Second order ordinary difference equations with symmetries: Lagrangian formalism, first integrals and exact solutions

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Abstract

This is the second part of the paper [1], where a group classification of invariant difference models, i.e. difference equations and meshes, was presented. In the continuous limit the results go over into Lie's classification of second order ordinary differential equations. The discrete model is a three point one and we have shown that it can be invariant under Lie groups of dimension $0 \leq n \leq 6$.

In the present paper we are developing and illustrating by examples the Lagrange formalism and difference analog of Noether-type construction for the second order difference equations and meshes admitting Lie point symmetries.

1 Introduction

A recent article was devoted to a symmetry classification of second order difference equation [1]. This classification was modeled on a classification paper of Sophus Lie, in which he provided a symmetry classification of second order differential equations (ODEs) [2]. As a matter of fact, the classification of difference schemes goes over into Lie's classification of ODEs in the continuous limit [1].

S. Lie showed that a second order ODE can be invariant under a group $\mathfrak{g}$ of dimension $n = 0, 1, 2, 3, \text{or } 8$. Lie's classification of equations with point symmetries into equivalence classes under the action of the infinite dimensional group $\mathrm{Diff}(2, \mathbb{C})$ of all local diffeomorphisms of a complex plane $\mathbb{C}^2$ is given in the Table 1. For $n \geq 2$ the equation can be integrated in quadratures. This can be done by transforming the equation to one of the "canonical" forms, integrated by Lie himself [2]. Virtually all standard methods of integrating second order ODEs analytically can be interpreted in this manner (though this is not mentioned in most elementary textbooks).

The situation with difference equations is much less developed. This is not surprising, since applications of Lie group theory to difference equations are much more recent [3,...,21]. Several different approaches are being pursued. One possibility is to consider the difference equations on a fixed lattice [3,...,13] and consider only transformations that do not act on the lattice. In order to obtain physically interesting symmetries in this approach, it is necessary to go beyond point symmetries and to let the transformations act on more than one point of the lattice. Lie algebra contractions occur in the continuous limit and some “generalized” symmetries may “contract” to point ones [10].

The second possibility is to consider group transformations acting both on the difference equations and on the lattice [1,14,...,21]. Technically this is achieved by considering a difference scheme, consisting of two equations, one representing the actual difference equation, the other the lattice.

This is the approach that we will follow in the present article. More specifically, we will consider the same three–point scheme as in our previous article [1]. The continuous limit of the scheme, if it exists, will be a second order ODE.

Thus, we consider two variables, x and y, with x the independent one and y dependent. The variable x runs through an infinite set of values $\{x=x_k,\ k\in\mathbb{Z}\}$ that are not necessarily equally spaced and are not prescribed a priory. Instead, we give two relation between any three neighboring points

$$F(x,x_-,x_+,y,y_-,y_+)=0, \tag{1.1}$$

$$\Omega(x,x_-,x_+,y,y_-,y_+)=0 \tag{1.2}$$

and also specify some initial conditions like x_0, x_1, $y_0=y(x_0)$, $y_1=y(x_1)$. In the continuous limit Eq. (1.1) goes into an ODE, (1.2) into an identity (like $0=0$), if the continuous limit exists.

The group transformations considered in this approach are of the same type as for ODEs. They are generated by a Lie algebra of vector fields of the form

$$X=\xi(x,y)\frac{\partial}{\partial x}+\eta(x,y)\frac{\partial}{\partial y}. \tag{1.3}$$

The corresponding transformations are purely point ones, since the coefficients ξ and η depend only on (x,y), not on the shifted points (x_+,y_+) or (x_-,y_-).

In Ref. [1] we showed how Lie group theory can be used to classify such pairs of equations as (1.1) and (1.2). Possible dimensions of the symmetry group G of Eq. (1.1), (1.2) are $n=0,1,2,3,4,5,\text{and }6$. The highest dimension, $n=6$, occurs only for difference schemes equivalent to

$$\frac{y_+-2y+y_-}{(x_+-x)^2}=0, \qquad x_+-2x+x_-=0.$$

The purpose of this article is to provide a Lagrange formalism and difference analog of Noether's theorem for second order difference schemes of the form (1.1) and (1.2), admitting Lie point symmetry groups. The Lagrangians will be used to obtain first integrals and in some cases to obtain exact analytic solutions of the difference schemes.

2 General theory

2.1 Definitions and notations

We study the difference system (1.1) and (1.2). In general, we assume that these equations can be solved to express x_+ and y_+ explicitly in terms of (x,y,x_-,y_-) and also vice versa, i.e. (x_-,y_-) in terms of the other quantities. We also make use of the following quantities

$$\begin{gathered} h_+=x_+-x, \qquad h_-=x-x_-, \qquad y_x=\frac{y_+-y}{h_+}, \qquad y_{\bar{x}}=\frac{y-y_-}{h_-}, \\ y_{x\bar{x}}=\frac{2}{h_++h_-}(y_x-y_{\bar{x}}), \end{gathered} \tag{2.1}$$

i.e. the up and down spacings in x, the right and left discrete first derivatives and the discrete second derivative, respectively. Continuous first and second derivatives are denoted y' and y'', respectively.

When acting on differential equations, the vector fields (1.3) must be prolonged to act on derivatives. For difference schemes, the prolongation of a vector field acts on variables at other points of the lattice. It is obtained by shifting the coefficients to the corresponding points. For three point schemes we have

$$\mathbf{pr}X = X + \xi(x_-, y_-)\frac{\partial}{\partial x_-} + \xi(x_+, y_+)\frac{\partial}{\partial x_+} + \eta(x_-, y_-)\frac{\partial}{\partial y_-} + \eta(x_+, y_+)\frac{\partial}{\partial y_+}. \tag{2.2}$$

2.2 Lagrangian formulation for second order ODEs

It has been known since E. Noether's fundamental work that conservation laws for differential equations are connected with their symmetry properties [22,23,24,25]. For convenience we present here some well–known results adapted to the case of second order ODEs.

Let us consider the functional

$$\mathbb{L}(y) = \int_I L(x, y, y')dx, \quad I \subset \mathbb{R}^1, \tag{2.3}$$

where $L(x, y, y')$ is called a first order Lagrangian. The functional (2.3) achieves its extremal values when $y(x)$ satisfies the Euler–Lagrange equation

$$\frac{\delta L}{\delta y} = \frac{\partial L}{\partial y} - D\left(\frac{\partial L}{\partial y'}\right) = 0, \qquad D = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + y''\frac{\partial}{\partial y'} + \cdots, \tag{2.4}$$

where D is the total derivative operator.

Let us consider a Lie point transformation G generated by the vector field (1.3). The group G is a "variational symmetry" of the functional $\mathcal{L}$ if and only if the Lagrangian satisfies

$$X(L) + LD(\xi) = 0. \tag{2.5}$$

We will actually need a weaker invariance condition than given by Eq. (2.5). The vector fields X is an "infinitesimal divergence symmetry" of the functional $\mathcal{L}(y)$ if there exists a function $B(x, y)$ such that

$$X(L) + LD(\xi) = D(B). \tag{2.6}$$

Furthermore, we can consider $B(x, y, y', ...)$. In this case we need to consider divergence invariance on the solutions of the Euler–Lagrange equations, i.e. relation (2.6) must hold on the Euler–Lagrange equation and its differential consequences.

The two important statements for us are:

1. If X is an infinitesimal divergence symmetry of the functional $\mathcal{L}$, it generates a symmetry group of the corresponding Euler–Lagrange equation.

2. Noether's theorem [22,...,26]
If X is an in infinitesimal divergence symmetry of the functional $\mathcal{L}$, then

$$N(L) - B = const, \qquad N = \xi + (\eta - \xi y')\frac{\partial}{\partial y'} \tag{2.7}$$

is a first integral of the corresponding Euler–Lagrange equation.

As a matter of fact, condition (2.5) is not necessarily to obtain the first integral (2.7). It suffices that the invariance condition (2.5) be satisfied on solutions of the Euler–Lagrange equations.

The above considerations tell us how to obtain invariant ODEs and conservation laws from divergence invariant Lagrangians. They do not tell us how to obtain invariant Lagrangians for invariant equations. This amounts to "variational integration", as opposed to variational differentiation.

A procedure that we shall use below to find invariant Lagrangians can be summed up as follows.

1. Start from a given ODE $y'' = f(x, y, y')$ and its symmetry algebra with basis

$$X_\alpha = \xi_\alpha(x,y)\frac{\partial}{\partial x} + \eta_\alpha(x,y)\frac{\partial}{\partial y}, \qquad \alpha = 1, ..., k.$$

Find the invariants of X_α in the space $\{x, y, y', \Lambda\}$, where Λ is the Lagrangian. The appropriate prolongation in this case is

$$\mathbf{pr}X = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \zeta^1\frac{\partial}{\partial y'} - (D\xi)\Lambda\frac{\partial}{\partial \Lambda}, \qquad \zeta^1 = D(\eta) - y'D(\xi) \tag{2.8}$$

and we require $L(x, y, y')$

$$\mathbf{pr}X(\Lambda - L)|_{\Lambda=L} = 0 \tag{2.9}$$

Each basis element X_α provides us with an equation of the form

$$\xi_\alpha\frac{\partial L}{\partial x} + \eta_\alpha\frac{\partial L}{\partial y} + \zeta^1_\alpha\frac{\partial L}{\partial y'} - (D\xi_\alpha)L = 0. \tag{2.10}$$

Solve the partial differential equations (2.10). This will provide us with the general form of an invariant Lagrangian. It may involve arbitrary functions of the invariants of X.

2. Request that the Euler–Lagrange equation (2.4) should coincide with the equation we started from. This will further restrict the invariant Lagrangian and determine whether one exists.

If this procedure does not yield suitable Lagrangian, step 1 can be weakened. We can request that the Lagrangian be invariant under some subgroup of the symmetry group of the given ODE, rather then the entire group. We then go though step 2, then verify whether the obtained Lagrangian is divergence invariant under the entire group, or at least a larger subgroup. In any case, each divergence symmetry of the Lagrangian will provide a first integral of the ODE.

2.3 Lagrangian formalism for second order difference equations

The variational formulation of discrete equations and a discrete analog of Noether's theorem are much more recent [21]. Here we briefly overview the results that we shall need below.

Let us consider a finite difference functional

$$\mathbb{L}_h = \sum_\Omega \mathcal{L}(x, x_+, y, y_+)h_+, \tag{2.11}$$

defined on some one–dimensional lattice Ω with step h_+ that can depend on the solution

$$h_+ = \varphi(x, y, x_+, y_+). \tag{2.12}$$

We are considering "two point" Lagrangians and lattices here, by analogy with first order Lagrangians for second order ODEs, where we had $L = L(x, y, y')$.

Let us again consider a Lie group of point transformations, generated by a Lie algebra of vector fields X_α of the form (1.3). The infinitesimal invariance condition of the functional (2.11) on the lattice (2.12) is given by two equations:

$$\begin{gathered} \xi\frac{\partial \mathcal{L}}{\partial x} + \xi^+\frac{\partial \mathcal{L}}{\partial x_+} + \eta\frac{\partial \mathcal{L}}{\partial y} + \eta^+\frac{\partial \mathcal{L}}{\partial y_+} + \mathcal{L}\underset{+h}{D}(\xi) = 0, \\ \underset{+h}{S}(\xi) - \xi = X(\varphi), \end{gathered} \tag{2.13}$$

where

$$\xi^+ = \xi(x_+, y_+), \quad \eta^+ = \eta(x_+, y_+), \quad \underset{\pm h}{S} f(x) = f(x_\pm), \quad \underset{+h}{D} = \frac{\underset{+h}{S} - 1}{h_+}. \tag{2.14}$$

Thus, $\underset{+h}{D}$ in the right discrete derivative.

A stationary value of the difference functional (2.11) is reached on a pair of equations, called *quasiextremals*. They are

$$\frac{\delta \mathcal{L}}{\delta x} = h_+ \frac{\partial \mathcal{L}}{\partial x} + h_- \frac{\partial \mathcal{L}^-}{\partial x} + \mathcal{L}^- - \mathcal{L}, \tag{2.15}$$

$$\frac{\delta \mathcal{L}}{\delta y} = h_+ \frac{\partial \mathcal{L}}{\partial y} + h_- \frac{\partial \mathcal{L}^-}{\partial y}, \tag{2.16}$$

where $\mathcal{L}^- = \underset{-h}{S}(\mathcal{L})$. These two equations can be viewed as providing both a discrete Euler–Lagrange equation and a lattice.

Now let us assume that the functional (2.11) is invariant under some Lie group G and let X be an element of the corresponding Lie algebra. Let us consider the variation of the functional along the orbit of the one–dimensional Lie group generated by X. The stationary value of the functional on the orbit is given by the quasiextremal equation

$$\xi \frac{\delta \mathcal{L}}{\delta x} + \eta \frac{\delta \mathcal{L}}{\delta y} = 0. \tag{2.17}$$

If the Lie algebra is at least two dimensional, we obtain two such equations and hence both quasiextremal equations (2.15) and (2.16) must be verified simultaneously. Moreover, the quasiextremal equations (2.15), (2.16) are not necessarily compatible with an arbitrary invariant lattice (2.12). Thus, the quasiextremal equations prescribe the choice of the lattice.

It has been shown elsewhere [21] that if the functional (2.11) is invariant under some group G, then the quasiextremal equations are also invariant.

A useful operator identity, valid for any Lagrangian $\mathcal{L}(x, x_+, y, y_+)$ and any vector field X is [21]

$$\begin{aligned} &\xi \frac{\partial \mathcal{L}}{\partial x} + \xi^+ \frac{\partial \mathcal{L}}{\partial x_+} + \eta \frac{\partial \mathcal{L}}{\partial y} + \eta^+ \frac{\partial \mathcal{L}}{\partial y_+} + \mathcal{L} \underset{+h}{D}(\xi) = \\ &= \xi \left(\frac{\partial \mathcal{L}}{\partial x} + \frac{h_-}{h_+} \frac{\partial \mathcal{L}^-}{\partial x} - \underset{+h}{D}(\mathcal{L}^-) \right) + \eta \left(\frac{\partial \mathcal{L}}{\partial y} + \frac{h_-}{h_+} \frac{\partial \mathcal{L}^-}{\partial y} \right) + \\ &+ \underset{+h}{D} \left(h_- \eta \frac{\partial \mathcal{L}^-}{\partial y} + h_- \xi \frac{\partial \mathcal{L}^-}{\partial x} + \xi \mathcal{L}^- \right). \end{aligned} \tag{2.18}$$

From Eq. (2.18) we obtain the following discrete analog of Noether's theorem.

Theorem 1. [21] Let the Lagrangian $\mathcal{L}$ be invariant under a Lie group G of dimension $n \geq 2$. Let the quasiextremal equations (2.15), (2.16) be satisfied. Then each basis element X_α of the Lie algebra generating the group G will provide us with a conservation law for the quasiextremal equations (2.15) and (2.16), namely

$$J_\alpha = h_- \eta_\alpha \frac{\partial \mathcal{L}^-}{\partial y} + h_- \xi_\alpha \frac{\partial \mathcal{L}^-}{\partial x} + \xi_\alpha \mathcal{L}^-, \qquad 1 \leq \alpha \leq n. \tag{2.19}$$

Similarly as in the continuous case, the invariance condition (2.13) for the Lagrangian can be weakened and the result of Theorem 1 strengthened. Indeed, let there exist a function $B = B(x, y, x_-, y_-, x_+, y_+, ...)$ such that the discrete Lagrangian satisfies

$$X(\mathcal{L}) + \mathcal{L} \underset{+h}{D}(\xi) = \underset{+h}{D}(B), \tag{2.20}$$

i.e. the Lagrangian is divergence invariant under X, rather than invariant. Then the symmetry X generates integral for the system (2.15), (2.16) given by

$$J = h_- \eta \frac{\partial \mathcal{L}^-}{\partial y} + h_- \xi \frac{\partial \mathcal{L}^-}{\partial x} + \xi \mathcal{L}^- - B. \tag{2.21}$$

Eq. (2.21) is a discrete analog of Eq. (2.7) for ODEs. The condition (2.20) should hold on the solutions of the quasiextremal equations (2.15) and (2.16). If $B = B(x, y)$, the condition should hold identically. In what follows we will consider only $B(x, y)$.

Theorem 2. If a Lagrangian $\mathcal{L}(x, x_+, y, y_+)$ is divergence invariant with respect to the operator X with some function $B = B(x, y)$, then the corresponding conservation law J is an invariant of the vector field X, i.e., $X(J) = 0$.

Proof From the equation (2.20) we can obtain the derivatives $B_x(x, y)$ and $B_y(x, y)$. Substituting them into the expression for $X(J)$, we obtain that $X(J) = 0$.

□

Theorem 3. Let quasiextremal equation (2.17) of the functional (2.11) correspond a group generated by an operator X. Let there exists an operator $\bar{X}$ which is a variational (or divergence) symmetry of the Lagrangian. Then the action of $\bar{X}$ transforms quasiextremal equation (2.17) into another quasiextremal equation of the same functional, namely the quasiextremal equation corresponding to the operator $X_* = [\bar{X}, X]$.

Proof The action of the operator $\bar{X}$ on the quasiextremal equation gives

$$\bar{X}(\xi) \frac{\delta \mathcal{L}}{\delta x} + \bar{X}(\eta) \frac{\delta \mathcal{L}}{\delta y} + \xi \bar{X} \left(\frac{\delta \mathcal{L}}{\delta x} \right) + \eta \bar{X} \left(\frac{\delta \mathcal{L}}{\delta y} \right) = 0. \tag{2.22}$$

Since $\bar{X}$ is a divergence symmetry we find with the help of the divergence invariance condition (2.20) that

$$\bar{X} \left(\frac{\delta \mathcal{L}}{\delta x} \right) = -\bar{\xi}_x \frac{\delta \mathcal{L}}{\delta x} - \bar{\eta}_x \frac{\delta \mathcal{L}}{\delta y}, \qquad \bar{X} \left(\frac{\delta \mathcal{L}}{\delta y} \right) = -\bar{\xi}_y \frac{\delta \mathcal{L}}{\delta x} - \bar{\eta}_y \frac{\delta \mathcal{L}}{\delta y},$$

where $\bar{\xi}$ and $\bar{\eta}$ are coefficients of the operator $\bar{X}$. Using these expressions, we rewrite the equation (2.22) as

$$(\bar{X}(\xi) - X(\bar{\xi})) \frac{\delta \mathcal{L}}{\delta x} + (\bar{X}(\eta) - X(\bar{\eta}) \frac{\delta \mathcal{L}}{\delta y} = 0.$$

Thus, it is the quasiextremal equation corresponding to the operator $[\bar{X}, X]$.

□

Lemma 1 If operators X and $\bar{X}$ are divergence symmetries of the Lagrangian (2.11) with functions B and $\bar{B}$ correspondingly, then the operator $X_* = [\bar{X}, X]$ is also a divergence symmetry with $B^* = \bar{X}(B) - X(\bar{B})$.

The same relation is valid for the conservation laws:

Theorem 4. Let a (divergence) variational symmetry X correspond to the conservation law J. Let there exists another (divergence) variational symmetry $\bar{X}$. Then the action of $\bar{X}$ transforms the conservation law J into the conservation law $J_* = \bar{X}(J)$ corresponding to the operator $X_* = [\bar{X}, X]$.

Proof Acting by the operator $\bar{X}$ on the conservation law J we obtain

$$\bar{X}(J)=\bar{X}(\eta)h_-\frac{\partial\mathcal{L}^-}{\partial y}+\eta\bar{X}\left(h_-\frac{\partial\mathcal{L}^-}{\partial y}\right)+\bar{X}(\xi)\left(h_-\frac{\partial\mathcal{L}^-}{\partial x}+\mathcal{L}^-\right)+\xi\bar{X}\left(h_-\frac{\partial\mathcal{L}^-}{\partial x}+\mathcal{L}^-\right)-\bar{X}(B).$$

Substituting found with the help of the divergence invariance condition for the operator $\bar{X}$ expressions

$$\bar{X}\left(h_-\frac{\partial\mathcal{L}^-}{\partial y}\right)=\bar{B}_y-\bar{\xi}_y\frac{\partial}{\partial x}(\mathcal{L}^-h_-)-\bar{\eta}_y\frac{\partial}{\partial y}(\mathcal{L}^-h_-),$$

$$\bar{X}\left(h_-\frac{\partial\mathcal{L}^-}{\partial x}+\mathcal{L}^-\right)=\bar{B}_x-\bar{\xi}_x\frac{\partial}{\partial x}(\mathcal{L}^-h_-)-\bar{\eta}_x\frac{\partial}{\partial y}(\mathcal{L}^-h_-),$$

we get

$$\bar{X}(J)=(\bar{X}(\eta)-X(\bar{\eta})h_-\frac{\partial\mathcal{L}^-}{\partial y}+(\bar{X}(\xi)-X(\bar{\xi}))\left(h_-\frac{\partial\mathcal{L}^-}{\partial x}+\mathcal{L}^-\right)-(\bar{X}(B)-X(\bar{B})).$$

This means that $\bar{X}(J)$ is the conservation law corresponding to the symmetry $[\bar{X},X]$.

□

Corollary On the solutions of the quasiextremal equations (2.15), (2.16) a (divergence) variational symmetry $\bar{X}$ transforms a conservation law J, corresponding to the symmetry X, into another conservation law $J_*=\bar{X}(J)$, corresponding to the symmetry $X_*=[\bar{X},X]$.

This property can be used to find all conservation laws from the so called *basic* conservation laws according to the action of the adjoint Lie algebra.

Let us compare the situation for second order ODEs and for three point difference schemes. For a second order ODE a Lagrangian that is divergence invariant under a two–dimensional symmetry group provides two integrals of motion. From them we can eliminate the remaining first derivative and obtain the general solution, depending on two arbitrary constants (the two first integrals). Moreover, we do not really need a Lagrangian. Once we have a two dimensional symmetry algebra, we can integrate in quadratures.

For three point difference schemes we have two equations to solve and their solution will involve four constants. The first problem is to find a discrete Lagrangian that provides the required difference scheme as quasiextremal equations. If such a Lagrangian exists and is invariant under some Lie group, then each basis element of the Lie algebra provides a first integral. Generally, there can be at most three functionally independent first integrals of the form

$$I(x,y,x_+,y_+)=const.$$

and four integrals of the form

$$\tilde{I}(i,x_i,y_i,x_{i+1},y_{i+1})=const.$$

If there are four first integrals of the latter from, we can completely integrate the discrete system. However, each first integral provides a simplification of the difference system.

3 Equations invariant under one and two–dimensional groups

We start with the simplest case of a symmetry group, namely the one–dimensional group. Its Lie algebra is generated by one vector field of the form (1.3). By an appropriate change of variables we take this vector field into its rectified form. Thus we have

$$\mathbf{D_{1,1}}:\qquad X_1=\frac{\partial}{\partial y}. \tag{3.1}$$

The most general second order ODE invariant under X_1 is

$$y'' = F(x, y'), \tag{3.2}$$

where F is an arbitrary function.

We are interested in the equations of this form which have a variational formulation and a variational symmetry. It is known that the Euler–Lagrange equation inherits all variational symmetries. Thus if there is a variational formulation and it has a symmetry it must be the symmetry X_1 and, consequently, the Lagrangian has the form $L = L(x, y')$. Therefore, the Euler–Lagrange equation is

$$\frac{\partial^2 L}{\partial x \partial y'} + y'' \frac{\partial^2 L}{\partial y'^2} = 0,$$

or

$$y'' = -\left(\frac{\partial^2 L}{\partial y'^2}\right)^{-1} \frac{\partial^2 L}{\partial x \partial y'}. \tag{3.3}$$

Since we can not provide a variational formulation for the general equation (3.2) we will consider its particular case, i.e. the equation (3.3) with an arbitrary function $L = L(x, y')$.

Noether's theorem provides us one first integral

$$J_1 = \frac{\partial L}{\partial y'}(x, y').$$

Let us remark that one can express y' from this first integral and write out the solution with the help of the quadrature so that one variational symmetry let us reduce the order of the equation by two.

Let us consider the discrete Lagrangian

$$L(x, x_+, y_x),$$

which is invariant under X_1. The corresponding quasiextremal equations are

$$\begin{aligned} \frac{\delta \mathcal{L}}{\delta y} &: \quad -L_3(x, x_+, y_x) + L_3(x_-, x, y_{\bar{x}}) = 0; \\ \frac{\delta \mathcal{L}}{\delta x} &: \quad L_1(x, x_+, y_x) h_+ - L(x, x_+, y_x) + L_2(x_-, x, y_{\bar{x}}) h_- + L(x_-, x, y_{\bar{x}}) = 0. \end{aligned} \tag{3.4}$$

Since the Lagrangian is invariant with respect to the operator (3.1) the system has the first integral

$$I_1 = L_3(x, x_+, y_x).$$

Remark Let us note that in the case of one–dimensional groups there is no need to consider the intersection of the quasiextremal equations (2.15), (2.16). To obtain a discrete model with a first integral it suffices to consider the system

$$\begin{aligned} &\xi \frac{\delta \mathcal{L}}{\delta x} + \eta \frac{\delta \mathcal{L}}{\delta y} = 0; \\ &\Omega(x_-, x, x_+, y_-, y, y_+) = 0. \end{aligned} \tag{3.5}$$

The first equation is the variational equation for the variation of the Lagrangian along the group orbit and the second equation is an invariant mesh which can be taken arbitrary. This system possesses the first integral given by (2.19). For the considered case of the symmetry (3.1) we get the discrete model

$$\frac{\delta\mathcal{L}}{\delta y} = 0; \qquad \Omega(x_-, x, x_+, y_x, y_{\bar{x}}) = 0; \tag{3.6}$$

which has the first integral I_1. For simplicity one can choose uniform mesh $h_- = h_+$.

It is interesting to note that the first integral I_1 let us express $y_x = g(x, x_+)$ so that we can reduce the system (3.6) (in particular the system (3.4)) to one second order discrete equation

$$\tilde{\Omega}(x_-, x, x_+) = 0.$$

Thus one variational symmetry let us reduce the order of the system by two. If we know the solution of the equation $\tilde{\Omega} = 0$, we can find the solution of the system solving one first order discrete equation $y_x = g(x, x_+)$.

$\mathbf{D_{2,1}}$ The Abelian Lie algebra with nonconnected basis elements

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y} \tag{3.7}$$

corresponds to the invariant ODE

$$y'' = F(y'), \tag{3.8}$$

where F is an arbitrary function. If

$$F(y') = \frac{1}{G''(y')} \tag{3.9}$$

the equation can be obtained from the Lagrangian

$$L = G(y') + y, \tag{3.10}$$

which admits symmetries X_1 and X_2:

$$X_1 L + LD(\xi_1) = 0;$$

$$X_2 L + LD(\xi_2) = 1 = D(x).$$

With the help of Noether's theorem we obtain the following first integrals:

$$J_1 = y + G(y') - y'G'(y'), \qquad J_2 = G'(y') - x.$$

As we mentioned in the proceeding section it is sufficient to have two first integrals to write out the general solution of a second order ODE without quadratures.

Now we are in a position to show how one can find a variational discrete model and its conservation laws by means of Lagrange-type technique. Let us choose a difference Lagrangian in the form

$$\mathcal{L} = \frac{y_+ + y}{2} + G(y_x),$$

then

$$X_1\mathcal{L} + \mathcal{L}\underset{+h}{D}(\xi_1) = 0;$$

$$X_2\mathcal{L} + \mathcal{L}\underset{+h}{D}(\xi_2) = 1 = \underset{+h}{D}(x).$$

The variations of $\mathcal{L}$ yield the following intersection of quasiextremal equations:

$$\begin{aligned} \frac{\delta \mathcal{L}}{\delta y} &: \quad G'(y_x) - G'(y_{\bar{x}}) = \frac{h_+ + h_-}{2}; \\ \frac{\delta \mathcal{L}}{\delta x} &: \quad -\frac{y_+ + y}{2} - G(y_x) + y_x G'(y_x) + \frac{y + y_-}{2} + G(y_{\bar{x}}) - y_{\bar{x}} G'(y_{\bar{x}}) = 0. \end{aligned} \tag{3.11}$$

Due to the invariance of the Lagrangian with respect to the operators X_1 and X_2, the difference analog of Noether's theorem yields two first integrals

$$I_1 = \frac{y_+ + y}{2} + G(y_x) - y_x G'(y_x), \qquad I_2 = -\frac{x + x_+}{2} + G'(y_x). \tag{3.12}$$

$\mathbf{D_{2,2}}$ The non–Abelian Lie algebra with nonconnected elements

$$X_1 = \frac{\partial}{\partial y}, \quad X_2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \tag{3.13}$$

yields the invariant ODE

$$y'' = \frac{1}{x} F(y'). \tag{3.14}$$

In the case

$$F(y') = \frac{G'(y')}{G''(y')} \tag{3.15}$$

this ODE is the Euler–Lagrangian equation for the Lagrangian

$$L = \frac{1}{x} G(y'), \tag{3.16}$$

which admits X_1 and X_2 as variational symmetries:

$$X_1 L + L D(\xi_1) = 0;$$

$$X_2 L + L D(\xi_2) = 0.$$

Noether's theorem provides us two first integrals:

$$J_1 = \frac{1}{x} G'(y'), \qquad J_2 = G(y') + \left(\frac{y}{x} - y'\right) G'(y').$$

Since $[X_1, X_2] = X_1$ the first integral J_2 is basic: $J_1 = X_1(J_2)$.

Let us take the difference Lagrangian

$$\mathcal{L} = \frac{2}{x + x_+} G(y_x),$$

which satisfies

$$X_1 \mathcal{L} + \mathcal{L} \underset{+h}{D}(\xi_1) = 0;$$

$$X_2 \mathcal{L} + \mathcal{L} \underset{+h}{D}(\xi_2) = 0.$$

Then the variations of $\mathcal{L}$ yield the following intersection of quasiextremal equations:

$$\begin{aligned}
&\frac{\delta\mathcal{L}}{\delta y}: \quad \frac{2}{x+x_+}G'(y_x) - \frac{2}{x+x_-}G'(y_{\bar{x}}) = 0;\\
&\frac{\delta\mathcal{L}}{\delta x}: \quad \begin{aligned}&-\frac{2h_+}{(x+x_+)^2}G(y_x) + \frac{2}{(x+x_+)}G'(y_x)y_x - \frac{2}{(x+x_+)}G(y_x)\\
&-\frac{2h_-}{(x+x_-)^2}G(y_{\bar{x}}) - \frac{2}{(x+x_-)}G'(y_{\bar{x}})y_{\bar{x}} + \frac{2}{(x+x_+)}G(y_{\bar{x}}) = 0.\end{aligned}
\end{aligned} \tag{3.17}$$

Since the Lagrangian is invariant with respect to the operators X_1 and X_2, we find the first integrals

$$I_1 = \frac{2G'(y_x)}{x+x_+}, \qquad I_2 = \frac{4xx_+}{(x+x_+)^2}G(y_x) + \frac{2G'(y')}{x+x_+}\left(\frac{y+y_+}{2} - \frac{x+x_+}{2}y_x\right) \tag{3.18}$$

for the solutions of (3.17). Similarly to the continuous case $I_1 = X_1(I_2)$.

4 Equations invariant under three-dimensional Lie groups

In this section we consider four cases of three–dimensional groups.

$\mathbf{D_{3,1}}$ The Lie algebra

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = \frac{\partial}{\partial y}, \qquad X_3 = x\frac{\partial}{\partial x} + (x+y)\frac{\partial}{\partial y} \tag{4.1}$$

corresponds to the invariant equation

$$y'' = \exp(-y'). \tag{4.2}$$

The equation can be obtained from the Lagrangian

$$L = \exp(y') + y, \tag{4.3}$$

which admits operators X_1 and X_2:

$$X_1L + LD(\xi_1) = 0;$$

$$X_2L + LD(\xi_2) = 1 = D(x);$$

but does not admit operator X_3. It is possible to show that there is no Lagrangian which admits all operators as divergence symmetries. Meanwhile, two different variational symmetries give two functionally independent first integrals that is enough to integrate a second order ODE.

Lemma 2. There is no Lagrangian function $L(x, y, y')$ which gives (4.2) as its Euler–Lagrange equation and is divergence invariant with $B = B(x, y)$ for all three symmetries (4.1).

Proof Let it not be true, then there exists a Lagrangian $L(x, y, y')$ which is divergence invariant for the operators (4.1):

$$X_i(L) + LD_x(\xi_i) = D_x(B^i(x,y)) = B^i_x(x,y) + B^i_y(x,y)y', \qquad i = 1,2,3$$

with some functions $B^1(x, y)$, $B^2(x, y)$ and $B^3(x, y)$, i.e.

$$\begin{cases}
\dfrac{\partial}{\partial x}L = B^1_x(x,y) + B^1_y(x,y)y';\\[2ex]
\dfrac{\partial}{\partial y}L = B^2_x(x,y) + B^2_y(x,y)y';\\[2ex]
x\dfrac{\partial}{\partial x}L + (x+y)\dfrac{\partial}{\partial y}L + \dfrac{\partial}{\partial y'}L + L = B^3_x(x,y) + B^3_y(x,y)y'.
\end{cases}$$

Note that the right hand sides of the equations are linear with respect to y'. Differentiating both sides of each equation with respect to y' twice, we get

$$\begin{cases} \dfrac{\partial^3 L}{\partial y'^2 \partial x} = 0; \\[2mm] \dfrac{\partial^3 L}{\partial y'^2 \partial y} = 0; \\[2mm] x\dfrac{\partial^3 L}{\partial y'^2 \partial x} + (x+y)\dfrac{\partial^3 L}{\partial y'^2 \partial y} + \dfrac{\partial^3 L}{\partial y'^3} + \dfrac{\partial^2 L}{\partial y'^2} = 0. \end{cases}$$

The equations require that

$$\frac{\partial^2}{\partial y'^2} L(x, y, y') = A \exp(-y'), \qquad A = const.$$

It follows that the solution of the system has the form

$$L(x, y, y') = A \exp(-y') + C(x, y)y' + D(x, y),$$

where $C(x, y)$ and $D(x, y)$ are arbitrary functions. It can be shown that no Lagrangian of this form can give the equation (4.2) as its Euler–Lagrange equation.

□

Two first integrals given by the operators X_1 and X_2

$$J_1 = \exp(y')(1 - y') + y, \qquad J_2 = \exp(y') - x$$

permit us to integrate the equation. Let us put $J_1 = A^0$, $J_2 = B^0$, where A^0 and B^0 are arbitrary constants, then

$$(B^0 + x)(1 - \ln(B^0 + x)) + y = A^0, \tag{4.4}$$

or

$$y = (B^0 + x)(\ln(B^0 + x) - 1) + A^0.$$

In the discrete case we can take the Lagrangian

$$\mathcal{L} = \exp(y_x) + \frac{y_+ + y}{2},$$

for which

$$X_1 \mathcal{L} + \mathcal{L} \underset{+h}{D}(\xi_1) = 0;$$

$$X_2 \mathcal{L} + \mathcal{L} \underset{+h}{D}(\xi_2) = 1 = \underset{+h}{D}(x).$$

The Lagrangian yields the quasiextremal equations:

$$\begin{aligned} \frac{\delta \mathcal{L}}{\delta y} &: \quad \frac{2}{h_- + h_+}(\exp(y_x) - \exp(y_{\bar{x}})) = 1; \\ \frac{\delta \mathcal{L}}{\delta x} &: \quad \exp(y_x)(y_x - 1) - \frac{y_+ + y}{2} - \exp(y_{\bar{x}})(y_{\bar{x}} - 1) + \frac{y + y_-}{2} = 0. \end{aligned} \tag{4.5}$$

This system of equations is invariant with respect to all three operators (4.1). Note that the invariance under X_3 is not inherited from the invariance of the Lagrangian. Due to the variational invariance with respect to the operators X_1 and X_2 the difference version of Noether's theorem let us find the first integrals

$$I_1 = \exp(y_x)(1 - y_x) + \frac{y_+ + y}{2}, \qquad I_2 = \exp(y_x) - \frac{x_+ + x}{2}. \tag{4.6}$$

$\mathbf{D_{3,2}}$ For the set of operators

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = \frac{\partial}{\partial y}, \qquad X_3 = x\frac{\partial}{\partial x} + ky\frac{\partial}{\partial y}, \qquad k \neq 0, \frac{1}{2}, 1, 2 \tag{4.7}$$

the invariant equation has the form

$$y'' = y'^{\frac{k-2}{k-1}}. \tag{4.8}$$

This equation can be obtained from the Lagrangian

$$L = \frac{(k-1)^2}{k}(y')^{\frac{k}{k-1}} + y, \tag{4.9}$$

which admits operators X_1 and X_2 for any parameter k and X_3 for $k = -1$:

$$X_1 L + L D(\xi_1) = 0;$$

$$X_2 L + L D(\xi_2) = 1 = D(x);$$

$$X_3 L + L D(\xi_3) = (k+1)L.$$

As for the previous case it is possible to show the following result.

Lemma 3. There is no Lagrangian function $L(x, y, y')$ which gives (4.8) with $k \neq -1$ as its Euler–Lagrange equation and is divergence invariant with $B = B(x, y)$ for all three symmetries (4.7).

For arbitrary k there are two first integrals

$$J_1 = \frac{(1-k)}{k}(y')^{\frac{k}{k-1}} + y, \qquad J_2 = (k-1)(y')^{\frac{1}{k-1}} - x.$$

They let us find the general solution setting $J_1 = A^0$, $J_2 = B^0$:

$$(A^0 - y)(B^0 + x)^{-k} = -\frac{1}{k}\left(\frac{1}{k-1}\right)^{(k-1)}, \tag{4.10}$$

or equivalently,

$$y = \frac{1}{k}\left(\frac{1}{k-1}\right)^{(k-1)}(B^0 + x)^k + A^0.$$

In the case $k = -1$ we can find the first integral corresponding to the symmetry X_3:

$$J_3 = \frac{2}{\sqrt{y'}}(y - xy') + xy.$$

It is functionally dependent on J_1 and J_2 since a second order ODE can possess only two functionally independent first integrals. Let us mention that the first integral J_3 is basic. Because of

$$[X_1, X_3] = X_1, \qquad [X_2, X_3] = -kX_2$$

Theorem 4 provides us the relationship between the first integrals

$$J_1 = X_1(J_3), \qquad J_2 = -X_2(J_3).$$

Let us choose

$$\mathcal{L} = \frac{(1-k)^2}{k}(y_x)^{\frac{k}{k-1}} + \frac{y_+ + y}{2}$$

as a discrete Lagrangian. It is invariant for X_1 and divergence invariant for X_2:

$$X_1\mathcal{L} + \mathcal{L}\underset{+h}{D}(\xi_1) = 0;$$

$$X_2\mathcal{L} + \mathcal{L}\underset{+h}{D}(\xi_2) = 1 = \underset{+h}{D}(x).$$

In the case $k = -1$ it is also invariant with respect to the operator X_3:

$$X_3\mathcal{L} + \mathcal{L}\underset{+h}{D}(\xi_3) = 0.$$

From the Lagrangian we obtain the quasiextremal equations

$$\begin{aligned} &\frac{\delta\mathcal{L}}{\delta y}: \quad \frac{2(k-1)}{h_- + h_+}\left((y_x)^{\frac{1}{k-1}} - (y_{\bar{x}})^{\frac{1}{k-1}}\right) = 1; \\ &\frac{\delta\mathcal{L}}{\delta x}: \quad \frac{(k-1)}{k}\left((y_x)^{\frac{k}{k-1}} - (y_{\bar{x}})^{\frac{k}{k-1}}\right) - \frac{y_+ + y}{2} + \frac{y + y_-}{2} = 0. \end{aligned} \tag{4.11}$$

This system of equations is invariant with respect to all three operators (4.7). For every solution of this system there are two first integrals

$$I_1 = \frac{(1-k)}{k}(y_x)^{\frac{k}{k-1}} + \frac{y_+ + y}{2}, \qquad I_2 = (k-1)(y_x)^{\frac{1}{k-1}} - \frac{x_+ + x}{2}. \tag{4.12}$$

Let us examine the particular case $k = -1$ in details. As for the underlying continuous equation we get an additional first integral

$$I_3 = \frac{2(x_+ y - y_+ x)}{h_+\sqrt{y_x}} + \frac{1}{2}(x_+ y + y_+ x) \tag{4.13}$$

corresponding to the symmetry X_3. In contrast to the continuous case three discrete first integrals I_1, I_2 and I_3 are functionally independent. The first integral I_3 is basic

$$I_1 = X_1(I_3), \qquad I_2 = -X_2(I_3).$$

Using all three first integrals, we can integrate the system of equation (4.11). Since

$$4 - I_1 I_2 - I_3 = \frac{1}{4}h_+^2 y_x$$

the expression $h_+^2 y_x$ is also a first integral of (4.11) that let us involve a special case of the mesh:

$$h_-^2 y_{\bar{x}} = h_+^2 y_x = \varepsilon, \quad \varepsilon = const, \quad 0 < \varepsilon \ll 1. \tag{4.14}$$

Setting $I_1 = A^h$, $I_2 = B^h$, we can find the solution of the quasiextremal equations (4.11) on the mesh (4.14)

$$(A^h - y)(B^h + x) = 4 - \left(\frac{\varepsilon}{2}\right)^2 \tag{4.15}$$

that well corresponds to the continuous solution (4.10). Thus three first integrals of the discrete model let us reduce the order of the discrete model by three. We integrated the difference ODE and reduced the order of the mesh equation by one.

In fact we can completely integrate the system (4.11). Let us substitute the mesh (4.14) into the system, then it gets reduced to the one variable equation

$$(4-\sqrt{\varepsilon})y_+ - 8y + (4+\sqrt{\varepsilon})y_- = 0. \tag{4.16}$$

This equation has the first integral

$$\tilde{I}_1 = -2\frac{y_+ - y}{\sqrt{\varepsilon}} + \frac{y_+ + y}{2},$$

which is the first integral I_1 inherited by the equation (4.16). The solution of (4.16) can be easily written as

$$y_i = C_1\left(\frac{4+\sqrt{\varepsilon}}{4-\sqrt{\varepsilon}}\right)^i + C_2, \qquad C_1, C_2 = const.$$

From the initial conditions we find

$$C_1 = \frac{4-\sqrt{\varepsilon}}{2\varepsilon}(y_1 - y_0), \qquad C_2 = \tilde{I}_1 = A^h, \qquad \varepsilon = (x_1 - x_0)(y_1 - y_0).$$

Having obtained the solution for y, we can easily find the mesh points with the help of (4.15) as

$$x_i = \left(4 - \frac{\varepsilon^2}{4}\right)\frac{1}{A^h - y_i} - B^h,$$

where

$$A^h = -2\sqrt{\frac{y_1 - y_0}{x_1 - x_0}} + \frac{y_1 + y_0}{2}, \qquad B^h = 2\sqrt{\frac{x_1 - x_0}{y_1 - y_0}} + \frac{x_1 + x_0}{2}.$$

Thus in the particular case $k = -1$ we completely integrated the quasiextremal equations.

$\mathbf{D_{3,3}}$ The group given by the operators

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \qquad X_3 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} \tag{4.17}$$

has the invariant equation

$$y'' = y^{-3}. \tag{4.18}$$

This equation can be obtained from the Lagrangian function

$$L = y'^2 - \frac{1}{y^2}, \tag{4.19}$$

which admits all three operators:

$$X_1 L + LD(\xi_1) = 0;$$

$$X_2 L + LD(\xi_2) = 0;$$

$$X_3 L + LD(\xi_3) = 2y'y = D(y^2).$$

Consequently, the symmetries let is find the following first integrals

$$J_1 = y'^2 + \frac{1}{y^2}, \quad J_2 = 2\frac{x}{y^2} - 2(y - y'x)y', \quad J_3 = \frac{x^2}{y^2} + (y - xy')^2.$$

As we noticed before only two first integrals of a second order ODE can be functionally independent. The structure of action of the adjoint Lie algebra

$$[X_1, X_2] = 2X_1, \qquad [X_1, X_3] = X_2, \qquad [X_2, X_3] = 2X_3$$

provides us the relations between the first integrals:

$$X_1(J_3) = J_2, \qquad X_1(J_2) = 2J_1, \qquad X_3(J_1) = -J_2, \qquad X_3(J_2) = -2J_3.$$

Therefore any first integral can be chosen as basic. Putting $J_1 = A^0$ and $J_2 = 2B^0$, we find the solution as

$$A^0 y^2 = (A^0 x - B^0)^2 + 1. \tag{4.20}$$

Let us consider the discrete Lagrangian function

$$\mathcal{L} = y_x^2 - \frac{1}{y_+ y}, \tag{4.21}$$

which admits all three operators (4.17):

$$X_1 \mathcal{L} + \mathcal{L} \underset{+h}{D}(\xi_1) = 0;$$

$$X_2 \mathcal{L} + \mathcal{L} \underset{+h}{D}(\xi_2) = 0;$$

$$X_3 \mathcal{L} + \mathcal{L} \underset{+h}{D}(\xi_3) = \underset{+h}{D}(y^2).$$

The Lagrangian generates the variational equations

$$\begin{aligned} \frac{\delta \mathcal{L}}{\delta y}: &\quad 2(y_x - y_{\bar{x}}) = \frac{h_+}{y^2 y_+} + \frac{h_-}{y^2 y_-}; \\ \frac{\delta \mathcal{L}}{\delta x}: &\quad (y_x)^2 + \frac{1}{y y_+} - (y_{\bar{x}})^2 - \frac{1}{y y_-} = 0; \end{aligned} \tag{4.22}$$

which are invariant with respect to the operators X_1, X_2, X_3 as it follows from the general theory and can easily be checked. The quasiextremal equations have three functionally independent first integrals

$$\begin{gathered} I_1 = {y_x}^2 + \frac{1}{y y_+}, \qquad I_2 = \frac{x + x_+}{y y_+} - y_x((y + y_+) - (x + x_+) y_x), \\ I_3 = \frac{x x_+}{y y_+} + \left(\frac{y + y_+}{2} - \frac{x + x_+}{2} y_x \right)^2. \end{gathered} \tag{4.23}$$

Similarly to the continuous case the discrete first integrals obey relations

$$X_1(I_3) = I_2, \qquad X_1(I_2) = 2I_1, \qquad X_3(I_1) = -I_2, \qquad X_3(I_2) = -2I_3$$

and, consequently, any first integral can be taken as basic.

To integrate (4.22) we involve a special case of the invariant mesh

$$\frac{h_-}{y_- y} = \frac{h_+}{y y_+} = \varepsilon, \quad \varepsilon = const, \quad 0 < \varepsilon \ll 1 \tag{4.24}$$

that is justified since $\frac{h_+}{yy_+}$ is a first integrals of the system as it is clearly seen from

$$\frac{1}{4}\left(\frac{h_+}{yy_+}\right)^2 = 1 + \frac{I_2^2}{4} - I_1 I_3.$$

Then a general solution of the discrete model (4.22) can be found with the help of the first integrals $I_1 = A^h$, $I_2 = 2B^h$ as

$$A^h y^2 = (A^h x - B^h)^2 + 1 - \frac{\varepsilon^2}{4}. \tag{4.25}$$

Notice, that the solution (4.25) contains three parameters and a third parameter ε springs from the mesh, where (4.25) is defined. This solution differs from the solution (4.20) of the underlying continuous equation by $\frac{\varepsilon^2}{4}$ and the estimation is homogeneous.

Substitution of the mesh (4.24) into the system of quasiextremal equations (4.22) reduces the latter to the equation

$$y_+ y_-(2 - \varepsilon^2) = y(y_+ + y_-), \tag{4.26}$$

which is a one–dimensional mapping. This mapping has the only first integral

$$\tilde{I}_1 = \left(\frac{y^+ - y}{\varepsilon y y^+}\right)^2 + \frac{1}{yy_+},$$

which is the first integral I_1 inherited from the system of quasiextremals.

Let us introduce the change of variable $z = \frac{1}{y}$, then the equation (4.26) becomes linear

$$z_+ - (2 - \varepsilon^2)z + z_- = 0$$

and can be integrated. Its solution can be presented in the form

$$z_i = C_1 \cos(i\varphi) + C_2 \sin(i\varphi), \qquad \varphi = \arctan\left(\frac{\varepsilon\sqrt{4 - \varepsilon^2}}{2 - \varepsilon^2}\right),$$

where C_1 and C_2 are constants which are to be determines from the initial condition

$$C_1 = z_0 = \frac{1}{y_0}, \qquad C_2 = \frac{1}{\sin(\varphi)}(z_1 - z_0\cos(\varphi)) = \frac{1}{\sin(\varphi)}\left(\frac{1}{y_1} - \frac{\cos(\varphi)}{y_0}\right).$$

Having found the general solution for $y_i = \frac{1}{z_i}$, we can obtain the mesh points x_i with the help of (4.25), where A^h and B^h are found using the initial conditions. Let as note that y_i reaches infinity in a finite number of steps: the discrete solution z_i jumps over zero that corresponds to the jump over infinity for the solution y_i.

So, we developed an invariant difference model of the original ODE, which possesses the same Lagrangian structure. The first integrals allow us completely integrate the discrete model.

$\mathbf{D_{3,4}}$ The group with operators

$$X_1 = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}, \qquad X_2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \qquad X_3 = x^2\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y} \tag{4.27}$$

corresponds to the invariant equation

$$y'' + 2\frac{y' + Cy'\sqrt{y'} + y'^2}{x - y} = 0 \tag{4.28}$$

which can be obtained by variation of the Lagrangian function

$$L = \frac{Cy' + 4\sqrt{y'} + C}{2(x-y)}.$$

It is easy to check that the operators (4.27) are admitted by the Lagrangian:

$$X_1 L + LD(\xi_1) = 0;$$

$$X_2 L + LD(\xi_2) = 0;$$

$$X_3 L + LD(\xi_3) = \frac{C}{2}(1-y') = D\left(\frac{C}{2}(x-y)\right).$$

Applying Noether's theorem, we get three first integrals

$$J_1 = \frac{y' + C\sqrt{y'} + 1}{(x-y)\sqrt{y'}}, \quad J_2 = \frac{xy' + C\frac{x+y}{2}\sqrt{y'} + y}{(x-y)\sqrt{y'}}, \quad J_3 = \frac{x^2y' + Cxy\sqrt{y'} + y^2}{(x-y)\sqrt{y'}}.$$

According to the structure of the adjoint Lie algebra

$$[X_1, X_2] = X_1, \qquad [X_1, X_3] = 2X_2, \qquad [X_2, X_3] = X_3$$

the first integrals are connected as follows

$$X_1(J_3) = 2J_2, \qquad X_1(J_2) = J_1, \qquad X_3(J_1) = -2J_2, \qquad X_3(J_2) = -J_3.$$

We can find the solution of the equation setting the first integrals $J_1 = A^0$, $J_2 = B^0$:

$$\left(B^0 - A^0 y - \frac{C}{2}\right)\left(A^0 x - B^0 - \frac{C}{2}\right) = 1$$

or

$$y = \frac{1}{A^0(B^0 + C/2 - A^0 x)} + \frac{2B^0 - C}{2A^0}.$$

Remark This function is a solution of (4.28) if a square root of the first derivative is chosen as

$$\sqrt{y'} = -\frac{1}{B^0 + C/2 - A^0 x}.$$

There is also a special solution

$$y = ax,$$

where the constant $a \neq 2$ is the solution of the algebraic equation

$$a + Ca\sqrt{a} + a^2 = 0.$$

To consider a variational model we choose the difference Lagrangian

$$\mathcal{L} = \frac{Cy_x + 4\sqrt{y_x} + C}{2\sqrt{(x-y)(x_+ - y_+)}}.$$

This Lagrangian is invariant only with respect to the operators X_1 and X_2:

$$X_1\mathcal{L} + \mathcal{L}\underset{+h}{D}(\xi_1) = 0;$$

$$X_2\mathcal{L} + \mathcal{L}\underset{+h}{D}(\xi_2) = 0;$$

By means of the variation we get the quasiextremal equations

$$
\begin{aligned}
\frac{\delta \mathcal{L}}{\delta y}: \quad & -\frac{C+2(y_x)^{-1/2}}{\sqrt{(x-y)(x_+-y_+)}}+\frac{C+2(y_{\bar{x}})^{-1/2}}{\sqrt{(x_--y_-)(x-y)}}+\\
&+\frac{1}{2(x-y)}\left(\frac{Cy_x+4\sqrt{y_x}+C}{\sqrt{(x-y)(x_+-y_+)}}h_+ +\frac{Cy_{\bar{x}}+4\sqrt{y_{\bar{x}}}+C}{\sqrt{(x_--y_-)(x-y)}}h_-\right)=0;\\
\frac{\delta \mathcal{L}}{\delta x}: \quad & -\frac{C+2\sqrt{y_x}}{\sqrt{(x-y)(x_+-y_+)}}+\frac{C+2\sqrt{y_{\bar{x}}}}{\sqrt{(x_--y_-)(x-y)}}+\\
&-\frac{1}{2(x-y)}\left(\frac{Cy_x+4\sqrt{y_x}+C}{\sqrt{(x-y)(x_+-y_+)}}h_+ +\frac{Cy_{\bar{x}}+4\sqrt{y_{\bar{x}}}+C}{\sqrt{(x_--y_-)(x-y)}}h_-\right)=0.
\end{aligned}
\tag{4.29}
$$

This equations are invariant only with respect to the symmetries X_1 and X_2 from (4.27), but not with respect to the third operator X_3. However, the linear combination $\frac{\delta\mathcal{L}}{\delta y}+\frac{\delta\mathcal{L}}{\delta x}=0$ is invariant with respect to X_3.

There are only two independent difference first integrals which hold on the quasiextremal equations

$$
I_1=\frac{y_x+C\sqrt{y_x}+1}{\sqrt{(x-y)(x_+-y_+)}\sqrt{y_x}}, \qquad I_2=\frac{\frac{x+x_+}{2}y_x+C\frac{x_++x+y+y_+}{4}\sqrt{y_x}+\frac{y+y_+}{2}}{\sqrt{(x-y)(x_+-y_+)}\sqrt{y_x}},
$$

and first integral I_2 is basic: $X_1(I_2)=I_1$.

Let us summarize that in the considered above cases of one, two and three–dimensional groups we had continuous Lagrangians invariant with respect to a subgroup of the considered group and possibly divergence invariant with respect to other operators. Starting from a continuous Lagrangian which is invariant with respect to a set of symmetries, we can construct a discrete Lagrangian which is invariant with respect to the same symmetries. However, if the original continuous Lagrangian is divergence invariant, the corresponding discrete Lagrangian may fail to be divergence invariant.

5 The other cases of the group classification of discrete models

The other types of three–point discrete models, which were found in the group classification [1], can be split into two groups: discrete models which can be transformed into the linear equation on some mesh

$$
\begin{cases} y_{x\bar{x}}=0; \\ \Omega(x_-,x,x_+,y_-,y,y_+)=0; \end{cases}
\tag{5.1}
$$

and discrete models without a continuous limit .

5.1 The linear discrete equation $y_{x\bar{x}}=0$

Many cases of the group classification of the three–point discrete models can be united into the form (5.1), where the mesh equations depends on the considered symmetry group. On the other hand, models of this form can be found as subcases of the more general invariant models when we specify arbitrary functions or constants.

Let us consider the discrete Lagrangian

$$
\mathcal{L}=\frac{y_x^2}{2}.
$$

It generates a degenerate system of quasiextremal equations providing only one equation $y_{x\bar{x}} = 0$. Invariance of the Lagrangian with respect to the operators

$$X_1 = \frac{\partial}{\partial y} \quad \text{and} \quad X_2 = 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$$

provides us two first integrals

$$I_1 = y_x, \qquad I_2 = y_x\left(\frac{y_+ + y}{2} - y_x\frac{x_+ + x}{2}\right),$$

which hold independently on the mesh equation. Setting $a = I_1$ and $b = I_2/I_1$, the solution of the linear equation can be easily written out

$$y = ax + b, \qquad a, b = const, \tag{5.2}$$

so that we are left with the mesh equation

$$\tilde{\Omega}(x_-, x, x_+) = \Omega(x_-, x, x_+, ax_- + b, ax + b, ax_+ + b) = 0.$$

Further order reduction depends on the possibilities to integrate the mesh equation $\tilde{\Omega} = 0$.

Remark Let us note that one can use Lagrangian approach for one variable to integrate the mesh equation. To do so we need to consider the Lagrangian of the form $L = L(x, x_+)$ that allows to use the theory presented in the beginning of the paper. If the mesh equation $\tilde{\Omega} = 0$ can be presented as an extremal equation, then a variational symmetry leads to a conservation law.

Classification of the system (5.1) over the mesh equation was singled out in [1]. Let is give the numbers of obtained models corresponding to group classification over Diff$(2, \mathbb{C})$ in Table 2.

Table 2. Number of invariant difference models of the form (5.1).

group dimension	1	2	3	4	6
number of difference models	1	2	6	3	1

Example 1. Here we consider only one discrete model with the linear difference equation, which admits the six–dimensional symmetry group (the largest possible for three–point discrete models)

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = x\frac{\partial}{\partial x}, \quad X_4 = y\frac{\partial}{\partial x}, \quad X_5 = x\frac{\partial}{\partial y}, \quad X_6 = y\frac{\partial}{\partial y}. \tag{5.3}$$

The group (5.3) is admitted by the system

$$\begin{cases} y_{x\bar{x}} = 0; \\ \\ h_+ = Ch_-; \end{cases} \tag{5.4}$$

where C is an arbitrary constant. It is reasonable to impose the restriction $C > 0$. Note that the mesh equation has one first integral $I_1 = x_+ - Cx$.

The mesh equation can be easily integrated

$$x_i = \begin{cases} \dfrac{x_1 - x_0}{C - 1}C^i + \dfrac{Cx_0 - x_1}{C - 1}, & \text{if} \quad C \neq 1, \\ \\ (x_1 - x_0)i + x_0, & \text{if} \quad C = 1, \end{cases}$$

providing together with (5.2) the general solution of (5.4). Let us remark that for $C \neq 1$ the mesh points $\{x_i\}$, $i \in \mathbb{Z}$ do not reach either ∞ or $-\infty$.

5.2 Discrete models without a continuous limit

In [1] going through the finite groups by dimension, we constructed invariant discrete models which involve arbitrary functions and constants. For each case we specified the choice of functions (or constants) which leads to an invariant ODE in the continuous limit. However, there are cases when there is no continuous limit. Obviously, such cases are not presented in the Lie's classification of ODEs.

Here we consider two examples of invariant discrete models without the continuous limit which possess variational formulations. For the considered so far cases we approximated Lagrangians of the underlying Euler–Lagrange equations to find discrete Lagrangians. This approach is of no use for models without the continuous limit. However, if we can find a discrete Lagrangian for such discrete systems, we can proceed to use the results of the discrete Lagrangian formalism.

Example 2. Let us consider the four–dimensional algebra spanned by the operators

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = \frac{\partial}{\partial y}, \qquad X_3 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \qquad X_4 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}. \tag{5.5}$$

There are no differential invariants in the space (x, y, y', y''), but $y'' = 0$ is an invariant manifold. There are two independent difference invariants, namely

$$I_1 = \frac{y_x - y_{\bar{x}}}{1 + y_x y_{\bar{x}}}, \qquad I_2 = \frac{h_+}{h_-}\left(\frac{1 + y_x^2}{1 + y_{\bar{x}}^2}\right)^{1/2}.$$

Hence we can write an invariant difference scheme as

$$\left\{\begin{array}{l} y_x - y_{\bar{x}} = C_1(1 + y_x y_{\bar{x}}); \\ \\ h_+ = C_2 h_- \sqrt{\dfrac{1 + y_{\bar{x}}^2}{1 + y_x^2}}; \end{array}\right. \tag{5.6}$$

where C_1 and C_2 are arbitrary constants. The continuous limit exists only if we have $C_1 = 0$. The limit is then $y'' = 0$.

Let us consider the discrete Lagrangian function

$$\mathcal{L} = y_x \arctan(y_x) - \frac{1}{2}\ln(h_+^2(1 + y_x^2)) + C\frac{y + y_+}{2h_+}, \tag{5.7}$$

which admits two operators X_1, X_2

$$X_1\mathcal{L} + \mathcal{L}\underset{+h}{D}(\xi_1) = 0;$$

$$X_2\mathcal{L} + \mathcal{L}\underset{+h}{D}(\xi_2) = \tfrac{C}{h_+} = \underset{+h}{D}(F(x)), \quad F(x_i) = Ci.$$

From this Lagrangian we get the following intersection of the quasiextremal equations

$$\begin{array}{ll} \dfrac{\delta\mathcal{L}}{\delta y}: & -\arctan(y_x) + \arctan(y_{\bar{x}}) + C = 0; \\ \\ \dfrac{\delta\mathcal{L}}{\delta x}: & h_+\sqrt{1 + y_x^2} = h_-\sqrt{1 + y_{\bar{x}}^2}; \end{array} \tag{5.8}$$

which is equivalent to (5.6) with $C_1 = C$, $C_2 = 1$. On these equations we have two first integrals

$$I_1 = -\frac{1}{2}\ln(h_+^2(1 + y_x^2)), \qquad I_2 = \arctan\left(\frac{y_{i+1} - y_i}{x_{i+1} - x_i}\right) - \frac{F(x_{i+1}) + F(x_i)}{2}.$$

With the help of the first integrals we can reduce the order of the system (5.8) by two. The reduced system can be presented as

$$\arctan\left(\frac{y_{i+1}-y_i}{x_{i+1}-x_i}\right) - \frac{F(x_{i+1})+F(x_i)}{2} = A, \qquad h_+\sqrt{1+y_x^2} = \varepsilon, \quad 0 \le \varepsilon \ll 1.$$

The reduced system can be integrated

$$y_n = y_1 + \varepsilon \sum_{i=1}^{n-1} \sin\left(A + \frac{C}{2}(2i+1)\right) \operatorname{sign}\left(\cos\left(A + \frac{C}{2}(2i+1)\right)\right),$$

$$x_n = x_1 + \varepsilon \sum_{i=1}^{n-1} \left|\cos\left(A + \frac{C}{2}(2i+1)\right)\right|,$$

where

$$\varepsilon = (x_1 - x_0)\sqrt{1 + y_x(0)}, \quad A = \arctan(y_x(0)) - \frac{C}{2}, \quad y_x(0) = \frac{y_1 - y_0}{x_1 - x_0}.$$

Example 3. The group with the infinitesimal generators

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = x\frac{\partial}{\partial x}, \qquad X_3 = y\frac{\partial}{\partial y}, \qquad X_4 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} \tag{5.9}$$

has no differential invariants, but does leave the manifold $y'' = 0$ invariant.

The expressions

$$I_1 = h_+h_-\frac{y_{x\bar{x}}}{y}, \qquad I_2 = \frac{y_-h_+}{y_+h_-}$$

generate the entire set of difference invariants and the general form of an invariant difference model is

$$\begin{cases} y_{x\bar{x}} = \dfrac{C_1}{h_+h_-}y; \\ \\ h_+y_- = C_2h_-y_+. \end{cases} \tag{5.10}$$

A continuous limit exists only for $C_1 = 0$.

Let us choose the discrete Lagrangian function

$$\mathcal{L} = y_x^2 + C\frac{yy_+}{h_+^2}, \tag{5.11}$$

which admits three operators X_1, $2X_2 + X_3$ and X_4:

$$X_1\mathcal{L} + \mathcal{L}\underset{+h}{D}(\xi_1) = 0;$$

$$(2X_2 + X_3)\mathcal{L} + \mathcal{L}\underset{+h}{D}(2\xi_2 + \xi_3) = 0;$$

$$X_4\mathcal{L} + \mathcal{L}\underset{+h}{D}(\xi_4) = \underset{+h}{D}(y^2).$$

Then we obtain the following system of the quasiextremal equations

$$\begin{aligned} &\frac{\delta\mathcal{L}}{\delta y}: \quad y_x - y_{\bar{x}} = C\frac{y_+}{2h_+} + C\frac{y_-}{2h_-}; \\ &\frac{\delta\mathcal{L}}{\delta x}: \quad (y_x)^2 + C\frac{yy_+}{h_+^2} - (y_{\bar{x}})^2 - \frac{Cyy_-}{h_-^2} = 0; \end{aligned} \tag{5.12}$$

which is invariant with respect to all four operators (5.9). This system is equivalent to the system (5.10) with $C_1 = \frac{2C}{2-C}$, $C_2 = 1$.

These quasiextremal equations have three first integrals

$$I_1 = {y_x}^2 + C\frac{yy_+}{h_+^2}, \qquad I_2 = C(x_+ + x)\frac{yy_+}{h_+^2} - y_x((y + y_+) - (x + x_+)y_x),$$
$$I_3 = Cxx_+\frac{yy_+}{h_+^2} + \left(\frac{y + y_+}{2} - \frac{x + x_+}{2}y_x\right)^2, \tag{5.13}$$

which obey the relations

$$X_1(I_3) = I_2, \qquad X_1(I_2) = 2I_1, \qquad X_3(I_1) = -I_2, \qquad X_3(I_2) = -2I_3.$$

To solve (5.13) we involve a special case of the invariant mesh

$$\frac{h_-}{yy_-} = \frac{h_+}{yy_+} = \varepsilon, \quad \varepsilon = const \tag{5.14}$$

that is possible since $\frac{h_+}{yy_+}$ is a first integral:

$$\frac{I_2^2}{4} - I_1 I_3 = \frac{C(C-4)}{4}\left(\frac{yy_+}{h_+}\right)^2.$$

Then a general solution of (5.13) can be found setting $I_1 = A^h$, $I_2 = 2B^h$ as

$$A^h y^2 = (A^h x - B^h)^2 + \left(C - \frac{C^2}{4}\right)\frac{1}{\varepsilon^2}. \tag{5.15}$$

Using the mesh (5.14), we reduce the system of quasiextremal equations to the one variable mapping

$$\frac{y_+ - y}{y_+} - \frac{y - y_-}{y_-} = C,$$

or

$$(2 - C)y_+y_- = y_+y + yy_-.$$

The mapping has one conservation law

$$\tilde{I}_1 = \varepsilon^2 I_1 = \left(\frac{y_+ - y}{yy_+}\right)^2 + \frac{C}{yy_+}.$$

Introducing the new variable $z = \frac{1}{y}$, we linearize the equation

$$z_+ - (2 - C)z + z_- = 0.$$

Let us note that a particular case of this equation $C = \varepsilon^2$, $0 < \varepsilon \ll 1$ was considered in the point $\mathbf{D_{3,3}}$. Since in the equations (5.12) C is an arbitrary constant we need to consider three cases:

a) If $C(C-4) > 0$, the solution has the form

$$z_i = a\left(\frac{2 - C + \sqrt{C(C-4)}}{2}\right)^i + b\left(\frac{2 - C + \sqrt{C(C-4)}}{2}\right)^i, \qquad a, b = const.$$

b) The case $C(C-4) < 0$, i.e. $0 < C < 4$, is covered by the results for $\mathbf{D_{3,3}}$ with $\varepsilon = \sqrt{C}$.

c) For $C = 0$ and $C = 4$ the system of quasiextremals is degenerate. If $C = 0$, we get the equation $y_{x\bar{x}} = 0$ with the solution $y_i = ax_i + b$. If $C = 4$, we get the equation which is transformed into $\tilde{y}_{x\bar{x}} = 0$ by the change of variables $\tilde{y}_i = (-1)^i y_i$. Thus, the solution is $y_i = (-1)^i(ax_i + b)$. To get a discrete model we need to propose some mesh equation.

In all three cases the constants are to be determined from the initial conditions. For $C \neq 0, 4$ the values x_i can be easily found with the help of (5.15).

Let us consider the particular case $C = 1$, then two initial values y_0, y_1 define the solution as

$$y_2 = -\frac{y_0 y_1}{y_1 - y_0}, \qquad y_{i+3} = -y_i, \quad i = 0, 1, 2, ...$$

so that we obtain a cyclic solution.

6 Concluding remarks

We constructed variational discrete models which approximate the equations from Lie's classification of second order ODEs. Variational formulation provides us possibilities to use Noether's theorem. With the help of symmetries one can find first integrals which can be used to reduce the order of the discrete model. We observed the following possibilities for order reduction corresponding to the different number of variational symmetries:

1. For one symmetry we can get only one first integral with the help of the Lagrangian approach. It holds on the extremal equation (2.17) considered on some invariant mesh. In this case there is no need to consider the intersection of the quasiextremal equations.

2. Two symmetries provide us two functionally independent first integrals I_1, I_2, which hold on the intersection of the quasiextremals. These first integrals reduce the system to the form

$$\tilde{F}(x, x_+, y, y_+, I_1, I_2) = 0,$$

$$\tilde{\Omega}(x, x_+, y, y_+, I_1, I_2) = 0.$$

3. Three symmetries let us find three independent first integrals I_1, I_2, I_3. It is possible to reduce the system of the quasiextremal equation to the mapping of the from

$$x_+ = f(x, I_1, I_2, I_3) \quad \text{or} \quad y_+ = g(y, I_1, I_2, I_3)$$

with an equation $H(x, y, I_1, I_2, I_3) = 0$ connecting x and y. If the mapping is integrable, we can write out the general solution of the discrete model (2.15), (2.16) in the form

$$x_i = X(i, x_0, y_0, x_1, y_1), \qquad y_i = Y(i, x_0, y_0, x_1, y_1),$$

satisfying the initial conditions

$$X(0, x_0, y_0, x_1, y_1) = x_0, \quad Y(0, x_0, y_0, x_1, y_1) = y_0,$$

$$X(1, x_0, y_0, x_1, y_1) = x_1, \quad Y(1, x_0, y_0, x_1, y_1) = y_1.$$

The approach can be extended in many directions. It is possible to extend the presented framework to the case of several dependent variables and higher order ODEs. Discrete models which do not have a continuous limit but can be presented as quasiextremal equations for an appropriate discrete Lagrangian are also covered by the discrete Lagrangian formalism. Work is in progress how to propose special types of meshes on which it is possible to achieve further integration of the discrete models reduced with the help of the first integrals.

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Appendix

Table 1. Lie's group classification of second order ODEs under the action of the infinite dimensional group Diff$(2,\mathbb{C})$ of all local diffeomorphisms of a complex plane $\mathbb{C}^2$.

dim L	Group	Operators	Continuous equation
1	$\mathbf{D_{1,1}}$	$X_1 = \dfrac{\partial}{\partial y}$	$y'' = F(x, y')$
2	$\mathbf{D_{2,1}}$	$X_1 = \dfrac{\partial}{\partial x}, \quad X_2 = \dfrac{\partial}{\partial y}$	$y'' = F(y')$
	$\mathbf{D_{2,2}}$	$X_1 = \dfrac{\partial}{\partial y}, \quad X_2 = x\dfrac{\partial}{\partial x} + y\dfrac{\partial}{\partial y}$	$y'' = \dfrac{1}{x}F(y')$
3	$\mathbf{D_{3,1}}$	$X_1 = \dfrac{\partial}{\partial x}, \quad X_2 = \dfrac{\partial}{\partial y}, \quad X_3 = x\dfrac{\partial}{\partial x} + (x+y)\dfrac{\partial}{\partial y}$	$y'' = \exp(-y')$
	$\mathbf{D_{3,2}}$	$X_1 = \dfrac{\partial}{\partial x}, \quad X_2 = \dfrac{\partial}{\partial y}, \quad X_3 = x\dfrac{\partial}{\partial x} + ky\dfrac{\partial}{\partial y}, \quad k \neq 0, \dfrac{1}{2}, 1, 2$	$y'' = y'^{\frac{k-2}{k-1}}$
	$\mathbf{D_{3,3}}$	$X_1 = \dfrac{\partial}{\partial x}, \quad X_2 = 2x\dfrac{\partial}{\partial x} + y\dfrac{\partial}{\partial y}, \quad X_3 = x^2\dfrac{\partial}{\partial x} + xy\dfrac{\partial}{\partial y}$	$y'' = y^{-3}$
	$\mathbf{D_{3,4}}$	$X_1 = \dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial y}, \quad X_2 = x\dfrac{\partial}{\partial x} + y\dfrac{\partial}{\partial y}, \quad X_3 = x^2\dfrac{\partial}{\partial x} + y^2\dfrac{\partial}{\partial y}$	$y'' + 2\dfrac{y' + Cy'\sqrt{y'} + y'^2}{x-y} = 0$
8	$\mathbf{D_{8,1}}$	$X_1 = \dfrac{\partial}{\partial x}, \quad X_2 = \dfrac{\partial}{\partial y}, \quad X_3 = x\dfrac{\partial}{\partial y}, \quad X_4 = x\dfrac{\partial}{\partial x}, \quad X_5 = y\dfrac{\partial}{\partial x},$ $X_6 = y\dfrac{\partial}{\partial y}, \quad X_7 = x^2\dfrac{\partial}{\partial x} + xy\dfrac{\partial}{\partial y}, \quad X_8 = xy\dfrac{\partial}{\partial x} + y^2\dfrac{\partial}{\partial y}$	$y'' = 0$