\big(V(h \otimes \overline{\eta})\big|k \otimes \overline{\xi}\big) = \big(V^*(h \otimes \xi)\big|k \otimes \eta\big) \quad \text{for } h,k \in \mathcal{H}_V \text{ and } \xi,\eta \in L^2(G). \tag{1.10}$$

We begin by observing that we can stick in an extra variable, which will give us room to manoevre; since we shall want to do this repeatedly, we state it as a separate lemma.

**Lemma 1.9.** *Let $V$ be an $\omega$-corepresentation of $G$. Then $V$ satisfies (1.10) if and only if*

$$\big(V_{12}(h\otimes\overline{\beta})\big|k\otimes\overline{\alpha}\big) = \big(V_{12}^*(h\otimes\alpha)\big|k\otimes\beta\big) \quad \text{for all } h,k\in\mathcal{H}_V \text{ and } \alpha,\beta\in L^2(G\times G). \tag{1.11}$$

**Proof.** Taking $\alpha=\xi\otimes f, \beta=\eta\otimes g$ in (1.11) gives

$$\big(V(h\otimes\overline{\eta})\big|k\otimes\overline{\xi}\big)\,\big(\overline{g}\big|\overline{f}\big) = \big(V^*(h\otimes\xi)\big|k\otimes\eta\big)\,\big(f\big|g\big),$$

which reduces to (1.10) (recall that we always have $(\overline{g}|\overline{f})=(f|g)$ in an $L^2$-space). This observation also shows that (1.10) implies (1.11) for elementary tensors $\alpha,\beta$, and the full generality of (1.11) follows, since both sides are linear in $\alpha$, antilinear in $\beta$, and continuous.

The next lemma is the analogue of the well-known absorbing property of the regular representation $L: L\otimes U\sim L\otimes 1$ for any unitary representation $U$ (cf. [7, XI.13.11] and [23, Lemma 11]). We shall use it to show that both (1.9) and (1.10) are equivalent to the analogous property of the regular $\omega$-representation $W_G^*\Sigma(\omega^*)$.

**Lemma 1.10.** *Suppose $\omega$ is a dual cocycle, $V$ is an $\omega$-corepresentation of $G$ and $W$ is an ordinary corepresentation of $G$. Then $U=W_{23}V_{13}$ is an $\omega$-corepresentation on $\mathcal{H}_V\otimes\mathcal{H}_W$. When $W=W_G^*$, $U$ is equivalent to $1\otimes W_G^*\Sigma(\omega^*)$: indeed,*

$$\operatorname{Ad}(V^*\otimes 1)\big((W_G^*)_{23}V_{13}\big)=\big(W_G^*\Sigma(\omega^*)\big)_{23}.$$

**Proof.** First of all,

$$\begin{aligned} U_{123}U_{124} &= W_{23}V_{13}W_{24}V_{14} = W_{23}W_{24}V_{13}V_{14} \\ &= i\otimes\delta_G(W)_{234}\, i\otimes\delta_G(V)_{134}\,\omega_{34}^* = i\otimes i\otimes\delta_G(W_{23}V_{13})\omega_{34}^*, \end{aligned}$$

so $U$ is an $\omega$-corepresentation. Next, using (1.7), we have

$$V_{12}^*(W_G^*)_{23}V_{13} = (W_G^*)_{23}\; i\otimes\delta_G(V^*)V_{13} = (W_G^*)_{23}\Sigma(\omega^*)_{23}V_{12}^*$$

which completes the proof.

**Proof of Proposition 1.8.** We claim that $V$ satisfies (1.11) if and only if $X=(W_G^*)_{23}V_{13}$ satisfies the appropriate version of (1.10) (i.e., equation (1.12) below). First suppose $V$ satisfies (1.11), and let $h,k\in\mathcal{H}$, $\gamma,\delta,\xi,\eta\in L^2(G)$. If we view the middle copy of $L^2(G)$ as the extra one in (1.11), Lemma 1.9 gives

$$\begin{aligned} \big(X^*(h\otimes\gamma\otimes\xi)\big|k\otimes\delta\otimes\eta\big) &= \big(V_{13}^*(W_G)_{23}(h\otimes\gamma\otimes\xi)\big|k\otimes\delta\otimes\eta\big) \\ &= \big(V_{13}(h\otimes\overline{\delta\otimes\eta})\big|k\otimes\overline{W_G(\gamma\otimes\xi)}\big). \end{aligned}$$

We trivially have $\overline{W_G(\gamma\otimes\xi)} = W_G(\overline{\gamma}\otimes\overline{\xi})$, so this is equal to

$$\big((W_G^*)_{23}V_{13}(h\otimes\overline{\delta}\otimes\overline{\eta})\big|k\otimes\overline{\gamma}\otimes\overline{\xi}\big).$$

In the middle variable, $(W_G^*)_{23}V_{13}$ acts as a multiplication operator, and from the equation $(M_f\overline{\delta}|\overline{\gamma}) = (M_f\gamma|\delta)$ we deduce that

$$\begin{aligned}\big(X^*(h\otimes\gamma\otimes\xi)\big|k\otimes\delta\otimes\eta\big) &= \big((W_G^*)_{23}V_{13}(h\otimes\gamma\otimes\overline{\eta})\big|k\otimes\delta\otimes\overline{\xi}\big)\\ &= \big(X(h\otimes\gamma\otimes\overline{\eta})\big|k\otimes\delta\otimes\overline{\xi}\big).\end{aligned}$$

Again, we use linearity and continuity to extend this equation from elementary tensors, and obtain

$$\big(X^*(\alpha\otimes\xi)\big|\beta\otimes\eta\big) = \big(X(\alpha\otimes\overline{\eta})\big|\beta\otimes\overline{\xi}\big) \quad\text{for } \alpha,\beta\in\mathcal{H}\otimes L^2(G). \tag{1.12}$$

On the other hand, if $X$ satisfies (1.12), we can take $\alpha = h\otimes\gamma$, $\beta = k\otimes\delta$, and apply the above reasoning to the left–hand side. We obtain

$$\big(V_{13}(h\otimes\overline{\delta\otimes\eta})\big|k\otimes\overline{W_G(\gamma\otimes\xi)}\big) = \big(V_{13}^*(h\otimes W_G(\gamma\otimes\xi))\big|k\otimes\delta\otimes\eta\big).$$

As usual, this extends from elementary tensors $\gamma\otimes\xi$, $\delta\otimes\eta$ to arbitrary functions in $L^2(G)\otimes L^2(G) = L^2(G\times G)$, and (1.11) follows since $W_G$ is surjective.

By Lemma 1.10, the unitary operator $V$ on $\mathcal{H}\otimes L^2(G)$ satisfies $\mathrm{Ad}\,(V^*\otimes 1)(X) = (W_G^*\Sigma(\omega^*))_{23}$. If we now replace $\alpha$, $\beta$ in (1.12) by $V(h\otimes\gamma)$, $V(k\otimes\delta)$, and suppress the first variable, we find that (1.12) is equivalent to

$$\big(W_G^*\Sigma(\omega^*)(\gamma\otimes\overline{\eta})\big|\delta\otimes\overline{\xi}\big) = \big(\Sigma(\omega)W_G(\gamma\otimes\xi)\big|\delta\otimes\eta\big) \quad\text{for } \gamma,\delta,\xi,\eta\in L^2(G).$$

Since this last statement is independent of the $\omega$-corepresentation $V$ we started with, this proves the proposition.

**Lemma 1.11.** *Let $\omega$ be a normalised dual cocycle on $G$. Then $\nu\otimes\nu(\omega^*) = \Sigma(\omega)$.*

**Proof.** (We would just apply $i\otimes\nu\otimes\nu$ to (1.6) if it made sense to do so.) Let $U$ be an $\omega$-corepresentation satisfying (1.9). Then two applications of (1.9) give

$$\begin{aligned}\big(U_{12}^*U_{13}^*(h\otimes\alpha\otimes\xi)\big|k\otimes\beta\otimes\eta\big) &= \big(U_{12}U_{13}(h\otimes\overline{\beta}\otimes\overline{\eta})\big|k\otimes\overline{\alpha}\otimes\overline{\xi}\big)\\ &= \big(i\otimes\delta_G(U)\omega_{23}^*(h\otimes\overline{\beta}\otimes\overline{\eta})\big|k\otimes\overline{\alpha}\otimes\overline{\xi}\big).\end{aligned}$$

Now

$$\begin{aligned}\big(\omega^*(\overline{\beta}\otimes\overline{\eta})\big|\delta\otimes\gamma\big) &= \big(\nu\otimes\nu(\omega^*)(\overline{\delta}\otimes\overline{\gamma})\big|\beta\otimes\eta\big)\\ &= \big(\overline{\delta}\otimes\overline{\gamma}\big|\nu\otimes\nu(\omega)(\beta\otimes\eta)\big)\\ &= \big(\overline{\nu\otimes\nu(\omega)(\beta\otimes\eta)}\ \big|\delta\otimes\gamma\big),\end{aligned}$$

and hence

$$\begin{aligned}
\big(U_{12}^*U_{13}^*(h\otimes\alpha\otimes\xi)\big|k\otimes\beta\otimes\eta\big) &= \big(i\otimes\delta_G(U)h\otimes\overline{\nu\otimes\nu(\omega)(\beta\otimes\eta)}\big|k\otimes\overline{\alpha\otimes\xi}\big)\\
&= \big(U_{12}(h\otimes W_G^*(\overline{\nu\otimes\nu(\omega)(\beta\otimes\eta)}))\big|k\otimes W_G^*(\overline{\alpha\otimes\xi})\big)\\
&= \big(U_{12}(h\otimes\overline{W_G^*(\nu\otimes\nu(\omega)(\beta\otimes\eta))})\big|k\otimes\overline{W_G^*(\alpha\otimes\xi)}\big) \quad \text{since } \overline{W_G^*\xi}=W_G^*\overline{\xi}\\
&= \big(U_{12}^*(h\otimes W_G^*(\alpha\otimes\xi))\big|k\otimes W_G^*(\nu\otimes\nu(\omega)(\beta\otimes\eta))\big) \quad \text{by Lemma 1.9}\\
&= \big(i\otimes\delta_G(U^*)h\otimes\alpha\otimes\xi\big|k\otimes(\nu\otimes\nu)(\omega)(\beta\otimes\eta)\big).
\end{aligned}$$

Thus we have $U_{12}^*U_{13}^*=\nu\otimes\nu(\omega^*)_{23}i\otimes\delta_G(U^*)$, and the result follows from Remark 1.4.

**Remark 1.12.** For compact $G$, this is a special case of [23, Lemma 13]; our proof avoids direct manipulation of $i\otimes\nu$. The lemma implies that condition (3.4) in the definition of normalised cocycle used in [12] is redundant. We point out that, even in the abelian case, the equation $\nu\otimes\nu(\omega)=\Sigma(\omega^*)$ is distinctly weaker than normalisation. For example, take $\widehat{G}=\mathbf{Z}_2=\{\pm1\}$, define $\rho:\widehat{G}\to\mathbf{T}$ by $\rho(1)=1,\rho(-1)=i$, and let $\sigma=\partial\rho$. Then $\mu\otimes\mu(\sigma)(s,t)=\sigma(s^{-1},t^{-1})=\sigma(s,t)=-1$ if $s=t=-1$, and is $=1$ otherwise. But

$$\sigma\big((-1)^{-1},(-1)\big)=\sigma(-1,-1)\neq 1.$$

**Proposition 1.13.** *If $\omega$ is a normalised dual cocycle, then $A(G,\omega)$ is a Banach ${}^*$-algebra with involution given by $f^*(t)=\overline{f(t)}$.*

**Proof.** Since this is the normal involution on $A(G)$, and we have not changed the Banach space structure, the only thing we have to check is $(f*_\omega g)^*=g^**_\omega f^*$. Recall that the involution on $A(G)$ can be described in terms of $\nu$ by $\langle T,f^*\rangle=\overline{\langle\nu(T^*),f\rangle}$. Thus for $T\in\mathcal{L}(G)$, we have

$$\begin{aligned}
\langle T,(f*_\omega g)^*\rangle &= \overline{\langle\nu(T^*),f*_\omega g\rangle}=\overline{\langle\delta_G(\nu(T^*))\omega^*,f\otimes g\rangle}\\
&= \overline{\langle\delta_G(\nu(T^*))\Sigma(\nu\otimes\nu(\omega)),f\otimes g\rangle} \quad \text{by Lemma 1.11}\\
&= \overline{\langle\Sigma(\delta_G(\nu(T^*))\nu\otimes\nu(\omega)),f\otimes g\rangle} \quad \text{since } \Sigma\circ\delta_G=\delta_G\\
&= \overline{\langle\delta_G(\nu(T^*))(\nu\otimes\nu(\omega)),g\otimes f\rangle}\\
&= \overline{\langle\nu\otimes\nu(\omega\delta_G(T^*)),g\otimes f\rangle} \quad \text{since } \delta_G\circ\nu=(\nu\otimes\nu)\circ\delta_G\\
&= \big\langle(\omega\delta_G(T^*))^*,g^*\otimes f^*\big\rangle\\
&= \langle T,g^**_\omega f^*\rangle,
\end{aligned}$$

as required.

**Theorem 1.14.** *Suppose $\omega$ is a normalised dual cocycle, and $V$ is an $\omega$-corepresentation of $G$ on $\mathcal{H}$. Then there is a norm-decreasing, nondegenerate ${}^*$-representation $\mu$ of $A(G,\omega)$ on $\mathcal{H}$ such that $\mu(f) = S_f(V)$ for $f \in A(G)$.*

**Proof.** We have already seen that $\mu$ is multiplicative and norm-decreasing in Lemma 1.5. Since

$$(\mu(f)h|k) = 0 \quad \text{for all } h \in \mathcal{H}, f \in A(G)$$

if and only if

$$(V(h \otimes \xi)|k \otimes \eta) = 0 \quad \text{for all } h \in \mathcal{H} \text{ and } \xi, \eta \in L^2(G)\ ,$$

the nondegeneracy of $\mu$ follows from the surjectivity of $V$. It remains to show that $\mu(f^*) = \mu(f)^*$. Suppose $f$ is the functional $T \mapsto (T\xi|\eta)$ in $L(G)_*$, so that the conjugate $f^*$ is the functional $T \mapsto (T\overline{\xi}|\overline{\eta})$. Then

$$\begin{aligned}(\mu(f^*)h|k) &= (V(h \otimes \overline{\xi})|k \otimes \overline{\eta}) = (V^*(h \otimes \eta)|k \otimes \xi) \quad \text{by Proposition 1.8}\\ &= \overline{(V(k \otimes \xi)|h \otimes \eta)} = \overline{(S_f(V)k|h)} = (h|S_f(V)k) = (h|\mu(f)k),\end{aligned}$$

as required.

**Problem.** We would very much like to know whether *every* ${}^*$-representation of $A(G,\omega)$ is given by slicing an $\omega$-corepresentation. This is true if $\omega = 1 \otimes 1$ by [14, Theorem A1], but the proof given there depends crucially on the commutativity of $A(G)$. In general, one can show that every ${}^*$-representation $\mu$ has the form $\mu(f) = S_f(V)$ for some operator $V$ satisfying (1.6) and (1.10): the problem is to show that $V$ has to be unitary.

**Definition 1.15.** Let $\omega$ be a normalised dual cocycle on a locally compact group $G$, and $\mu$ be the ${}^*$-representation of $A(G,\omega)$ given by $\mu(f) = S_f\big(W_G^*\Sigma(\omega^*)\big)$. The *(reduced) twisted dual-group algebra* $C_r^*(\widehat{G},\omega)$ is the closure of $\mu\big(A(G,\omega)\big)$ in $B(L^2(G))$.

**Lemma 1.16.** *When $G$ is a second countable abelian group, and the normalised dual cocycle $\omega$ is the Fourier transform of a normalised cocycle $\sigma$ on $\widehat{G}$, the Fourier transform $\mathcal{F}h(x) = \int \langle x, s\rangle h(s)\, ds$ of $L^1(\widehat{G})$ onto $A(G)$ induces an isomorphism of the twisted group algebra $C^*(\widehat{G},\overline{\sigma})$ onto $C_r^*(\widehat{G},\omega)$.*

**Proof.** We have already seen that $\mathcal{F}$ is a ${}^*$-isomorphism of $L^1(\widehat{G},\overline{\sigma})$ onto $A(G,\omega)$, so we just have to check that the $C^*$-norms match up. Moving the representation $\mu \circ \mathcal{F}$ over to $L^2(\widehat{G})$ via the Fourier transform $\mathcal{F} : L^2(\widehat{G}) \to L^2(G)$ gives the representation

$$\begin{aligned}\pi(f) &= \mathcal{F}^{-1}\mu(\mathcal{F}f)\mathcal{F} = \mathcal{F}^{-1}\big(S_{\mathcal{F}f}\big(W_G^*\Sigma(\omega^*)\big)\big)\mathcal{F}\\ &= S_f\big((\mathcal{F}^{-1} \otimes \mathcal{F}^{-1})W_G^*\Sigma(\omega^*)(\mathcal{F} \otimes \mathcal{F})\big)\\ &= S_f\big((\mathcal{F}^{-1} \otimes \mathcal{F}^{-1})W_G^*(\mathcal{F} \otimes \mathcal{F})\Sigma(\overline{\sigma})\big).\end{aligned}$$

Since $(\mathcal{F}^{-1}\otimes\mathcal{F}^{-1})W_G^*(\mathcal{F}\otimes\mathcal{F})$ is the operator $V_{\widehat{G}}^*$ given by $V_{\widehat{G}}^*h(s,t) = h(s-t,t)$, and every $f\in L^1(\widehat{G})$ is the pointwise product $\xi\overline{\eta}$ of two functions in $L^2(\widehat{G})$, we can compute

$$\begin{aligned}\big(\pi(f)h\big|k\big) &= \big(V_{\widehat{G}}^*\Sigma(\overline{\sigma})(h\otimes\xi)\big|k\otimes\eta\big)\\ &= \iint \overline{\sigma(t,s-t)}\,h(s-t)\xi(t)\overline{k(s)\eta(t)}\,ds\,dt\\ &= \int\Big(\int f(t)\,\overline{\sigma(t,s-t)}\,h(s-t)\,dt\Big)\overline{k(s)}\,ds.\end{aligned}$$

Thus $\pi$ is the integrated form of the representation $U:\widehat{G}\to U(L^2(\widehat{G}))$ defined by

$$(U_t h)(s) = \overline{\sigma(t,s-t)}\,h(s-t).$$

This is one realisation of the regular $\overline{\sigma}$-representation of $\widehat{G}$, and its integrated form is faithful on $C^*(\widehat{G},\overline{\sigma})$ because $\widehat{G}$ is amenable (e.g. [16, Theorem 3.11]). (This is where we use the second countability of $G$; we lack a reference for the general case.) Thus $\pi$ is isometric, and since $\mu$ is isometric by definition, $\mathcal{F}$ extends to an isometric $^*$-isomorphism of $C^*(\widehat{G},\overline{\sigma})$ onto $C_r^*(\widehat{G},\omega)$.

**Problem.** There is considerable evidence that the dual group algebra $C_0(G)$ is amenable in an appropriate sense (e.g. [5; 19]), and hence we conjecture that the regular representation always induces an isomorphism of the $C^*$-enveloping algebra $C^*(A(G,\omega))$ onto $C_r^*(\widehat{G},\omega)$.

**Proposition 1.17.** *Suppose $\omega$ is a normalised dual cocycle on a locally compact group $G$. For each $y\in G$, $\alpha_y(f)(x) = f(y^{-1}x)$ defines a $^*$-automorphism $\alpha_y$ of $A(G,\omega)$, and $\alpha$ is then a strongly continuous action of $G$ on $A(G,\omega)$. If $\mu$ is the representation of $A(G,\omega)$ obtained by slicing the regular $\omega$-corepresentation $W_G^*\Sigma(\omega^*)$, then $(\mu,R^G)$ is a covariant representation of $\big(A(G,\omega),G,\alpha\big)$. In particular, it follows that $\alpha$ extends to a strongly continuous action of $G$ on $C_r^*(\widehat{G},\omega)$.*

**Proof.** We first observe that

$$\langle L_{y^{-1}}T,f\rangle = \langle T,\alpha_y(f)\rangle \quad\text{for } T\in\mathcal{L}(G); \tag{1.13}$$

to see this, choose $\xi,\eta\in L^2(G)$ such that $\langle T,f\rangle = (T\xi|\eta)$, and expand both sides for $T=L_x$, $x\in G$. From (1.13) we easily verify that $\alpha_y(f*_\omega g) = \alpha_y(f)*_\omega\alpha_y(g)$; since we trivially have $\alpha_y(f^*) = \alpha_y(f)^*$, it follows that $\alpha_y$ is a $^*$-automorphism of $A(G,\omega)$. If again $f\in A(G)=\mathcal{L}(G)_*$ is given by $T\mapsto(T\xi|\eta)$, we also have

$$\big|\langle T,\alpha_y(f)-f\rangle\big| = \big|\big((L_{y^{-1}}T-T)\xi\big|\eta\big)\big| \le \|T\|\,\|\xi\|\,\|L_y\eta-\eta\|$$

and hence $\|\alpha_y(f) - f\|_{A(G)} \to 0$ as $y \to e$. Thus $\alpha$ is strongly continuous.

For the covariance, we compute:

$$\begin{aligned} R_y\mu(f)R_y^* &= R_y S_f\big(W_G^*\Sigma(\omega^*)\big)R_y^* \\ &= S_f\big((R_y\otimes 1)W_G^*\Sigma(\omega^*)(R_y^*\otimes 1)\big) \\ &= S_f\big((R_y\otimes 1)W_G^*(R_y^*\otimes 1)\Sigma(\omega^*)\big), \end{aligned}$$

because $R_y$ is in $\mathcal{R}(G) = \mathcal{L}(G)'$. A straightforward calculation shows that

$$(R_y\otimes 1)W_G^*(R_y^*\otimes 1) = (1\otimes L_y^*)W_G^*,$$

and (1.13) implies that $S_f((1\otimes L_y^*)T) = S_{\alpha_y(f)}(T)$, so we have

$$R_y\mu(f)R_y^* = S_f((1\otimes L_y^*)W_G^*\Sigma(\omega^*)) = S_{\alpha_y(f)}(W_G^*\Sigma(\omega^*)) = \mu(\alpha_y(f)).$$

It follows immediately that $\alpha_y$ is isometric for the $C^*$-norm, and hence extends to a $^*$-automorphism of $C_r^*(\widehat{G},\omega)$. Since $\mu : A(G,\omega) \to B(L^2(G))$ is norm-decreasing, the continuity of $\alpha$ on $A(G,\omega)$ implies that $\alpha : G \to \operatorname{Aut} C_r^*(\widehat{G},\omega)$ is strongly continuous.

## §2. Induced $C^*$-algebras and induced representations.

Suppose $H$ is a closed subgroup of a locally compact group $G$, and $\beta : H \to \operatorname{Aut} B$ is an action of $H$ on a $C^*$-algebra $B$. Then the *induced $C^*$-algebra* $\operatorname{Ind}_H^G(B,\beta)$ is the subalgebra

$$\{f \in C_b(G,B) : f(xh) = \beta_{h^{-1}}(f(x)) \text{ for } x\in G, h\in H, \text{ and } xH \mapsto \|f(x)\| \text{ is in } C_0(G/H)\}$$

of $C_b(G,B)$. The larger group $G$ acts by left translation on $\operatorname{Ind}_H^G B$: $\tau_x(f)(y) = f(x^{-1}y)$. We refer to $\big(\operatorname{Ind}_H^G B, G, \tau\big)$ as the *induced system.* These systems have been studied in, for example, [15] and [20]. We can easily construct elements of $\operatorname{Ind} B$:

**Lemma 2.1.** *Let $\beta : H \to \operatorname{Aut} B$, and for $f \in C_c(G)$, $b \in B$ define*

$$(f \odot b)(x) = \int_H f(xh)\beta_h(b)\,dh.$$

*Then the functions $f\odot b$ span a dense subspace of $\operatorname{Ind}_H^G(B,\beta)$ — indeed, $f$ and $b$ need only run through dense subspaces of $C_c(G)$ and $B$.*

**Proof.** A routine calculation shows that the functions $f\odot b$ do lie in $\operatorname{Ind}_H^G B$. To see that they span a dense subspace, suppose $a \in \operatorname{Ind} B$, and $x \in G$. Choose $f \in C_c(G)$ such that $f \geq 0$, $\int f(xh)\,dh = 1$, and then we have $\|\big(f\odot a(x)\big)(y) - a(y)\|$ small for $y$ in a neighbourhood $N$ of $x$. Because both $f \odot a(x)$ and $a$ are in $\operatorname{Ind} B$, it follows that $\|\big(f\odot a(x)\big)(y) - a(y)\|$ is small throughout the strip $N\cdot H$. We can now use a partition of unity on $G/H$, which pulls through the integral defining $f\odot a(x)$, to approximate $a$ uniformly on all of $G$.

We seek to construct covariant representations of $\left(\operatorname{Ind}_H^G B, G, \tau\right)$ from covariant representations of $(B, G, \beta)$. It turns out that, if $(\pi, S)$ is a covariant representation of $(B, G, \beta)$ on $\mathcal{H}$, then $\left(\operatorname{Ind}_H^G B, G, \tau\right)$ acts naturally in the Hilbert space of the induced representation $\operatorname{Ind}_H^G S$. To see this, it seems most convenient to use Blattner's formulation of $\operatorname{Ind}_H^G S$, which we briefly review (cf. [7, XI.10.8]).

Suppose $S$ is a unitary representation of $H$ on a Hilbert space $\mathcal{H} = \mathcal{H}_S$. We let $\Delta$ and $\delta$ denote the modular functions on $G$ and $H$, and fix a continuous rho-function $\rho : G \to (0, \infty)$, so that

$$\rho(xh) = \frac{\delta(h)}{\Delta(h)}\rho(x) \quad \text{for } x \in G, h \in H$$

(cf. [7, III.13.2 and III.14.5]). Then by [7, III.13.10], $\rho$ determines a regular Borel measure $\rho^\sharp$ on $G/H$ such that

$$\int_{G/H}\int_H \phi(xh)\, dh\, d\rho^\sharp(xH) = \int_G \rho(x)\phi(x)\, dx \quad \text{for } \phi \in C_c(G).$$

The space $\mathcal{H}_{\operatorname{Ind} S}$ then consists of the locally measurable functions $\phi : G \to \mathcal{H}_S$ such that

$$\phi(xh) = \delta(h)^{1/2}\Delta(h)^{-1/2} S_{h^{-1}}\phi(x) \quad \text{for } x \in G, h \in H,$$

and such that the function $xH \mapsto \rho(x)^{-1}\|\phi(x)\|^2$ is in $L^1(G/H, d\rho^\sharp)$; as usual, we identify functions which are equal locally almost everywhere, and then $\mathcal{H}_{\operatorname{Ind} S}$ is a Hilbert space with

$$(\phi|\psi) = \int_{G/H} \rho(x)^{-1}\big(\phi(x)|\psi(x)\big)\, d\rho^\sharp(xH).$$

The *induced representation* $\operatorname{Ind}_H^G S$ is then defined by

$$\left(\operatorname{Ind}_H^G S\right)_y(\phi)(x) = \phi(y^{-1}x) \quad \text{for } \phi \in \mathcal{H}_{\operatorname{Ind} S}, \text{ and } x, y \in G.$$

Again we need to be able to write down lots of elements of $\mathcal{H}_{\operatorname{Ind} S}$, which requires a standard lemma.

**Lemma 2.2.** *If $\psi \in C_c(G)$ and $\xi \in \mathcal{H}_S$, then*

$$(\psi \odot \xi)(x) = \int_H \psi(xh)\delta(h)^{-1/2}\Delta(h)^{1/2} S_h(\xi)\, dh$$

*defines an element of $\mathcal{H}_{\operatorname{Ind} S}$, and the elements of this form span a dense subspace of $\mathcal{H}_{\operatorname{Ind} S}$.*

**Proof.** Since $\psi \odot \xi$ is continuous, it is certainly measurable, and a straightforward change-of-variables argument shows that

$$(\psi \odot \xi)(xh) = \delta(h)^{1/2}\Delta(h)^{-1/2} S_{h^{-1}}(\psi \odot \xi)(x).$$

To see that $\rho(x)^{-1}\|\psi\odot\xi(x)\|^2$ is integrable, use a change of variables and Fubini's Theorem to compute

$$\begin{aligned}\int_{G/H}&\rho(x)^{-1}\|(\psi\odot\xi)(x)\|^2\,d\rho^\sharp(xH)\\ &=\int_H\int_{G/H}\int_H\rho(xk)^{-1}\psi(xkl)\overline{\psi(xk)}\Delta(l)^{1/2}\delta(l)^{-1/2}(S_l\xi|\xi)\,dk\,d\rho^\sharp(xH)\,dl\\ &=\int_H\delta(l)^{-1/2}\Delta(l)^{1/2}(\phi*\overline{\psi})(l^{-1})(S_l\xi|\xi)\,dl,\end{aligned}$$

where $\phi(y)=\psi(y^{-1})$. Since $\phi,\psi$ are in $C_c(G)$, the function

$$f:l\mapsto\delta(l)^{-1/2}\Delta(l)^{1/2}\phi*\overline{\psi}(l^{-1})$$

is continuous and has compact support. Thus

$$\int_{G/H}\rho(x)^{-1}\|\psi\odot\xi(x)\|^2\,d\rho^\sharp(xH)=\big(S(f)\xi\big|\xi\big)\le\|f\|_1\|\xi\|^2,$$

and $\psi\odot\xi\in\mathcal{H}_{\operatorname{Ind}S}$, as claimed. To see that these elements span a dense subspace, verify that

$$(\psi\odot\xi|\phi)=\int_G\psi(x)(\xi|\phi(x))\,dx,$$

so that no element $\phi$ of $\mathcal{H}_{\operatorname{Ind}S}$ can be orthogonal to $\operatorname{sp}\{\psi\odot\xi\}$.

**Proposition 2.3.** *Suppose $(\pi,S)$ is a covariant representation of $(B,G,\beta)$. Then there is a representation $\widetilde{\pi}$ of $\operatorname{Ind}_H^G(B,\beta)$ on $\mathcal{H}_{\operatorname{Ind}S}$ such that*

$$\big(\widetilde{\pi}(f)\phi\big)(x)=\pi(f(x))\phi(x),$$

*and $\big(\widetilde{\pi},\operatorname{Ind}_H^G S\big)$ is then a covariant representation of $\big(\operatorname{Ind}_H^G B,G,\tau\big)$. The representation $\widetilde{\pi}$ is faithful if and only if $\pi$ is.*

**Proof.** We first need to check that $\widetilde{\pi}(f)\phi$ belongs to $\mathcal{H}_{\operatorname{Ind}S}$. Using the characterisation of locally $\lambda$-measurable functions as the $\lambda$-a.e. limit of simple functions, and the continuity of $f$, one can verify that $\widetilde{\pi}(f)\phi$ is locally measurable. If $\phi\in\mathcal{H}_S$, we have

$$\begin{aligned}\big(\widetilde{\pi}(f)\phi\big)(xh)&=\pi\big(f(xh)\big)\phi(xh)\\ &=\pi\big(\beta_h^{-1}(f(x))\big)\big(\delta(h)/\Delta(h)\big)^{1/2}S_{h^{-1}}\phi(x)\\ &=\big(S_h^{-1}\pi(f(x))S_h\big)\big(\delta(h)/\Delta(h)\big)^{1/2}S_{h^{-1}}\phi(x)\\ &=\big(\delta(h)/\Delta(h)\big)^{1/2}S_{h^{-1}}\big(\widetilde{\pi}(f)\phi\big)(x).\end{aligned}$$

Next, we have

$$\begin{aligned}\int \rho(x)^{-1}\|\widetilde{\pi}(f)\phi(x)\|^2\,d\rho^\sharp(xH) &\le \int \rho(x)^{-1}\|f(x)\|^2\|\phi(x)\|^2\,d\rho^\sharp(xH)\\ &\le \|f\|^2\int \rho(x)^{-1}\|\phi(x)\|^2\,d\rho^\sharp(xH),\end{aligned}$$

which implies both that $\widetilde{\pi}(f)\phi$ is in $\mathcal{H}_{\operatorname{Ind} S}$, and that $\widetilde{\pi}(f)$ is a bounded operator. We deduce that $\widetilde{\pi}$ is a representation of $\operatorname{Ind} B$ on $\mathcal{H}_{\operatorname{Ind} S}$. Now a trivial calculation gives

$$\big(\operatorname{Ind} S\big)_x\widetilde{\pi}(f) = \widetilde{\pi}\big(\tau_x(f)\big)\big(\operatorname{Ind} S\big)_x,$$

so that $(\widetilde{\pi}, \operatorname{Ind} S)$ is a covariant representation of $(\operatorname{Ind} B, G, \tau)$.

Next suppose $\pi$ is faithful and $\widetilde{\pi}(f) = 0$, and fix $\xi \in \mathcal{H}_S$. For any $\psi \in C_c(G)$ we can form $\psi \odot \xi$ as in Lemma 2.2, and then $\widetilde{\pi}(f)(\psi \odot \xi) = 0$ says

$$\int_H \psi(xh)\delta(h)^{-1/2}\Delta(h)^{1/2}\pi(f(x))(S_h\xi)\,dh = 0 \quad x-\text{a.e..}$$

Since the integrand is continuous in $x$ and $h$, it must vanish identically for all $x$, and by varying $\psi \in C_c(G)$ we can deduce that $\pi(f(x))\xi = 0$ for all $x$, and $f \equiv 0$. Conversely, suppose that $\widetilde{\pi}$ is faithful and $b \in B$ is non-zero. If $f$ has support in a neighbourhood of the identity in which $\beta_h(b) \sim b$, then $f \odot b(e) \sim b$, and $\widetilde{\pi}(f \odot b) \neq 0$. But then there exists $x$ such that

$$0 \neq \pi\big(f \odot b(x)\big) = \int f(xh)\pi\big(\beta_h(b)\big)\,dh = \int f(xh)S_h\pi(b)S_h^*\,dh,$$

and hence $\pi(b) \neq 0$.

It is well-known that inducing the left regular representation $L^H$ of $H$ to $G$ gives the left regular representation $L^G$ of $G$ (e.g. [7, XI.12.18]), and of course the same is true for the right regular representations $R^H, R^G$. But we have chosen our conventions to match left rather than right (we are using $\mathcal{L}(G)$, for example), so we have to work to write down an explicit operator intertwining $R^G$ and $\operatorname{Ind} R^H$.

**Lemma 2.4.** *The operator $Q : L^2(G) \to \mathcal{H}_{\operatorname{Ind} R^H}$ defined by*

$$(Q\phi(x))(h) = \Delta(x)^{-1/2}\phi(hx^{-1})$$

*is unitary, and satisfies $QR^GQ^* = \operatorname{Ind} R^H$.*

**Proof.** Two routine calculations show that for $\phi \in L^2(G)$, we have

$$(Q\phi)(xh)(k) = \delta(h)^{1/2}\Delta(h)^{-1/2}R^H_{h^{-1}}(Q\phi(x))(k),$$

and that $\|Q\phi\|^2 = \|\phi\|^2$: indeed, for the latter, we have

$$\begin{aligned}
(Q\phi|Q\phi) &= \int_{G/H}\int_H \rho(x)^{-1}\Delta(x)^{-1}|\phi(hx^{-1})|^2\,dh\,d\rho^\sharp(xH)\\
&= \int_{G/H}\int_H \rho(x)^{-1}\Delta(x)^{-1}\delta(h)^{-1}|\phi(h^{-1}x^{-1})|^2\,dh\,d\rho^\sharp(xH)\\
&= \int_{G/H}\int_H \rho(xh)^{-1}\Delta(xh)^{-1}|\phi(h^{-1}x^{-1})|^2\,dh\,d\rho^\sharp(xH)\\
&= \int_G \Delta(x)^{-1}|\phi(x^{-1})|^2\,dx = \|\phi\|^2,
\end{aligned}$$

which shows that $Q\phi \in \mathcal{H}_{\operatorname{Ind} R^H}$, that $Q$ is isometric, and, by reversing the calculation, that for any measurable function $\phi$ on $G$,

$$\|\phi\|^2 = \int_{G/H}\int_H \rho(x)^{-1}\Delta(x)^{-1}|\phi(hx^{-1})|^2\,dh\,d\rho^\sharp(xH). \tag{2.1}$$

Thus to see that $Q$ is unitary, we need to show it has dense range: we show that if $\psi \in C_c(G)$ and $\xi \in C_c(H) \subset L^2(H)$, then $\psi \odot \xi$ belongs to the range of $Q$. For we can define a continuous function $\phi$ on $G$ by

$$\phi(x) = \Delta(x)^{-1/2}(\psi \odot \xi)(x^{-1})(e) = \Delta(x)^{-1/2}\int_H \psi(x^{-1}k)\Delta(k)^{1/2}\xi(k)\,dk,$$

and then

$$\begin{aligned}
\Delta(x)^{-1/2}\phi(hx^{-1}) &= \Delta(h)^{-1/2}\int \psi(xh^{-1}k)\Delta(k)^{1/2}\xi(k)\,dk\\
&= \Delta(h)^{-1/2}\int \psi(xl)\Delta(hl)^{1/2}\xi(hl)\,dl\\
&= \int \psi(xl)\Delta(l)^{1/2}\delta(l)^{-1/2}\big(R_l^H\xi\big)(h)\,dl\\
&= (\psi \odot \xi)(x)(h).
\end{aligned} \tag{2.2}$$

Plugging this into (2.1) gives

$$\|\phi\|_2^2 = \int_{G/H} \rho(x)^{-1}\|\psi \odot \xi(x)\|^2\,\rho^\sharp(xH) = \|\psi \odot \xi\|^2,$$

so that $\phi \in L^2(G)$, and (2.2) says $Q\phi = \psi \odot \xi$. Since $\{\psi \odot \xi\}$ is dense in $\mathcal{H}_{\operatorname{Ind} R^H}$ by Lemma 2.2, we deduce that $Q$ is unitary, as claimed.

Finally, we check that $QR_x^G = (\operatorname{Ind} R^H)_x Q$: for $\phi \in L^2(G)$, we have

$$\begin{aligned}
\big((\operatorname{Ind} R^H)_x Q\phi\big)(y)(h) &= \Delta(x^{-1}y)^{-1/2}\phi(hy^{-1}x)\\
&= \Delta(y)^{-1/2}\big[\Delta(x)^{1/2}\phi\big((hy^{-1})x\big)\big]\\
&= \Delta(y)^{-1/2}\big(R_x^G\phi\big)(hy^{-1})\\
&= \big(QR_x^G\phi\big)(y)(h).
\end{aligned}$$

## §3. The twisted dual-group algebras of inflated cocycles.

One way to construct cocycles on a group $G$ is to start with a cocycle on a quotient $G/N$, and inflate it to a cocycle on $G$ by composing with the quotient map $q \times q$ of $G \times G$ onto $G/N \times G/N$. Correspondingly, we can construct dual cocycles on $G$ from ones on a subgroup $H$. For this, we shall need to know that $\mathcal{L}(H)$ embeds in $\mathcal{L}(G)$, just as $L^\infty(G/N)$ embeds via $q^*$ in $L^\infty(G)$; after we have this, it is quite easy to inflate dual cocycles (Proposition 3.2). We shall then prove our main theorem.

**Lemma 3.1.** (C. Herz [9]) *Suppose $G$ is a second countable locally compact group and $H$ a closed subgroup. Then there is a unique normal $^*$-monomorphism $I$ of $\mathcal{L}(H)$ into $\mathcal{L}(G)$ such that $I(L_h^H) = L_h^G$ for $h \in H$. Indeed, if $c : H \setminus G \to G$ is a Borel section, there is a unitary operator $V$ on $L^2(G \times H)$ such that*

$$V\xi(s,h) = \xi(h^{-1}c(Hs), sc(Hs)^{-1}h),$$

*and then $I$ satisfies*

$$V(1 \otimes T)V^* = I(T) \otimes 1 \text{ for } T \in \mathcal{L}(H). \tag{3.1}$$

*Dually, restricting functions to $H$ is a norm-decreasing homomorphism of $A(G) = \mathcal{L}(G)_*$ onto $A(H) = \mathcal{L}(H)_*$ such that*

$$\langle I(T), f\rangle = \langle T, f\big|_H\rangle \text{ for } T \in \mathcal{L}(H), f \in A(G).$$

**Proof.** (This is basically Herz' proof.) We first note that $I \otimes L^H$, $L^G \otimes I$ extend to faithful normal representations of $\mathcal{L}(H)$ and $\mathcal{L}(G)$ on $L^2(G \times H)$, so (3.1) will uniquely determine a normal homomorphism $I$ with the required properties if

$$V(1 \otimes L_h^H)V^* = L_h^G \otimes 1 \text{ for } h \in H.$$

But this is an easy calculation, and the first part follows.

For the second part, we just need to observe that for $f \in A(G)$, the functional $T \mapsto \langle I(T), f\rangle$ is weak$^*$ continuous, and hence lies in $\mathcal{L}(H)_*$. Since $\langle I(L_h^H), f\rangle = \langle L_h^G, f\rangle = f(h)$, the element of $A(H)$ corresponding to this functional is $f\big|_H$, and since $I$ is norm-preserving, we have

$$\|f\big|_H\|_{A(H)} = \|f\big|_H\|_{\mathcal{L}(H)_*} \le \|f\|_{\mathcal{L}(G)_*} = \|f\|_{A(G)}.$$

The surjectivity of $f \to f\big|_H$ follows from the Hahn-Banach Theorem because $I$ embeds $\mathcal{L}(H)$ isometrically as a weak$^*$ closed subspace of $\mathcal{L}(G)$.

**Proposition 3.2.** *Let $H$ be a closed subgroup of a second countable group $G$, and suppose $\omega$ is a dual cocycle on $H$. Then $I \otimes I(\omega)$ is a dual cocycle on $G$. If $\omega$ is normalised, so is $I \otimes I(\omega)$.*

**Proof.** For the first part, we just need to observe that because both $\delta_G \circ I$ and $(I \otimes I) \circ \delta_H$ are normal extensions of the map $L_h^H \to L_h^G \otimes L_h^G$, they agree on $\mathcal{L}(H)$, and thus the cocycle identities

$$I \otimes I(\omega)_{12}\, \delta_G \otimes i(I \otimes I(\omega)) = I \otimes I(\omega)_{23}\, i \otimes \delta_G(I \otimes I(\omega))$$

follows easily from the corresponding property of $\omega$. Now suppose that $W$ is an $\omega$-corepresentation of $H$ on $\mathcal{H}$. We claim that $U = i \otimes I(W)$ is then an $I \otimes I(\omega)$-corepresentation of $G$ on $\mathcal{H}$ satisfying (1.9). First of all, the identity

$$U_{12}U_{13} = i \otimes \delta_G(U) I \otimes I(\omega)^*_{23}$$

follows easily from the corresponding property of $W$ and the equation $\delta_G \circ I = (I \otimes I) \circ \delta_H$. For the normalisation condition, we recall that $I$ is characterised by $I(T) \otimes 1 = V(1 \otimes T)V^*$, and have a quick look at the definition of $V$ to see that $\overline{V^*\xi} = V^*\overline{\xi}$. Then for $h, k \in \mathcal{H}$, $\xi, \eta \in L^2(G)$ and $\alpha, \beta \in L^2(H)$, an application of Lemma 1.9 gives

$$\begin{aligned}
\big(U(h \otimes \overline{\eta})\big|k \otimes \overline{\xi}\big)(\alpha|\beta) &= \big(U_{12}(h \otimes \overline{\eta} \otimes \overline{\beta})\big|k \otimes \overline{\xi} \otimes \overline{\alpha}\big) \\
&= \big(W_{13}(h \otimes V^*(\overline{\eta} \otimes \overline{\beta}))\big|k \otimes V^*(\overline{\xi} \otimes \overline{\alpha})\big) \\
&= \big(W_{13}(h \otimes \overline{V^*(\eta \otimes \beta)})\big|k \otimes \overline{V^*(\xi \otimes \alpha)}\big) \\
&= \big(W^*_{13}(h \otimes V^*(\xi \otimes \alpha))\big|k \otimes V^*(\eta \otimes \beta)\big) \\
&= \big(U^*(h \otimes \xi)|k \otimes \eta\big)(\alpha|\beta),
\end{aligned}$$

which proves that $U$ satisfies (1.9), and $\omega$ is normalised.

**Theorem 3.3.** *Let $H$ be a closed subgroup of a locally compact group $G$, $\omega$ a normalised dual cocycle on $H$, and $\operatorname{Inf}\omega = I \otimes I(\omega)$ its inflation to a normalised dual cocycle on $G$. Let $\alpha^H$ denote the action of $H$ by left translation on $A(H, \omega)$ and $C_r^*(\widehat{H}, \omega)$, as in Proposition 1.17. For $f \in A(G, \operatorname{Inf}\omega)$, define $\Phi(f) : G \to A(H)$ by $\Phi(f)(x) = \alpha^G_{x^{-1}}(f)\big|_H$. Then $\Phi$ extends to an equivariant isomorphism of $\big(C_r^*(\widehat{G}, \operatorname{Inf}\omega), G, \alpha^G\big)$ onto $\big(\operatorname{Ind}_H^G\big(C_r^*(\widehat{H}, \omega), \alpha^H\big), G, \tau\big)$.*

Most of this section is devoted to the proof of this theorem. We begin by noting from Lemma 3.1 that the restriction $\alpha^G_{x^{-1}}(f)\big|_H$ lies in $A(H)$, and hence in $C_r^*(\widehat{H}, \omega)$. The map $\Phi(f) : G \to C_r^*(\widehat{H}, \omega)$ is continuous because

$$\big\|\Phi(f)(x)\big\| \leq \big\|\alpha^G_{x^{-1}}(f)\big|_H\big\|_{A(H)} \leq \big\|\alpha^G_{x^{-1}}(f)\big\|_{A(G)} \tag{3.2}$$

(using Lemma 3.1), and translation is continuous on $A(G)$. If $h \in H$, then

$$\Phi(f)(xh) = \alpha^G_{h^{-1}x^{-1}}(f)\big|_H = \alpha^H_{h^{-1}}\big(\alpha^G_{x^{-1}}(f)\big|_H\big) = \alpha^H_{h^{-1}}(\Phi(f)(x)).$$

Since $A_c(G)$ is dense in $A(G)$, and $xH \mapsto \|\Phi(f)(x)\|$ has compact support when $f$ does, the inequality (3.2) shows that $xH \mapsto \|\Phi(f)(x)\|$ lies in $C_0(G/H)$ for all $f \in A(G)$. Thus $\Phi(f)$ does belong to $\operatorname{Ind}_H^G\big(C_r^*(\widehat{H},\omega)\big)$ for every $f \in A(G)$.

It is clear that $\Phi$ is *-linear, so we need to check multiplicativity. We compute:

$$\begin{aligned}
\big(\Phi(f)(x) *_\omega \Phi(g)(x)\big)(h) &= \langle L_h^H, \alpha_x^{-1}(f)\big|_H *_\omega \alpha_x^{-1}(g)\big|_H\rangle \\
&= \langle (L_h^H \otimes L_h^H)\omega^*, \alpha_x^{-1}(f)\big|_H \otimes \alpha_x^{-1}(g)\big|_H\rangle \\
&= \langle I \otimes I((L_h^H \otimes L_h^H)\omega^*), \alpha_x^{-1}(f) \otimes \alpha_x^{-1}(g)\rangle \\
&= \langle (L_x^G \otimes L_x^G)(L_h^G \otimes L_h^G)(\operatorname{Inf}\omega)^*, f \otimes g\rangle \\
&= \langle L_x^G L_h^G, f *_{\operatorname{Inf}\omega} g\rangle \\
&= \langle I(L_h^H), \alpha_{x^{-1}}(f *_{\operatorname{Inf}\omega} g)\rangle \\
&= \langle L_h^H, \alpha_{x^{-1}}(f *_{\operatorname{Inf}\omega} g)\big|_H\rangle \\
&= \big(\Phi(f *_{\operatorname{Inf}\omega} g)(x)\big)(h).
\end{aligned}$$

We shall complete the proof of Theorem 3.3 by showing that the homomorphism $\Phi$ has dense range and is isometric.

By Lemma 2.1, $\Phi$ will have dense range if the element $f \odot \phi$ has the form $\Phi(g)$ whenever $f \in A_c(G)$ and $\phi \in A_c(H)$. Given such $f$ and $\phi$, we define

$$g(x) = \int_H f(xh)\phi(h^{-1})\,dh;$$

since $h \mapsto f(\cdot h)\phi(h^{-1})$ is a continuous, compactly supported function with values in $A_c(G)$, $g$ belongs to $A_c(G)$. For $k \in H$ we have

$$\alpha_x^{-1}(g)(k) = \int_H f(xkh)\phi(h^{-1})\,dh = \int_H f(xh)\phi(h^{-1}k)\,dh,$$

so that $\alpha_x^{-1}(g)\big|_H$ is given by the $A(H)$–valued integral $\int_H f(xh)\alpha_h^H(\phi)\,dh$ defining $(f \odot \phi)(x)$. Thus $\Phi(g) = f \odot \phi$, and the range of $\Phi$ is dense.

We shall prove that $\Phi$ is isometric by constructing a faithful representation $\pi$ of $\operatorname{Ind} C_r^*(\widehat{H},\omega)$ such that $\pi \circ \Phi$ is equivalent to the regular representation $\mu^G$ of $C_r^*(\widehat{G},\operatorname{Inf}\omega)$. The representation $\pi$ is the one induced from the regular representation $\mu^H$ of $C_r^*(\widehat{H},\omega)$: since $(\mu^H, R^H)$ is covariant (Proposition 1.17), Proposition 2.3 implies that $\big(\widetilde{\mu^H}, \operatorname{Ind}_H^G R^H\big)$ is a faithful covariant representation of $\big(\operatorname{Ind} C_r^*(\widehat{H},\omega), G, \tau\big)$. Because $Q^*(\operatorname{Ind} R^H)Q = R^G$, and we know $(\mu^G, R^G)$ is covariant, it is natural to guess that $Q^*(\widetilde{\mu^H} \circ \Phi)Q = \mu^G$, and this is precisely what happens. We need a technical lemma.

**Lemma 3.4.** *For $T \in \mathcal{L}(H) \otimes \mathcal{L}(H)$ and $\xi, \eta, \phi, \psi \in C_c(G)$, we have*

$$\begin{aligned}\int_{G/H} \rho(x)^{-1} \Big( i \otimes I(T)(Q\xi(x) \otimes \phi) \big| i \otimes I(W_H)(Q\eta(x) \otimes L_{x^{-1}}\psi) \Big) \, d\rho^\sharp(xH) \\ = \Big( I \otimes I(T)(\xi \otimes \phi) \big| W_G(\eta \otimes \psi) \Big).\end{aligned} \tag{3.3}$$

**Proof.** We first claim that it is enough to prove the equality for $T = L_a^H \otimes L_b^H$, where $a, b \in H$. The right-hand side is normal in $T$ because $I \otimes I$ is, and we have to show that the same is true of the left-hand side. The integrand $f(T, x)$, say, is normal in $T$ for each fixed $x$, and

$$\begin{aligned}|f(T,x)| &\leq \rho(x)^{-1} \|T\| \, \|Q\xi(x) \otimes \phi\| \, \|Q\eta(x) \otimes \psi\| \\ &\leq \rho(x)^{-1} \|T\| \|\phi\| \, \|\psi\| \, \|Q\xi(x)\| \, \|Q\eta(x)\|.\end{aligned} \tag{3.4}$$

Because $Q\xi$ and $Q\eta$ are in $\mathcal{H}_{\operatorname{Ind} R}$, both $\rho(x)^{-1/2}\|Q\xi(x)\|$ and $\rho(x)^{-1/2}\|Q\eta(x)\|$ are in $L^2(d\rho^\sharp)$, and the right-hand side of (3.4) is integrable. Now the Dominated Convergence Theorem implies that the left-hand side of (3.3) is normal in $T$.

So we suppose $T = L_a^H \otimes L_b^H$. Then the left-hand side of (3.3) is

$$\begin{aligned}\int_{G/H} \int_G \int_H \rho(x)^{-1} \big(L_a^H \otimes L_b^G\big)(Q\xi(x) \otimes \phi)(h, y) \overline{(Q\eta(x) \otimes L_{x^{-1}}\psi)(h, h^{-1}y)} \, dh \, dy \, d\rho^\sharp(xH) \\ = \iiint \frac{\Delta(x^{-1})}{\rho(x)} \xi(a^{-1}hx^{-1}) \phi(b^{-1}y) \overline{\eta(hx^{-1}) \psi(xh^{-1}y)} \, dh \, dy \, d\rho^\sharp(xH) \\ = \int_{G/H} \int_H \frac{\Delta(x^{-1})}{\rho(x)} F(hx^{-1}) \, dh \, d\rho^\sharp(xH),\end{aligned}$$

where

$$F(z) = \xi(a^{-1}z)\overline{\eta(z)} \int_G \phi(b^{-1}y)\overline{\psi(z^{-1}y)} \, dy.$$

Thus the left-hand side of (3.3) is

$$\begin{aligned}\int_{G/H} \int_H & \frac{\Delta(x^{-1})\delta(h^{-1})}{\rho(x)} F(h^{-1}x^{-1}) \, dh \, d\rho^\sharp(xH) \\ &= \int_{G/H} \int_H \rho(xh)^{-1} \Delta(h^{-1}x^{-1}) F(h^{-1}x^{-1}) \, dh \, d\rho^\sharp(xH) \\ &= \int_G \rho(x) \Delta(x^{-1}) F(x^{-1}) \rho(x)^{-1} \, dx \\ &= \int_G F(x) \, dx \\ &= \int_G \int_G \xi(a^{-1}x)\overline{\eta(x)} \phi(b^{-1}y) \overline{\psi(x^{-1}y)} \, dx \, dy \\ &= \Big( L_a^G \otimes L_b^G(\xi \otimes \phi) \big| W_G(\eta \otimes \psi) \Big) \\ &= \Big( I \otimes I\big(L_a^H \otimes L_b^H\big)(\xi \otimes \phi) \big| W_G(\eta \otimes \psi) \Big),\end{aligned}$$

as required.

As foreshadowed above, we now prove that

$$Q^*(\widetilde{\mu^H} \circ \Phi)Q = \mu^G. \tag{3.5}$$

In the proof of Lemma 2.4 we showed that $Q$ is unitary, so it is enough to check that, for $f \in A(G)$ and $\xi, \eta \in L^2(G)$, we have

$$\left(\widetilde{\mu^H}(\Phi(f))Q\xi \middle| Q\eta\right) = \left(\mu^G(f)\xi \middle| \eta\right). \tag{3.6}$$

The left-hand side of (3.6) is

$$\begin{aligned}
\int_{G/H} & \rho(x)^{-1}\left(\mu^H\left(\Phi(f)(x)\right)Q\xi(x)\middle|Q\eta(x)\right)\,d\rho^\sharp(xH) \\
&= \int_{G/H} \rho(x)^{-1}\left(\mu^H\left(\alpha^G_{x^{-1}}(f)\big|_H\right)Q\xi(x)\middle|Q\eta(x)\right)\,d\rho^\sharp(xH) \\
&= \int_{G/H} \rho(x)^{-1}\left\langle W_H^*\Sigma(\omega^*), \omega_{Q\xi(x),Q\eta(x)} \otimes \alpha^G_{x^{-1}}(f)\big|_H\right\rangle\,\rho^\sharp(xH) \\
&= \int_{G/H} \rho(x)^{-1}\left\langle i \otimes I\left(W_H^*\Sigma(\omega^*)\right), \omega_{Q\xi(x),Q\eta(x)} \otimes \alpha^G_{x^{-1}}(f)\right\rangle\,\rho^\sharp(xH),
\end{aligned}$$

because restriction : $A(G) \to A(H)$ is dual to $I$. If $f \in A(G)$ is the functional $\omega_{\phi,\psi} : T \mapsto (T\phi|\psi)$, then a calculation on elements of the form $L_y$ shows that $\alpha_{x^{-1}}(f)$ is the functional $T \mapsto (T\phi|L_{x^{-1}}\psi)$, and the left-hand side of (3.6) is

$$\int_{G/H} \rho(x)^{-1}\left(i \otimes I\left(W_H^*\Sigma(\omega^*)\right)\left(Q\xi(x) \otimes \phi\right)\middle|Q\eta(x) \otimes L^G_{x^{-1}}\psi\right)\,d\rho^\sharp(xH).$$

By Lemma 3.4, this is

$$\begin{aligned}
\left(I \otimes I(\Sigma(\omega^*))(\xi \otimes \phi)\middle|W_G(\eta \otimes \psi)\right) &= \left(W_G^*\Sigma(\operatorname{Inf}\omega^*)(\xi \otimes \phi)\middle|(\eta \otimes \psi)\right) \\
&= \left\langle W_G^*\Sigma(\operatorname{Inf}\omega^*), \omega_{\xi,\eta} \otimes \omega_{\phi,\psi}\right\rangle \\
&= \left(S_f\left(W_G^*\Sigma(\operatorname{Inf}\omega^*)\right)\xi\middle|\eta\right) \\
&= \left(\mu^G(f)\xi\middle|\eta\right),
\end{aligned}$$

and we have proved (3.6), hence also (3.5).

From Equation (3.5) we deduce that $\Phi$ is isometric for the norm on $C_r^*(\widehat{G}, \operatorname{Inf}\omega)$, which is pulled back from the norm on $B(L^2(G))$ via $\mu^G$, and the norm on $\operatorname{Ind} C_r^*(\widehat{H}, \omega)$ induced from the norm on $C_r^*(\widehat{H}, \omega)$. Since Proposition 1.17 says that $(\mu^G, R^G)$ is a covariant representation of $\big(C_r^*(\widehat{G}, \operatorname{Inf}\omega), G, \alpha^G\big)$, Proposition 2.3 that $(\widetilde{\mu^H}, \operatorname{Ind} R^H)$ is a covariant representation of $\big(\operatorname{Ind} C_r^*(\widehat{H}, \omega), G, \tau\big)$, and Lemma 2.4 that $Q^*(\operatorname{Ind} R^H)Q = R^G$, Equation (3.5) also implies that the isomorphism $\Phi$ is equivariant. This completes the proof of Theorem 3.3.

**Remark 3.5.** If we start with an abelian group $G$ and a cocycle on $G$ inflated from a cocycle $\sigma$ on a quotient $G/H$, we can apply the theorem to the dual group $\widehat{G}$. We first have to note that the cocycle $\sigma$ is equivalent to a normalised one, and then let $\omega$ be the corresponding dual cocycle on $(G/H)^\frown = H^\perp$. Since the Fourier transform carries the embedding $I : \mathcal{L}(H^\perp) \to \mathcal{L}(G)$ into the map $q^* : f \mapsto f \circ q$ of $L^\infty(G/H)$ into $L^\infty(G)$, the inflated cocycle $\operatorname{Inf}\sigma$ on $G$ corresponds to the inflated dual cocycle $\operatorname{Inf}\omega$ on $\widehat{G}$. Thus from Theorem 3.3 and Lemma 1.16 we deduce

$$C^*(G, \operatorname{Inf}\sigma) \cong C_r^*(\widehat{\widehat{G}}, \operatorname{Inf}\omega) \cong \operatorname{Ind}_{H^\perp}^{\widehat{G}}\big(C_r^*(\widehat{H^\perp}, \omega), \alpha^{H^\perp}\big) \cong \operatorname{Ind}_{H^\perp}^{\widehat{G}}\big(C^*(G/H, \sigma), \widehat{\alpha}\big). \quad (3.7)$$

This result about abelian groups contains a good deal of known information about the twisted group algebras of abelian groups. For example, since by [2, Theorem 3.1] every cocycle on $G$ is equivalent to one inflated from a totally skew cocycle $\sigma$ on a quotient $G/H$, and $\operatorname{Prim} C^*(G/H, \sigma)$ is then trivial [2], we deduce that $\operatorname{Prim} C^*(G, \operatorname{Inf}\sigma) \cong H^\perp$. This is a theorem of Green [8, Proposition 34] in general, and Baggett-Kleppner in the type I case [2, p.310]. Indeed, the isomorphism (3.7) may itself be known: the second author vaguely remembers being told something of the sort by Dorte Olesen in 1986. We do not know where a proof of (3.7) has previously been written down, although a stable isomorphism could be easily obtained by converting $C^*(G/H, \sigma) \otimes \mathcal{K}$ into a crossed product $\mathcal{K} \times G/H$, as in [16, §3], and applying [15, Theorem 2.4]. It might be interesting to know if there is a similar stabilisation trick which will realise $C_r^*(\widehat{G}, \omega) \otimes \mathcal{K}$ as a crossed product of $\mathcal{K}$ by a coaction of $G$; if so, the analogue of [15, Theorem 2.4] in [18] would presumably give a stable version of Theorem 3.3.

**Examples 3.6.** **(1)** Let $G$ be the real Heisenberg group, that is, $G = \mathbf{R}^3$ with multiplication given by

$$(x_1, y_1, z_1) = (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2 + y_1 x_2).$$

We fix a pair of real numbers $\mu, \nu$, and set

$$H_{\mu,\nu} = \big\{(\mu s, \nu s, t) : s, t \in \mathbf{R}\big\};$$

then $H_{\mu,\nu}$ is a closed abelian subgroup of $G$, isomorphic to $\mathbf{R}^2$ via $(\mu s, \nu s, t) \mapsto (s, t - \mu\nu s^2/2)$. (Indeed, $H_{\mu,\nu}$ is normal, and every proper normal, connected subgroup of $G$ is of this form.) We can therefore take $\omega$ to be the dual cocycle on $H_{\mu,\nu}$ corresponding to the normalised cocycle $\sigma_\theta$ on $\mathbf{R}^2$ given by

$$\sigma_\theta\big((s_1,t_1),(s_2,t_2)\big) = \exp 2\pi i\theta(t_1 s_2 - s_1 t_2),$$

and our theorem says

$$C^*(\widehat{G}, \operatorname{Inf}\omega) \cong \operatorname{Ind}_{H_{\mu,\nu}}^G \big(C^*(\widehat{H}_{\mu,\nu},\omega),\alpha^H\big) \cong \operatorname{Ind}_{H_{\mu,\nu}}^G \big(C^*(\mathbf{R}^2,\sigma_\theta),\widehat{\alpha}\big),$$

where $\widehat{\alpha}$ is the dual action of $H_{\mu,\nu} \cong \mathbf{R}^2$ on $C^*(\mathbf{R}^2,\sigma_\theta)$. Provided $\theta \neq 0$, $C^*(\mathbf{R}^2,\sigma_\theta) \cong \mathcal{K}(L^2(\mathbf{R}))$, and $\operatorname{Ind}_{H_{\mu,\nu}}^G C^*(\mathbf{R}^2,\sigma_\theta)$ is a separable continuous-trace $C^*$-algebra with spectrum $G/H_{\mu,\nu}$, and primitive quotients isomorphic to $\mathcal{K}(L^2(\mathbf{R}))$ [20]. Since the map $(x,y,z) \mapsto \nu x - \mu y$ induces an isomorphism of $G/H_{\mu,\nu}$ onto $\mathbf{R}$, and $\mathbf{R}$ is finite-dimensional with $H^3(\mathbf{R},\mathbf{Z}) = 0$, it follows from [4], [17, 1.12] and the Dixmier-Douady Theorem [3, Chap. 10] that the algebra $\operatorname{Ind}_{H_{\mu,\nu}}^G C^*(\mathbf{R}^2,\sigma_\theta)$ is isomorphic to $C_0\big(\mathbf{R},\mathcal{K}(L^2(\mathbf{R}))\big)$.

**(2)** Let $G$ be the quotient of the real Heisenberg group by the central copy of $\mathbf{Z}$, so that $G = \mathbf{R}^2 \times \mathbf{T}$ with multiplication given by

$$(x_1,y_1,z_1)(x_2,y_2,z_2) = \big(x_1+x_2, y_1+y_2, z_1 z_2 \exp 2\pi i y_1 x_2\big),$$

and let $H$ be the quotient of $H_{\mu,\nu}$ by the same copy of $\mathbf{Z}$. So now $H \cong \mathbf{R}\times\mathbf{T}$, and we can take $\omega_\theta$ to be the dual cocycle on $H$ corresponding to the cocycle $\sigma_\theta$ on $\widehat{H} = \mathbf{R}\times\mathbf{Z}$ defined by

$$\sigma_\theta\big((s_1,n_1),(s_2,n_2)\big) = \exp 2\pi i\theta(n_1 s_2 - s_1 n_2).$$

This dual cocycle is also inflated from the subgroup $\theta^{-1}\mathbf{Z}\times\mathbf{T}$, where the corresponding cocycle $\tau$ on $\big(\theta^{-1}\mathbf{Z}\times\mathbf{T}\big)^\wedge \cong \mathbf{T}\times\mathbf{Z}$ has $C^*(\mathbf{T}\times\mathbf{Z},\tau) \cong \mathcal{K}(L^2(\mathbf{T}))$; if we apply the analysis of the preceding example to this subgroup, we find that

$$C^*_r(\widehat{G},\operatorname{Inf}\omega_\theta) \cong \operatorname{Ind}_{\theta^{-1}\mathbf{Z}\times\mathbf{T}}^G \mathcal{K} \cong C_0(\mathbf{R}\times\mathbf{T},\mathcal{K}).$$

However, if we want to view $\omega_\theta$ as a one parameter family of dual cocycles, it makes more sense to apply Theorem 3.3 to the fixed subgroup $H$, and then we get

$$C^*_r(\widehat{G},\operatorname{Inf}\omega_\theta) \cong \operatorname{Ind}_H^G C^*(\widehat{H},\sigma_\theta) \cong \operatorname{Ind}_H^G C^*(\mathbf{R}\times\mathbf{Z},\sigma_\theta).$$

**(3)** Let $G = U_2(\mathbf{C})$, and take $H$ to be the diagonal subgroup

$$H = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} : |\lambda| = |\mu| = 1 \right\} \cong \mathbf{T}^2.$$

The one-parameter family $\sigma_\theta$ of cocycles on $\mathbf{T}^2 = \mathbf{Z}^2$ given by

$$\sigma_\theta\big((m_1, n_1), (m_2, n_2)\big) = \exp 2\pi i\theta(n_1 m_2 - m_1 n_2)$$

gives a one-parameter family $\omega_\theta$ of dual cocycles on $U_2(\mathbf{C})$. Theorem 3.3 describes the dual-group algebra $C_r^*(\widehat{U}_2, \operatorname{Inf}\omega_\theta)$ as $\operatorname{Ind}_H^{U_2} A_{2\theta}$, where $A_{2\theta} \cong C^*(\mathbf{Z}^2, \sigma_\theta)$ is the usual rotation algebra. (The factor of 2 comes in because we are using normalised cocycles; see §4.1 below.) So for irrational $\theta$, this algebra has primitive ideal space $U_2/H$, but is definitely not type I.

## §4. Explicit formulas for dual cocycles inflated from abelian subgroups.

The results of the previous sections show that we can construct a normalised dual cocycle on $G$ from a normalised cocycle $\sigma$ on the dual $\widehat{H}$ of an abelian subgroup $H$, by first Fourier transforming $\sigma$ to give a dual cocycle $\omega = \widehat{\sigma}$ on $\widehat{H}$, and then inflating $\omega$ to a dual cocycle $\operatorname{Inf}\omega$ on $G$. So far, this is the only known method of producing dual cocycles, and there is considerable evidence that they all arise this way. Indeed, the longstanding conjecture of [10] that a compact simple group $G$ cannot act ergodically on an infinite-dimensional factor is by [23, 12] equivalent to showing that all dual cocycles are inflated from abelian subgroups. This is difficult stuff, though: Wassermann had to work very hard to establish this conjecture for $G = SU(2)$ [24] and other low-dimensional compact Lie groups [25]. In an appendix, we show that a dual cocycle on a connected Lie group which lies on a one-parameter group of dual cocycles with a reasonable infinitesimal generator, is necessarily inflated from an abelian subgroup.

So it makes sense to look in more detail at the stucture of the algebras $A(G, \operatorname{Inf}_H^G\omega)$ and $C_r^*(\widehat{G}, \operatorname{Inf}_H^G\omega)$ — and since we know a great deal about 2-cocycles on abelian groups, we can obtain quite explicit formulas in this special case. Virtually every 2-cocycle on a locally compact abelian group $\widehat{H}$ is equivalent to a continuous skew-symmetric bicharacter on $\widehat{H}\times\widehat{H}$, and all are if $s \mapsto 2s$ is an automorphism of $\widehat{H}$ [11, p.33], or if $\widehat{H} = \mathbf{R}^p \times \mathbf{Z}^q$. The continuous bicharacters $\sigma$ on $\widehat{H}\times\widehat{H}$ are in one-to-one correspondence with the continuous homomorphisms $\pi : s \mapsto \sigma(s, \cdot)$ of $\widehat{H}$ into $H = H^{\widehat{\ }}$ [11, 1.1], and the skew-symmetry of $\sigma$ is then equivalent to

$$\langle \pi(s), s\rangle = 1 \quad \text{for all } s \in \widehat{H}. \tag{4.1}$$

We shall say that a continuous homomorphism $\pi : \widehat{H} \to H$ is *skew* if it satisfies (4.1). Note that (4.1) implies

$$\langle \pi(s), t\rangle = \overline{\langle \pi(t), s\rangle} \quad \text{for all } s, t \in \widehat{H};$$

to see this, expand the identity $\langle \pi(s + t), s + t\rangle = 1$.

**Examples 4.1.** **(1)** For $H = \mathbf{T}^2$ and $\theta \in \mathbf{R}$, we define $\pi_\theta : \widehat{H} = \mathbf{Z} \times \mathbf{Z} \to H = \mathbf{T} \times \mathbf{T}$ by

$$\pi(m,n) = \big(\exp 2\pi i n\theta, \exp(-2\pi i m\theta)\big).$$

The corresponding cocycle on $\mathbf{Z}^2 = \widehat{H}$ is then given by

$$\sigma_\theta\big((m,n),(p,q)\big) = \exp 2\pi i\theta(np - mq).$$

These are not all inequivalent: a quick calculation shows that if $\rho(m,n) = i^{m^2-n^2}$, then

$$\begin{aligned}\sigma_{\theta+\frac{1}{2}}\big((m,n),(p,q)\big) &= \big((d\rho)\sigma_\theta\big)\big((m,n),(p,q)\big)\\ &:= \frac{\rho(m,n)\rho(p,q)}{\rho(m+p,n+q)}\sigma_\theta\big((m,n),(p,q)\big).\end{aligned}$$

Thus the family $\big\{\sigma_\theta : \theta \in [0,\frac{1}{2})\big\}$ parametrises $H^2(\widehat{H},\mathbf{T})$, but the map $\theta \mapsto \sigma_\theta$ does not provide a canonical realisation of the 2-cocycles as skew-symmetric bicharacters. (The multiplier $\sigma_\theta$ is equivalent to the more common $\omega_{2\theta}\big((m,n),(p,q)\big) := \exp 2\pi i(2\theta)np$: take $\rho(m,n) = \exp 2\pi i\theta mn$, and then $\sigma_\theta = (d\rho)\omega_{2\theta}$. We do not use the representatives $\omega_\theta$ for $H^2$ because they are not normalised.)

**(2)** Similar remarks apply to $H = \mathbf{T}^n$, $\widehat{H} = \mathbf{Z}^n$: the skew-symmetric bicharacters are given by

$$\sigma_A\big((m,n),(p,q)\big) = \exp 2\pi i(n^tAp - m^tAq),$$

where $A$ is a skew-symmetric matrix with real entries, and the class of $\sigma_A$ in $H^2(\mathbf{Z}^n,\mathbf{T})$ is unchanged if we add multiples of $\frac{1}{2}$ to any entry of $A$ [1].

**(3)** Let $G$ be the quotient of the real Heisenberg group by the central copy of $\mathbf{Z}$, as in Example 3.6(2), and let $H$ be the subgroup $H_{\mu,\nu} \cong \mathbf{R} \times \mathbf{T}$. Then, as in Example (1), the skew homomorphisms $\pi : \widehat{H} = \mathbf{R} \times \mathbf{Z} \to H = \mathbf{R} \times \mathbf{T}$ are given by

$$\pi_\theta\big((r,n)\big) = \big(n\theta, \exp(-2\pi i\theta r)\big)$$

for some $\theta \in \mathbf{R}$.

Suppose that $G$ is a locally compact group, $H$ is a closed abelian subgroup, $\pi : \widehat{H} \to H$ is a continuous skew homomorphism, and $\sigma$ is the cocycle on $\widehat{H}$ defined by

$$\sigma(s,t) = \langle \pi(s), t\rangle \quad \text{for } s,t \in \widehat{H}.$$

The corresponding dual cocycle $\omega$ on $H$ is given by

$$\langle \omega, \phi \otimes \psi\rangle = \int_{\widehat{H}}\int_{\widehat{H}} \sigma(s,t)\widehat{\phi}(s)\widehat{\psi}(t)\,ds\,dt \quad \text{for } \phi,\psi \in A(H),$$

where $\widehat{\phi}$ is by definition the unique element of $L^1(\widehat{H})$ whose inverse Fourier transform is $\phi$; if $\phi \in A(H) \cap L^1(H)$ (if $\phi \in A_c(H)$, for example), then $\widehat{\phi}$ is given by the usual formula

$$\widehat{\phi}(s) = \int_H \phi(h)\overline{\langle h, s\rangle}\, dh.$$

The multiplication on $A(G, \operatorname{Inf}\omega)$ is given by

$$\begin{aligned}
f *_{\operatorname{Inf}\omega} g(x) &= \langle L_x^G, f *_{\operatorname{Inf}\omega} g\rangle \\
&= \big\langle \big(L_x^G \otimes L_x^G\big)(\operatorname{Inf}\omega)^*, f \otimes g\big\rangle \\
&= \big\langle I \otimes I(\omega^*), \alpha_{x^{-1}}(f) \otimes \alpha_{x^{-1}}(g)\big\rangle \\
&= \big\langle \omega^*, \alpha_{x^{-1}}(f)\big|_H \otimes \alpha_{x^{-1}}(g)\big|_H\big\rangle \\
&= \int_{\widehat{H}}\int_{\widehat{H}} \overline{\sigma(s,t)}\, \widehat{f}(x,s)\widehat{g}(x,t)\, ds\, dt,
\end{aligned}$$

where now $\widehat{f}(x,\cdot)$ is the unique element of $L^1(\widehat{H})$ whose inverse Fourier transform is the function $h \mapsto f(xh)$; $\widehat{f}$ exists because the restriction $\alpha_{x^{-1}}(f)\big|_H$ is in $A(H)$, and if it is also in $L^1(H)$, then $\widehat{f}$ is the partial Fourier transform

$$\widehat{f}(x,s) = \int_H f(xh)\overline{\langle h, s\rangle}\, dh.$$

Thus we have

$$f *_{\operatorname{Inf}\omega} g(x) = \int_{\widehat{H}}\int_{\widehat{H}} \overline{\langle \pi(s), t\rangle}\, \widehat{f}(x,s)\widehat{g}(x,t)\, ds\, dt. \tag{4.2}$$

Since $\overline{\langle \pi(s), t\rangle} = \langle s, \pi(t)\rangle$, the inside integral is an inverse Fourier transform, and we can rewrite (4.2) as

$$f *_{\operatorname{Inf}\omega} g(x) = \int_{\widehat{H}} f(x\pi(t))\widehat{g}(x,t)\, dt.$$

Alternatively, we can use Fubini's Theorem to change the order of integration in (4.2), and obtain

$$f *_{\operatorname{Inf}\omega} g(x) = \int_{\widehat{H}} \widehat{f}(x,s) g(x\pi(-s))\, ds.$$

When the subgroup $H$ is a direct product $H_1 \times H_2$, as in our examples, the skew homomorphism $\pi : \widehat{H} \to H$ has the form

$$\pi(h_1, h_2) = \big(\pi_{11}(h_1)\pi_{12}(h_2), \pi_{21}(h_1)\pi_{22}(h_2)\big),$$

where $\pi_{ij} : \widehat{H}_j \to H_i$, $\pi_{11}$ and $\pi_{22}$ are skew, and

$$\langle \pi_{21}(s), t\rangle = \overline{\langle s, \pi_{12}(t)\rangle} = \langle s, \pi_{12}(-t)\rangle \quad \text{for } s \in \widehat{H}_1, t \in \widehat{H}_2; \tag{4.3}$$

we write $\pi_{21}(s) = \pi_{12}^t(-s)$ as shorthand for (4.3). In our examples, the diagonal terms $\pi_{ii}$ are absent, and $\pi$ is determined by a continuous homomorphism $\rho : \widehat{H}_2 \to H_1$ via

$$\pi(s_1, s_2) = \big(\rho(s_2), \rho^t(-s_1)\big).$$

For such $\pi$ the formula (4.2) can again be substantially simplified:

$$\begin{aligned} f *_{\mathrm{Inf}\,\omega} g(x) &= \int_{\widehat{H}_2} \int_{\widehat{H}_2} \int_{\widehat{H}_1} \int_{\widehat{H}_1} \langle \rho^t(s_1), t_2 \rangle \overline{\langle \rho(s_2), t_1 \rangle}\, \widehat{f}(x, s_1, s_2) \widehat{g}(x, t_1, t_2)\, ds_1\, dt_1\, ds_2\, dt_2 \\ &= \int_{\widehat{H}_2} \int_{\widehat{H}_2} \widehat{f}\big(x(\rho(t_2), 0), s_2\big) \widehat{g}\big(x(\rho(-s_2), 0), t_2\big)\, ds_2\, dt_2, \end{aligned}$$

where now $\widehat{f}(y, \cdot)$ is the unique function in $L^1(H_2)$ with inverse Fourier transform $h_2 \mapsto f(y(0, h_2))$. Notice that, by Fubini's Theorem, the integrand in the last integral is integrable because the integrand in the previous line is integrable in $(s_1, s_2, t_1, t_2)$.

For our final simplification, we supress all mention of $H_1$, by starting with an abelian subgroup $K$ of $G$ and a continuous homomorphism $\rho : \widehat{K} \to G$ such that $\rho(s)k = k\rho(s)$ for all $s \in \widehat{K}$ and $k \in K$; we can then take $H_2 = K$, $H_1 = \overline{\rho(\widehat{K})}$ and, for example, replacing $(\rho(t_2), 0)$ by $\rho(t_2)$. We sum up our calculations:

First, a convention. If $K$ is a closed abelian subgroup of $G$, and $f \in A(G)$, then we denote by $\widehat{f}(x, \cdot)$ the unique element of $L^1(\widehat{K})$ whose inverse Fourier transform is the function $\alpha_{x^{-1}}(f)\big|_K : k \mapsto f(xk)$. Our earlier comments about (3.2) show that the map $x \mapsto \alpha_{x^{-1}}(f)\big|_K$ is continuous from $G$ from $A(K) \cong L^1(\widehat{K})$. We therefore have:

**Lemma 4.2.** *If $f \in A(G)$ there is a function $\widehat{f} : G \times \widehat{K} \to \mathbf{C}$ such that $x \mapsto \widehat{f}(x, \cdot)$ is in $C_b(G, L^1(\widehat{K}))$, $k \mapsto f(xk)$ is the inverse Fourier transform of $s \mapsto \widehat{f}(x, s)$, and $\sup_x \|\widehat{f}(x, \cdot)\|_{L^1(\widehat{K})} \leq \|f\|_{A(G)}$. (If $\alpha_{x^{-1}}(f)\big|_K \in A(K) \cap L^1(K)$, then*

$$\widehat{f}(x, s) = \int_K f(xk) \overline{\langle k, s \rangle}\, dk.)$$

*In any case, we refer to $\widehat{f}(x, \cdot)$ as the partial Fourier transform of $f$ along $xK$.*

**Theorem 4.3.** *Suppose $G$ is a locally compact group, $K$ a closed abelian subgroup, and $\rho : \widehat{K} \to G$ a continuous homomorphism such that $\rho(s)k = k\rho(s)$ for all $s \in \widehat{K}, k \in K$. Then $H = \overline{\rho(\widehat{K})} \times K$ is a closed abelian subgroup of $G$; let $\omega$ be the normalised dual cocycle on $H$ corresponding to the normalised cocycle $\sigma$ on $\widehat{H}$ given by*

$$\sigma\big((s_1, s_2), (t_1, t_2)\big) = \langle \rho(s_2), t_1 \rangle \overline{\langle s_1, \rho(t_2) \rangle} \quad \text{for } (s_1, s_2), (t_1, t_2) \in \widehat{H}.$$

*Then the multiplication on $A(G, \operatorname{Inf}\omega)$ is given by*

$$f *_{\operatorname{Inf}\omega} g(x) = \int_{\widehat{K}} \int_{\widehat{K}} \widehat{f}(x\rho(t), s)\widehat{g}(x\rho(-s), t)\, ds\, dt,$$

*where $\widehat{f}(y, \cdot)$ denotes the partial Fourier transform of $f$ along $yK$, as above.*

**Example 4.4.** Return to the Heisenberg group example of 3.6(1), with $K$ the central copy of $\mathbf{R}$, and $\rho : \mathbf{R} \to G$ given by $\rho(t) = (\mu t, \nu t, \mu\nu t^2/2)$. The identity $\widehat{f}(xl, s) = \langle l, s\rangle \widehat{f}(x, s)$ means we can view the partial Fourier transform as a function on $\mathbf{R}^2 \times \mathbf{R} = \mathbf{R}^2 \times \widehat{\mathbf{R}}$ via

$$\widehat{f}(x, y, t) = \int_{\mathbf{R}} f(x, y, z)e^{-2\pi izt}\, dz,$$

and the same identity allows us to express the multiplication on $A(G)$ as

$$f * g(x, y, z) = \int_{\mathbf{R}} \int_{\mathbf{R}} \exp 2\pi i \left( z(s + t) + s\left(\frac{\mu\nu t^2}{2} + y\mu t\right) + t\left(\frac{\mu\nu s^2}{2} - y\mu s\right)\right) \times$$
$$\widehat{f}(x + \mu t, y + \nu t, s)\widehat{g}(x - \mu s, y - \nu s, t)\, ds\, dt.$$

**Remark.** We have seen that $G$ always acts by left translation on $A(G, \omega)$, but since the definition of $*_\omega$ involves putting $\omega^*$ on the right of the variable, we would not necessarily expect $G$ to act by right translation on $A(G, \omega)$. However, we can identify some situations in which it does, and these will be important later. We begin by looking at the effect of automorphisms, and obtain information about right translation by $x$ on $G$ by observing that $y \mapsto yx = x(x^{-1}yx)$ is the composition of an inner automorphism and a left translation.

If $\alpha$ is a bicontinuous automorphism of $G$ (write $\alpha \in \operatorname{Aut} G$), there is a constant $\Delta(\alpha)$ such that $\int_G f \circ \alpha = \Delta(\alpha) \int_G f$, and then $U_\alpha \xi = \Delta(\alpha)^{1/2} \xi \circ \alpha$ defines a unitary operator $U_\alpha$ on $L^2(G)$. Since we trivially have $L_x U_\alpha = U_\alpha L_{\alpha(x)}$, the map $T \mapsto U_\alpha^* T U_\alpha$ is a normal isomorphism $\overline{\alpha}$ of $\mathcal{L}(G)$ onto $\mathcal{L}(G)$ such that $\overline{\alpha}(L_x) = L_{\alpha(x)}$.

**Lemma 4.5.** *Suppose $\omega$ is a normalised dual cocycle and $\alpha \in \operatorname{Aut} G$. Then $\alpha^* : f \mapsto f \circ \alpha$ is an automorphism of $A(G, \omega)$ if and only if $\overline{\alpha} \otimes \overline{\alpha}(\omega) = \omega$. In this case, $\alpha^*$ extends to a ${}^*$-automorphism of $C_r^*(\widehat{G}, \omega)$, implemented by $U_\alpha^*$.*

**Proof:** Since $\alpha^*$ is always a bijection, we have only to show that it is multiplicative iff $\overline{\alpha} \otimes \overline{\alpha}(\omega) = \omega$. But it is easy to check that $\langle \overline{\alpha}(T), f\rangle = \langle T, f \circ \alpha\rangle$, so

$$\begin{aligned}
(f \circ \alpha) *_\omega (g \circ \alpha)(x) &= (f *_\omega g) \circ \alpha(x) \\
&\iff \big\langle (L_x \otimes L_x)\omega^*, (f \circ \alpha) \otimes (g \circ \alpha)\big\rangle = \big\langle (L_{\alpha(x)} \otimes L_{\alpha(x)})\omega^*, f \otimes g\big\rangle \\
&\iff \big\langle \overline{\alpha} \otimes \overline{\alpha}(L_x \otimes L_x)\overline{\alpha} \otimes \overline{\alpha}(\omega^*), f \otimes g\big\rangle = \big\langle (L_{\alpha(x)} \otimes L_{\alpha(x)})\omega^*, f \otimes g\big\rangle \\
&\iff \big\langle (L_{\alpha(x)} \otimes L_{\alpha(x)})\overline{\alpha} \otimes \overline{\alpha}(\omega^*), f \otimes g\big\rangle = \big\langle (L_{\alpha(x)} \otimes L_{\alpha(x)})\omega^*, f \otimes g\big\rangle.
\end{aligned}$$

This is true for all $x \in G$ and $f, g \in A(G)$ precisely when $\overline{\alpha} \otimes \overline{\alpha}(\omega) = \omega$.

For the last part, we let $\mu$ denote the regular representation of $A(G, \omega)$; then we need to show that $(\mu, U_\alpha)$ is a covariant representation of $(A(G, \omega), \alpha^*)$. Observe that $\langle f \circ \alpha, T \rangle = \langle f, \overline{\alpha}(T) \rangle = \langle f, U_\alpha^* T U_\alpha \rangle$, and that $U_\alpha \otimes U_\alpha$ commutes with $W_G^*$. Then, writing $U$ for $U_\alpha$, we have

$$\begin{aligned} U^* \mu(f \circ \alpha) U &= S_{f \circ \alpha}\big((U^* \otimes 1) W_G^* \Sigma(\omega^*)(U \otimes 1)\big) \\ &= S_f\big((U^* \otimes U^*) W_G^* \Sigma(\omega^*)(U \otimes U)\big) \\ &= S_f\big(W_G^* (U^* \otimes U^*) \Sigma(\omega^*)(U \otimes U)\big) \\ &= S_f\big(W_G^* \Sigma(\overline{\alpha} \otimes \overline{\alpha}(\omega^*))\big) \\ &= S_f\big(W_G^* \Sigma(\omega^*)\big) \\ &= \mu(f), \end{aligned}$$

as required.

**Lemma 4.6.** *Suppose $G$ is a second countable locally compact group, $H$ a closed abelian subgroup, and $\omega$ is the normalised dual cocycle on $H$ given by a skew homomorphism $\pi : \widehat{H} \to H$. Suppose $\alpha \in \operatorname{Aut} G$ satisfies $\alpha(H) = H$ and $\alpha \circ \pi(s \circ \alpha) = \pi(s)$ for all $s \in \widehat{H}$. Then $f \mapsto f \circ \alpha$ is an automorphism of $A(G, \operatorname{Inf} \omega)$, and extends to a $^*$-automorphism of $C_r^*(\widehat{G}, \operatorname{Inf} \omega)$.*

**Proof.** Since $\overline{\alpha}$, $\overline{\alpha_H} := \overline{\alpha|_H}$ and $I$ are all normal, the calculation

$$\overline{\alpha} \circ I(L_h^H) = \overline{\alpha}(L_h^G) = L_{\alpha(h)}^G = I\big(L_{\alpha(h)}^H\big) = I \circ \overline{\alpha_H}(L_h^H)$$

implies that

$$\overline{\alpha} \otimes \overline{\alpha}(\operatorname{Inf} \omega) = \overline{\alpha} \otimes \overline{\alpha}(I \otimes I(\omega)) = I \otimes I(\overline{\alpha_H} \otimes \overline{\alpha_H}(\omega)) = (I \otimes I) \circ (\overline{\alpha_H} \otimes \overline{\alpha_H}) \circ \operatorname{Ad}(\mathcal{F} \otimes \mathcal{F})(\sigma),$$

where $\mathcal{F} : L^2(\widehat{H}) \to L^2(H)$ is the Fourier transform as before, and $\sigma(s, t) = \langle \pi(s), t \rangle$. Now $\mathcal{F}^{-1} L_h \mathcal{F}$ is the function $f_h : t \mapsto \overline{\langle h, t \rangle}$, and hence

$$\overline{\alpha_H}(\mathcal{F} f_h \mathcal{F}^{-1}) = \overline{\alpha_H}(L_h^H) = L_{\alpha(h)}^H = \mathcal{F} f_{\alpha(h)} \mathcal{F}^{-1} = \mathcal{F}(f_h \circ \alpha^t) \mathcal{F}^{-1},$$

where $\langle \alpha^t(t), h \rangle := \langle t, \alpha(h) \rangle$. Again, we can use the normality of $\overline{\alpha}$ and $(\alpha^t)^*$ to deduce that $\overline{\alpha_H} \circ \operatorname{Ad} \mathcal{F} = \operatorname{Ad} \mathcal{F} \circ (\alpha^t)^*$, and

$$\overline{\alpha} \otimes \overline{\alpha}(\operatorname{Inf} \omega) = (I \otimes I) \operatorname{Ad}(\mathcal{F} \otimes \mathcal{F})\big(\sigma \circ (\alpha^t \otimes \alpha^t)\big).$$

But a simple calculation shows that $\alpha \circ \pi \circ \alpha^t = \pi \iff \sigma \circ (\alpha^t \otimes \alpha^t) = \sigma$, so

$$\overline{\alpha} \otimes \overline{\alpha}(\operatorname{Inf} \omega) = (I \otimes I) \operatorname{Ad}(\mathcal{F} \otimes \mathcal{F})(\sigma) = \operatorname{Inf} \omega,$$

and the result follows from Lemma 4.5.

**Corollary 4.7.** *If $x \in G$ satisfies $x^{-1}Hx = H$ and $\pi(s \circ \operatorname{Ad} x^{-1}) = \operatorname{Ad} x \circ \pi(s)$, then $\beta_x(f)(y) = f(yx)$ is an automorphism of $A(G, \operatorname{Inf}\omega)$.*

**Proof.** Under these circumstances, $\beta_x$ is the composition of $f \mapsto f \circ \operatorname{Ad} x^{-1}$ and the left translation $\alpha_x$.

**Example 4.8.** We return to Example 3.6(2), where $G$ is the modified Heisenberg group, $H = H_{\mu,\nu}$, and $\pi : \mathbf{R} \times \mathbf{Z} = \widehat{H} \to H$ is defined via the isomorphism of $H$ with $\mathbf{R} \times \mathbf{T}$ by $\pi(r,n) = (n\theta, \exp(-2\pi i r\theta))$. We identify $(s,z) \in \mathbf{R} \times \mathbf{T}$ with $\big(\mu s, \nu s, z\exp(-\pi i \mu\nu s^2)\big)$, and compute

$$\begin{aligned}(x,y,1)^{-1}(s,z)(x,y,1) &= \big(\mu s, \nu s, z\exp(-\pi i \mu\nu s^2 - 2\pi i s(\mu y - \nu x))\big) \\ &= \big(s, z\exp(-2\pi i s(\mu y - \nu x))\big).\end{aligned}$$

Thus

$$\begin{aligned}\big\langle (s,z), (r,n) \circ \operatorname{Ad}(x,y,1)^{-1} \big\rangle &= \exp(2\pi i r s) z^n \exp(-2\pi i s(\mu y - \nu x)n) \\ &= \big\langle (s,z), (r - (\mu y - \nu x)n, n) \big\rangle,\end{aligned}$$

so that

$$\pi\big((r,n) \circ \operatorname{Ad}(x,y,1)^{-1}\big) = \big(n\theta, \exp -2\pi i(r - n(\mu y - \nu x))\theta\big).$$

Now one easily checks that this is $\operatorname{Ad}(x,y,1)^{-1}(\pi(r,n))$. Thus according to Corollary 4.7, right translation gives an action of $G$ on $A(G, \operatorname{Inf}\omega)$, which extends to an action $\beta$ of $G$ on $C_r^*(\widehat{G}, \operatorname{Inf}\omega)$; it is continuous because we can write down a unitary representation on $L^2(G)$ implementing it.

## §5. Dense subalgebras of $C^*(\widehat{G}, \operatorname{Inf}\omega)$.

When forming deformations of $C_0(X)$, one wants a $^*$-subspace $\mathcal{A}$ of $C_0(X)$ and a one-parameter family of multiplications on $\mathcal{A}$. We have seen that, when $X = G$, $A(G)$ is a suitable subspace. To deform algebras of the form $C_0(G/\Gamma)$, it is tempting to look for a suitable $\Gamma$-invariant subalgebra of $C_0(G)$, but in general there will be no $\Gamma$-invariant functions which vanish at infinity, and one is forced to look in $C_b(G)$ rather than $C_0(G)$. The obvious algebra of bounded functions associated to the Fourier algebra $A(G)$ is $B(G)$, but unfortunately $B(G) \cap C_0(G/\Gamma)$ need not be large enough to separate points of $G/\Gamma$ (see Example 5.4 below). Here we propose another $^*$-space of functions in $C_0(G)$, which is a Banach $^*$-algebra with respect to the same multiplication as $A(G, \operatorname{Inf}\omega)$, but whose bounded analogue will contain more $\Gamma$-invariant functions. The basic observation is that, when the abelian subgroup $K$ in Theorem 4.3 is compact, the partial Fourier coefficients make sense for any continuous function, and asking that $f\big|_K$ be in $A(K)$ is equivalent to asserting that the partial Fourier series of $f$ be summable.

**Lemma 5.1.** *Suppose $G$ is a locally compact group and $K$ a compact abelian subgroup. For any $f \in C_b(G)$, we define the partial Fourier transform $\widehat{f} : G \times \widehat{K} \to \mathbf{C}$ by*

$$\widehat{f}(x, s) = \int_K \overline{\langle k, s\rangle}\, f(xk)\, dk.$$

*Then*

*(1) $\widehat{f}(\cdot, s) \in C_b(G)$ for all $s \in \widehat{K}$;*

*(2) $\widehat{f}(xk, s) = \langle k, s\rangle\, \widehat{f}(x, s)$ for $k \in K$, $s \in \widehat{K}$;*

*(3) if $f(xk) = \langle k, s\rangle\, f(x)$ for $k \in K$ and $s \in \widehat{K}$, then $\widehat{f}(\cdot, t) = 0$ for $t \neq s$, and $\widehat{f}(\cdot, s) = f$;*

*(4) if $s \in G$ is fixed and $k \mapsto f(xk)$ is in $A(K)$ (in particular, for all $x$ if $f \in A(G)$), then $f(x) = \sum_{s\in\widehat{K}} \widehat{f}(x, s)$.*

**Proof.** (1) follows from a standard compactness argument, (2) and (3) from routine computations, and (4) from the Fourier Inversion Theorem.

**Proposition 5.2.** *Let $G$ be a locally compact group, $K$ a compact abelian subgroup and $\rho : \widehat{K} \to G$ a continuous homomorphism such that $\rho(s)k = k\rho(s)$ for $k \in K$, $s \in \widehat{K}$. Then*

$$C_{b,1}(G) := \big\{f \in C_b(G) : \|f\|_{\infty,1} := \sum_s \sup_x |\widehat{f}(x, s)\| < \infty\big\}$$

*is a Banach $^*$-algebra with norm $\|\cdot\|_{\infty,1}$ and operations*

$$\begin{aligned} f * g(x) &= \sum_{s,t} \widehat{f}(x\rho(t), s)\widehat{g}(x\rho(-s), t), \\ f^*(x) &= \overline{f(x)}. \end{aligned} \tag{5.1}$$

*Further, $C_{0,1}(G) := C_{b,1}(G) \cap C_0(G)$ is a closed 2-sided ideal in $C_{b,1}(G)$.*

**Proof.** If $f \in C_{b,1}(G)$, we can view $\widehat{f}$ as an element $s \mapsto \widehat{f}(\cdot, s)$ of $l^1(\widehat{K}, C_b(G))$, and $f \mapsto \widehat{f}$ is then an isometric embedding of $C_{b,1}(G)$ in the Banach space $l^1(\widehat{K}, C_b(G))$. Thus if $\{f_n\}$ is Cauchy in $C_{b,1}(G)$, the sequence $\{\widehat{f_n}\}$ converges in $l^1(\widehat{K}, C_b(G))$, say $\widehat{f_n} \to g$. We define $f(x) = \sum_s g(x, s)$, which converges absolutely and uniformly to a continuous function $f$ with $\|f\|_\infty \leq \|g\|_{\infty,1}$. It follows easily from Lemma 5.1(2) that $g(xk, s) = \langle k, s\rangle\, g(x, s)$, and since $\sum_s g(x, s)$ converges uniformly in $x$, we deduce from Lemma 5.1(3) that

$$\widehat{f}(x, s) = \sum_t \big(g(\cdot, t)\big)^\wedge(x, s) = \widehat{g}(x, s).$$

Thus $f \in C_{b,1}(G)$, and $\widehat{f_n} \to g$ in $l^1(\widehat{K}, C_b(G))$ is equivalent to $f_n \to f$ in $C_{b,1}(G)$.

Next, observe that

$$\begin{aligned}\sum_{s,t} \left|\widehat{f}(x\rho(t),s)\widehat{g}(x\rho(-s),t)\right| &\leq \sum_s \Big(\|\widehat{f}(\cdot,s)\|_\infty \sum_t |\widehat{g}(x\rho(-s),t)|\Big)\\ &\leq \sum_s \|\widehat{f}(\cdot,s)\|_\infty \sum_t \|\widehat{g}(\cdot,t)\|_\infty\\ &= \|f\|_{\infty,1}\|g\|_{\infty,1},\end{aligned}$$

so the series defining $f * g$ converges absolutely, uniformly in $x$, to a continuous function $f*g$ on $G$. We can then pull the summation through the integral defining $(f*g)\widehat{\ }(x,r)$ to see that

$$(f*g)\widehat{\ }(x,r) = \sum_p \widehat{f}(x\rho(r-p),p)\widehat{g}(x\rho(-p),r-p).$$

Thus we have

$$\begin{aligned}\|f*g\|_{\infty,1} &= \sum_r \sup_x \Big|\sum_p \widehat{f}(x\rho(r-p),p)\widehat{g}(x\rho(-p),r-p)\Big|\\ &\leq \sum_r \sum_p \|\widehat{f}(\cdot,p)\|_\infty \|\widehat{g}(\cdot,r-p)\|_\infty\\ &= \|f\|_{\infty,1}\|g\|_{\infty,1}.\end{aligned}$$

A messy and boring calculation verifies that $(f*g)*h = f*(g*h)$, and we have shown that $C_{b,1}(G)$ is a Banach algebra. That $f \mapsto f^*$ is an involution follows from the easily checked identity $\widehat{f^*}(x,s) = \overline{\widehat{f}(x,-s)}$.

We have already seen that the series defining $f*g$ converges uniformly in $x$; if $f$ or $g$ is in $C_0(G)$, then each summand is also in $C_0(G)$, and hence so is the sum $f*g$. Thus $C_{0,1}(G)$ is an ideal. To see that it is closed, note first that for every $f \in C_{b,1}(G)$ and $x \in G$, we can recover the function $k \mapsto f(xk)$ as the sum of the absolutely convergent Fourier series $\sum_s \widehat{f}(x,s)\langle k,s\rangle$, and we have $\|f\|_\infty \leq \|f\|_{\infty,1}$. Thus if $f_n \to g$ in $C_{b,1}(G)$, and each $f_n$ is in $C_0(G)$, we have $f_n \to g$ uniformly, and $g$ also lies in $C_0(G)$.

**Remark.** The subspace $C_{c,1}(G) = C_{b,1}(G) \cap C_c(G)$ is *not* an ideal in $C_{b,1}(G)$. To see this, suppose $f$ has compact support $C$. Then $x \mapsto \widehat{f}(x\rho(t),s)$ has support in $C\rho(t)^{-1}$, so the summands in (5.1) have different supports. If $\rho(\widehat{K})$ is not compact in $G$, which is easy to arrange if $K$ is the subgroup $\{0\} \times \mathbf{T}$ of $\mathbf{R} \times \mathbf{T}$, for example, then $\cup_{t\in\widehat{K}} C\rho(t)^{-1}$ will not be compact in $G$. Thus to force $f*g$ into $C_c(G)$, we would need to insist that $\widehat{f}$ and $\widehat{g}$ have finite support in the $\widehat{K}$-variable.

**Proposition 5.3.** *Suppose $G$, $K$ and $\rho : \widehat{K} \to G$ are as in Proposition 5.2. Then $A_1(G) = A(G) \cap C_{b,1}(G)$ is a dense $^*$-subalgebra of both $A(G, \operatorname{Inf}\omega)$ and $C_{0,1}(G, \operatorname{Inf}\omega)$. If $K$ is central in $G$, then $A(G)$ is itself a dense $^*$-subalgebra of $C_{0,1}(G, \operatorname{Inf}\omega)$.*

**Proof.** Since the multiplication formulas are the same, we have only to show that $A_1(G)$ is dense in $A(G)$ and $C_{0,1}(G)$. For the first, we start with $\xi, \eta \in C_c(G)$, $f(x) = (L^G_x\xi|\eta)$, and we choose $\phi \in L^1(K)$ such that $\widehat{\phi} \in l^1(\widehat{K})$ and $L^G|_K(\phi)\xi$ approximates $\xi$ in $L^2(G)$. If we now set $g(x) = \big(L^G_x L^G|_K(\phi)\xi|\eta\big)$, then $g$ approximates $f$ in $A(G)$. But a quick calculation shows

$$g(x) = \int_K \phi(h)f(xh)\,dh,$$

and hence

$$\begin{aligned}\widehat{g}(x,s) &= \int_K \overline{\langle k,s\rangle} \int_K \phi(h)f(xkh)\,dh\,dk\\ &= \int_K\int_K \overline{\langle kh,s\rangle}\,\overline{\langle h,-s\rangle}\,\phi(h)f(xkh)\,dh\,dk\\ &= \widehat{\phi}(-s)\widehat{f}(x,s).\end{aligned}$$

Thus for any $x \in G$, we have

$$|\widehat{g}(x,s)| \le |\widehat{\phi}(-s)| \int_K |f(xk)|\,dk \le |\widehat{\phi}(-s)|\|f\|_\infty,$$

so that $\|g\|_{\infty,1} \le \|\widehat{\phi}\|_1\|f\|_\infty < \infty$. Thus $g \in A_1(G)$, and we have shown $A_1(G)$ dense in $A(G)$.

To see that $A_1(G)$ is dense in $C_{0,1}(G)$, we start with $f \in C_{0,1}(G)$. Since we can approximate $f$ by a finite sum $\sum_s \widehat{f}(\cdot,s)$, it will be enough to approximate one fixed $\widehat{f}(\cdot,s)$ by something in $A_1(G)$. We can certainly find $g \in A(G)$ such that $\|g - \widehat{f}(\cdot,s)\|_\infty < \epsilon$. Then $g_1(x) = \widehat{g}(x,s)$ is in $A(G)$ because it is the integral of the continuous $A(G)$-valued function $k \mapsto \overline{\langle k,s\rangle} g(\cdot k)$, and a trivial estimate shows $\|g_1 - \widehat{f}(\cdot,s)\|_\infty < \epsilon$. Since $\widehat{g}_1(\cdot,t) = 0$ unless $t = s$, Lemma 5.1(3) implies that $\|g_1 - \widehat{f}(\cdot,s)\|_{\infty,1} = \|g_1 - \widehat{f}(\cdot,s)\|_\infty$, and we are done.

We now show that $A(G) \subset C_{0,1}(G)$ if $K$ is central. First note that if $f(x) = (L_x\xi|\eta)$, then

$$\widehat{f}(x,s) = \int_K \overline{\langle k,s\rangle}\,(L^G_k\xi|L^G_{x^{-1}}\eta)\,dk = (L^G_x P_s\xi|\eta), \tag{5.2}$$

where $P_s$ is the family of mutually orthogonal projections on $L^2(G)$ defined by

$$(P_s\xi|\eta) = \int_K \overline{\langle k,s\rangle}\,(L^G_k\xi|\eta)\,dk.$$

Because $K$ is central, and the operators $P_s$ are projections, we have $L^G_x P_s = L^G_x P_s^2 = P_s L^G_x P_s$, and

$$\widehat{f}(x,s) = (L^G_x P_s\xi|\eta) = (P_s L^G_x P_s\xi|\eta) = (L^G_x P_s\xi|P_s\eta).$$

It follows that

$$\sum_s \sup_x |\widehat{f}(x,s)| \le \sum_s \|P_s\xi\|\,\|P_s\eta\| \le \Big(\sum_s \|P_s\xi\|^2\Big)^{1/2}\Big(\sum_s \|P_s\eta\|^2\Big)^{1/2} \le \|\xi\|\,\|\eta\|.$$

Thus $\|f\|_{\infty,1} \le \|\xi\|\,\|\eta\|$, and $f \in C_{0,1}(G)$. This completes the proof of the proposition.

**Concluding remarks.** This construction suggests that we need not have started with a group $G$: a space $X$ with a right action of (say) $\mathbf{T}^2$ would have been enough. However, even this extra generality does not cover some of the potentially interesting examples. For example, if $G$ is the real Heisenberg group and $\Gamma$ its integer subgroup, [21] suggests we should be able to deform $C(G/\Gamma)$ by taking $K$ to be the copy of $\mathbf{T}$ coming from the centre of $G$, and $\rho : \mathbf{Z} = \widehat{\mathbf{T}} \to G/\Gamma$ to be the map $n \mapsto (\mu n\theta, \nu n\theta, \mu\nu n^2\theta^2/2)\Gamma$. Although this does not fit the outline we have given above, our formulas do give $^*$-algebras based on $C_{b,1}(G) \cap C(G/\Gamma)$ which are deformations of $C(G/\Gamma)$. We leave the details to a sequel, but close by showing that it is necessary to use $C_{b,1}(G)$ rather than $B(G)$: $B(G) \cap C(G/\Gamma)$ is not big enough.

**Example 5.4.** *Let $G$ be the Heisenberg group based on $\mathbf{R}\times\mathbf{R}\times\mathbf{T}$, as in Example 3.6(2), and let $\Gamma$ be the subgroup $\{(m,n,1) : m,n \in \mathbf{Z}\}$. Then every function in $B(G) \cap C(G/\Gamma)$ is constant on cosets of the subgroup $K = \{(0,0,z) : z \in \mathbf{T}\}$ of $G$.*

**Proof.** Suppose $\phi \in B(G) \cap C(G/\Gamma)$. There is a representation $U$ of $G$ on a Hilbert space $\mathcal{H}$ such that $\phi(x) = (U_x\xi|\eta)$. Let $P$ be the orthogonal projection onto $\mathcal{H}_0 = \overline{\mathrm{sp}}\{U_x\eta : x \in G\}$; we may assume that $\xi \in \mathcal{H}_0$ without changing $\phi$. For $u \in \Gamma$, we have

$$0 = \phi(xu) - \phi(x) = \big((U_{xu} - U_x)\xi\big|\eta\big) = \big((U_u - 1)\xi\big|U_{x^{-1}}\eta\big)$$

for all $x \in G$, so $(U_u - 1)\xi \perp \mathcal{H}_0$. Since we know $U_u(\xi)$ is in $\mathcal{H}_0$, it follows that $U_u\xi = \xi$ for all $u \in \Gamma$.

By Fourier analysis of the representation $U\big|_K$, we can expand $\xi$ in a sum $\sum \xi_k$ in which $U_{(0,0,z)}\xi_k = z^k\xi_k$, by taking $\xi_k = \int_K z^{-k}U_{(0,0,z)}\xi\,dz$. Since $(0,0,z)$ is in the centre of $G$, we then have $U_u\xi_k = \xi_k$ for all $u \in \Gamma$. Now take $u = (0,1,1) \in \Gamma$, $v(x) = (x,0,1)$, and $w(x) = (0,0,\exp 2\pi i x)$. Then we have $uv(x) = v(x)uw(x)$, and hence

$$\big(U_{v(x)}\xi_k\big|\xi_k\big) = \big(U_{uv(x)}\xi_k\big|\xi_k\big) = \big(U_{v(x)uw(x)}\xi_k\big|\xi_k\big) = \exp 2\pi i k x\big(U_{v(x)}\xi_k\big|\xi_k\big).$$

Thus $\big(U_{v(x)}\xi_k\big|\xi_k\big) = 0$ whenever $kx \notin \mathbf{Z}$, so by letting $x \to 0$ we see that $\xi_k = 0$ for $k \neq 0$. Thus $\xi = \xi_0$, $U_{(0,0,z)}\xi = U_{(0,0,z)}\xi_0 = \xi_0 = \xi$, and

$$\phi(x(0,0,z)) = \big(U_xU_{(0,0,z)}\xi\big|\eta\big) = \big(U_x\xi\big|\eta\big) = \phi(x),$$

justifying the claim.

**Appendix: One-parameter groups of dual cocycles.**

If $H = \mathbf{R}^m \times \mathbf{T}^n$, then $\widehat{H} = \mathbf{R}^m \times \mathbf{Z}^n$, and every cocycle on $H$ is equivalent to one of the form $\sigma(p,q) = \exp i\theta(p,q)$ for some skew-symmetric bilinear form $\theta : \widehat{H} \times \widehat{H} \to \mathbf{R}$. Each such cocycle lies on a 1-parameter group $\sigma_t : (p,q) \mapsto \exp it\theta(p,q)$, and the Fourier transform of $\sigma_t$ is then a 1-parameter group of normalised dual cocycles on $H$. If $H$ is a closed subgroup of a Lie group $G$, inflating up to $G$ gives a 1-parameter group $\omega(t)$ of normalised dual cocycles on $G$, which satisfies

$$\Sigma(\omega(t)) = \omega(t)^*. \tag{A1}$$

We shall see in this appendix that, under some restrictions on the infinitesimal generator, every 1-parameter group of normalised dual cocycles on a connected Lie group $G$ satisfying (A1) is inflated in this way from some such abelian subgroup $H$.

Suppose $\omega(t)$ is a strongly continuous 1-parameter group of normalised dual cocycles on a connected Lie group $G$ satisfying (A1). By Stone's Theorem, there is an (unbounded) self-adjoint operator $\theta$ on $L^2(G \times G)$ such that $\omega(t) = \exp it\theta$. Let $\mathcal{U}(\mathfrak{g})$ be the universal enveloping algebra of the Lie algebra $\mathfrak{g}$ of $G$, so that $\mathcal{U}(\mathfrak{g})$ consists of the right invariant differential operators on $G$. We shall assume that the generator $\theta$ lies in $\mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$. (In the first paragraph, $\theta$ is multiplication by a second degree polynomial on $L^2(\widehat{G} \times \widehat{G})$, and its Fourier transform is a second order differential operator in $\mathfrak{g} \otimes \mathfrak{g}$.) We shall need three basic facts about $\mathcal{U}(\mathfrak{g})$: that $\mathcal{U}(\mathfrak{g})$ is the linear span of $\{X_1 X_2 \ldots X_k : X_i \in \mathfrak{g}\}$; that $\delta_G$ extends to a map $\delta_G : \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$ satisfying

$$\delta_G(X) = X \otimes 1 + 1 \otimes X \tag{A2}$$

for $X \in \mathfrak{g}$; and that every $X \in \mathcal{U}(\mathfrak{g})$ satisfying (A2) lies in $\mathfrak{g}$. Thus our infinitesimal generator $\theta$ maps $L^2(G \times G) \cap C^\infty(G \times G)$ into itself, and the operators $\delta_G \otimes i(\theta)$, $\theta_{12} = \theta \otimes 1$, etc., all lie in $\mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$.

We now look at how the algebraic properties of $\omega(t)$ are reflected in $\theta$. First of all, differentiating the cocycle identity gives

$$\theta_{12} + \delta_G \otimes i(\theta) = \theta_{23} + i \otimes \delta_G(\theta). \tag{A3}$$

We can similarly differentiate (A1) to obtain

$$\Sigma(\theta) = -\theta. \tag{A4}$$

Since $\omega(t)\Sigma(\omega(t))^* = \omega(2t)$, the argument of [12, 4.4], which does not use the compactness of $G$ assumed there, gives

$$\omega(2t)_{12}\,\omega(2t)_{13} = \omega(t)_{23}\, i \otimes \delta_G(\omega(2t))\,\omega(-t)_{23}.$$

Differentiating this gives

$$\theta_{12} + \theta_{13} = i \otimes \delta_G(\theta). \tag{A5}$$

The next lemma shows that (A4) and (A5) force $\theta$ to lie in the Lie algebra rather than the enveloping algebra. This is at least partly attributable to our assumption (A1): in view of Lemma 11, it implies $\nu \otimes \nu(\omega(t)) = \omega(t)$, and hence, formally at least, $\nu \otimes \nu(\theta) = \theta$. Since $\nu(X) = -X$ for $X \in \mathfrak{g}$, this is true if $\theta \in \mathfrak{g} \otimes \mathfrak{g}$, but for more general $T = X_1 X_2 \ldots X_k$ in $\mathcal{U}(\mathfrak{g})$ the relationship between $T$ and $\nu(T) = \nu(X_k) \ldots \nu(X_1)$ could be complicated.

**Lemma A1.** *If $\theta \in \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(g)$ satisfies Equations (A4) and (A5), then $\theta \in \mathfrak{g} \otimes \mathfrak{g}$.*

**Proof.** We can write $\theta = \sum_i A_i \otimes B_i$ for some linearly independent set $\{A_i\}$ in $\mathcal{U}(\mathfrak{g})$. Then (A5) gives

$$\sum A_i \otimes B_i \otimes 1 + A_i \otimes 1 \otimes B_i = \sum A_i \otimes \delta_G(B_i),$$

so $\delta_G(B_i) = B_i \otimes 1 + 1 \otimes B_i$, and $B_i \in \mathfrak{g}$. Now use $\theta = -\Sigma(\theta)$ and the same argument to see that $A_i \in \mathfrak{g}$.

If we take the adjoint of the cocycle identity for $\omega(t)$, and replace $t$ by $-t$, we obtain

$$\delta_G \otimes i(\omega(t))\,\omega(t)_{12} = i \otimes \delta_G(\omega(t))\,\omega(t)_{23};$$

combining this with the original cocycle identity gives

$$\big[\omega(t)_{12}, \delta_G \otimes i(\omega(t))\big] = \big[\omega(t)_{23}, i \otimes \delta_G(\omega(t))\big],$$

and differentiating twice gives

$$\big[\theta_{12}, \delta_G \otimes i(\theta)\big] = \big[\theta_{23}, i \otimes \delta_G(\theta)\big]. \tag{A6}$$

Now putting (A3) and (A5) into (A6) gives

$$\big[\theta_{12}, \theta_{13} + \theta_{23}\big] = \big[\theta_{23}, \theta_{12} + \theta_{13}\big]. \tag{A7}$$

Next we apply $\Sigma_{12}$ and $\Sigma_{23}$ to (A7), to obtain

$$\begin{aligned}\big[-\theta_{12}, \theta_{23} + \theta_{13}\big] &= \big[\theta_{13}, -\theta_{12} + \theta_{23}\big]\\ \big[\theta_{13}, \theta_{12} - \theta_{23}\big] &= \big[-\theta_{23}, \theta_{13} + \theta_{12}\big],\end{aligned}$$

and add the last three equations:

$$\big[\theta_{13}, \theta_{12} - \theta_{23}\big] = 0. \tag{A8}$$

Finally, putting (A8) into (A7) shows that

$$[\theta_{12}, \theta_{23}] = 0. \tag{A9}$$

Similar arguments show that all the operators $\theta_{12}$, $\theta_{13}$, $i \otimes \delta_G(\theta)$, etc, commute, and hence so do the ones appearing in the cocycle identity.

**Theorem A2.** *Suppose $G$ is a connected Lie group, and $\omega(t) = \exp it\theta$ is a one parameter group of normalised dual cocycles on $G$ whose infinitesimal generator $\theta$ is in $\mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$ and satisfies $\theta = -\Sigma(\theta)$. Then there is a closed, connected abelian subgroup $H$ of $G$ such that $\omega(t)$ is inflated from a one-parameter group of normalised dual cocycles on $H$, as described at the beginning of this appendix.*

**Proof.** By Lemma A1, there are $Y_i \in \mathfrak{g}$ and a linearly independent subset $\{X_i\}$ of $\mathfrak{g}$ such that $\theta = \sum X_i \otimes Y_i = -\sum Y_i \otimes X_i$. Equations (A4) and (A9) imply that

$$\sum_{i,j} X_i \otimes [Y_i, Y_j] \otimes X_j = 0.$$

Thus we have $[Y_i, Y_j] = 0$ for all $i$, $j$, and $\mathfrak{h} = \mathrm{sp}\,\{Y_i\}$ is an abelian subalgebra of $\mathfrak{g}$. We may therefore suppose that $\{X_i\}$ is a basis for $\mathfrak{h}$, and that $\theta = \sum \theta_{ij} X_i \otimes X_j$ with $\theta_{ij} = -\theta_{ji}$. Let $H$ be the connected subgroup of $G$ corresponding to $\mathfrak{h}$; then $\omega(t)$ is inflated from a dual cocycle on $H$, because its infinitesimal generator $\theta$ lies in the Lie algebra $\mathfrak{h} \otimes \mathfrak{h}$. We can view $\widehat{H}$ as a subgroup of $\mathfrak{h}$, with the pairing between $\widehat{H}$ and $H$ given in terms of the Euclidean product on $\mathfrak{h}$ corresponding to the basis $\{X_i\}$ by $\langle \exp tX, p\rangle = \exp it(X \cdot p)$. Thus $X \in \mathfrak{h}$ is the Fourier transform of the function $p \mapsto iX \cdot p$, $X \otimes Y$ the Fourier transform of $(p, q) \mapsto -(X \cdot p)(Y \cdot q)$, and $\theta$ the Fourier transform of

$$(p, q) \mapsto -\sum \theta_{ij} (X_i \cdot p)(X_j \cdot q).$$

We can choose the basis for $\mathfrak{h}$ so that $\mathfrak{h} \cong \mathbf{R}^m \times \mathbf{R}^n$, $H \cong \mathbf{R}^m \times \mathbf{T}^n$, and $\widehat{H} \mapsto \mathfrak{h}$ is the usual embedding of $\mathbf{R}^m \times \mathbf{Z}^n$ in $\mathbf{R}^{m+n}$, and then the dual cocycle on $H$ has precisely the form we started with.

**Remarks. (1)** The assumption that $\theta \in \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$ is strong, but not unreasonable. We do not know whether we need to asumme $\Sigma(\theta) = -\theta$: it is possible that any 1-parameter group $\{\omega(t)\}$ of dual cocycles will be equivalent to one whose generator satisfies this.

**(2)** Is Theorem A2 also true for a 1-parameter family $\{\omega(t)\}$ of dual cocycles which is not a group? If $\omega(t)$ is normalised and satisfies $\omega(t)^* = \omega(-t)$, $\omega(0) = 1$, then the arguments above show that $\theta = -i\omega'(0)$ still satisfies (A4) and (A5). So if we again assume $\theta \in \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$, we deduce that $\theta \in \mathfrak{h} \otimes \mathfrak{h}$ for some abelian subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. (This does not prove the desired theorem, because we cannot in general recover $\omega(t)$ as $\exp it\theta$.) Our observation should be compared with the last part of [25], where Wassermann argues that a 1-parameter family $\{B(t)\}$ of operators satisfying the classical Yang-Baxter equations always has derivative $b = B'(0)$ coming from an abelian Lie algebra.

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