Conservation laws of semidiscrete Hamiltonian equations

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Abstract

Many evolution partial differential equations can be cast into Hamiltonian form. Conservation laws of these equations are related to one–parameter Hamiltonian symmetries admitted by the PDEs [1]. In this paper we consider symmetries and Noether's theorem for semidiscrete Hamiltonian equations which are obtained by space discretization of Hamiltonian PDEs. Using symmetries, one can find conservation laws of these equations. Several applications including a transfer equation and the Korteweg-de Vries equation are presented.

I Introduction

Many partial differential equations (PDEs) possess conservation laws. Solutions of the equations of nondissipative continuum mechanics can often be presented as solutions of the Euler–Lagrange equations for the relevant functional. Conservation laws of such equations are given by Noether's theorem [2] if the admitted symmetries are known. The first discrete analog of Noether's theorem was presented by S. Maeda [3],[4], who considered transformations which do not change the independent variables and hence the mesh. Discrete versions of Noether's theorem for transformations which may also involve the independent variables have been developed by V. A. Dorodnitsyn [5],[6],[7].

However, there are evolution equations without appropriate or natural Lagrangian formulation. For these equations one may prefer to use the Hamiltonian viewpoint. Much work has been done recently in preserving structure and conservation laws of Hamiltonian PDEs under discretization. One approach is based on using Hamiltonian formulation of PDEs on multi–symplectic structures, that generalizes classical Hamiltonian structure by assigning a distinct symplectic operator for each space direction and time [8]. Variational integrators [9] generalize Veselov–type [10],[11] discretizations of PDEs in variational form. They conserve the corresponding discrete multi–symplectic form as well as the discrete momentum mappings corresponding to symmetries which do not transform the mesh. Different symplectic ODE solvers can also be used to build up multi–symplectic integrators for PDEs [12],[13],[14] which preserve multi–symplectic structure and may preserve some conservation laws.

In this paper we consider Hamiltonian formulation of evolution equations based on the Poisson bracket approach. Conservation laws of these equations are related to one–parameter Hamiltonian symmetries admitted by the PDEs [1],[15]. We introduce a discrete Poisson bracket, which provides us an infinite dimensional system of semidiscrete equations. Our main interest is to find conservation laws of such systems with the help of Noether's theorem in the Hamiltonian formulation. It will transpire that Noether's theorem can be carried over from the continuous case to the semidiscrete case so that one can use Hamiltonian symmetries of semidiscrete equations to find their conservation laws. As one can expect not all symmetries and conservation laws can be preserved under discretization of the bracket.

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It is not known how to find all symmetries of a set of discrete equations. An overview of different methods and references can be found, for example, in [16],[17]. Some Lie point symmetries of the original continuous equations can be preserved under discretization. These symmetries are realized by the same vector field in the continuous and discrete cases [6]. Discrete equations can also have symmetries which have no continuous counterparts. Because our main interest is to see what happens to conservation laws of continuous equations under space discretization we do not consider symmetries which disappear in the continuous limit.

In the paper we consider semidiscrete Hamiltonian equations and their conservation laws. The space discretization of Hamiltonian PDEs can be viewed as a first step of discretization that leads from a PDE to a difference scheme. Therefore the consideration of semidiscrete Hamiltonian equations can explain how space discretizations affects the conservation laws.

The paper is organized as follows: In Section 2 we consider continuous and semidiscrete Hamiltonian equations. Symmetries and their connection with conservation laws are reviewed in Section 3, where we construct the ideal of the Lie–Bäcklund operators on a uniform space mesh and present the Hamiltonian form of Noether's theorem. Section 4 contains applications. Finally, we summarize the results of the paper in the concluding Section 5.

II Hamiltonian equations

In what follows (except section **3.2**) we will consider the case of one dependent variable u and two independent variables: time t and space coordinate x. We assume that the solutions are sufficiently smooth, all variational derivatives tend to zero as $u \to 0$ and that u and a number of space derivatives of u tend to zero as $|x| \to \infty$. For simplicity we suppose that the solution decays fast enough so that all integrals in the continuous case and all sums in the semidiscrete case make sense. All integrals are assumed to be taken over the whole real line.

II.1 Hamiltonian partial differential equations

Many evolution partial differential equations of the form

$$u_t = K(x, u^{(n)}), \tag{2.1}$$

where $u^{(n)} = (u, u_1, u_2, ..., u_n)$ represents u and a finite set of derivatives of u with respect to space coordinate x, can be represented in the Hamiltonian form

$$u_t = \mathcal{D}\left(\frac{\delta \mathcal{H}}{\delta u}\right), \quad \mathcal{H}[u] = \int H(x, u^{(n)}) dx \tag{2.2}$$

with the help of the Hamiltonian functional $\mathcal{H}[u]$, variational operator $\delta \cdot / \delta u$ and a linear operator $\mathcal{D}$ [1]. Denote by $\mathcal{F}$ the space of functionals of the type

$$\int P(t, x, u^{(m)}) dx, \quad m \in \mathbb{N}.$$

The operator $\mathcal{D}$ is required to be Hamiltonian, i.e., it forms the Poisson bracket

$$\{\mathcal{P}, \mathcal{L}\} = \int \frac{\delta \mathcal{P}}{\delta u} \mathcal{D}\left(\frac{\delta \mathcal{L}}{\delta u}\right) dx \tag{2.3}$$

satisfying the conditions of *skew-symmetry*

$$\{\mathcal{P}, \mathcal{L}\} = -\{\mathcal{L}, \mathcal{P}\} \tag{2.4}$$

and the *Jacobi identity*

$$\{\{\mathcal{P}, \mathcal{L}\}, \mathcal{R}\} + \{\{\mathcal{R}, \mathcal{P}\}, \mathcal{L}\} + \{\{\mathcal{L}, \mathcal{R}\}, \mathcal{P}\} = 0 \tag{2.5}$$

for all functionals $\mathcal{P}, \mathcal{L}, \mathcal{R} \in \mathcal{F}$.

The variational derivative of a functional can be found by the action of the Euler operator on the integrand

$$\frac{\delta \mathcal{H}}{\delta u} = E(H), \quad E = \frac{\partial \cdot}{\partial u} - D_x\left(\frac{\partial \cdot}{\partial u_1}\right) + D_x^2\left(\frac{\partial \cdot}{\partial u_2}\right) + \cdots + (-1)^n D_x^n\left(\frac{\partial \cdot}{\partial u_n}\right) + \cdots, \tag{2.6}$$

where D_x is the total differentiation operator:

$$D_x = \frac{\partial}{\partial x} + u_1\frac{\partial}{\partial u} + u_2\frac{\partial}{\partial u_1} + \cdots + u_{n+1}\frac{\partial}{\partial u_n} + \cdots \tag{2.7}$$

Example 2.1 Let us illustrate the described above approach on Korteweg-de Vries equation [1],[18]

$$u_t = uu_x + u_{xxx}. \tag{2.8}$$

Korteweg-de Vries equation can be presented in the Hamiltonian form (2.2) by several ways [1]. One of them is the following:

$$\mathcal{D} = D_x, \qquad \mathcal{H} = \int\left(\frac{u^3}{6} - \frac{u_x^2}{2}\right)dx. \tag{2.9}$$

$\lozenge$

Example 2.2 The same bracket can be used to cast the transfer equation

$$u_t = h''(u)u_x \tag{2.10}$$

in the the Hamiltonian form with the Hamiltonian functional

$$\mathcal{H} = \int h(u)dx. \tag{2.11}$$

$\lozenge$

II.2 Semidiscrete Hamiltonian equations

Given a Hamiltonian PDE, it is desirable to discretize both the Poisson bracket and the Hamiltonian functional so that we preserve Poisson structure.

To do so let us introduce a two–dimensional mesh which is uniform (regular) in space and continuous in time. We denote the mesh points as $\{x_i(t)\}$, $i = 0, \pm 1, \pm 2, ...$, $t \geq 0$ and define the mesh by two conditions:

$$\begin{cases} \Omega_1 = (x_{i+1}(t) - x_i(t)) - (x_i(t) - x_{i-1}(t)) & = \ 0 \quad i = 0, \pm 1, \pm 2, ..., \quad t \geq 0; \\ \Omega_2 = x_i(t+\tau) - x_i(t) & = \ 0 \quad i = 0, \pm 1, \pm 2, ..., \quad t, \tau \geq 0. \end{cases} \tag{2.12}$$

We will use the short notation $\Omega = 0$ for this mesh. The first equation means that for any fixed time the space mesh is uniform. The second equation requires that we consider only vertical mesh lines in the time–space plane. Figure 1 shows the type of the mesh we specified.

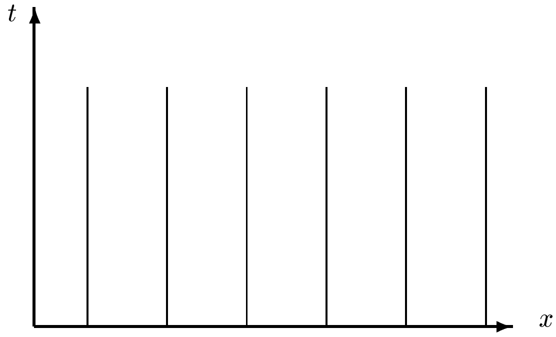

Fig. 1. The uniform mesh for the semidiscrete equations.

At the mesh points we introduce discrete space derivatives $u_1 = \underset{+h}{D} u$, $u_2 = \underset{-h}{D} u_1$, $u_{2k+1} = \underset{+h}{D} u_{2k}$, $u_{2k+2} = \underset{-h}{D} u_{2k+1}$, $k = 1, 2, ...,$ where $\underset{+h}{D}$ and $\underset{-h}{D}$ are the right and left discrete differentiation operators. They can be defined with the help of the right shift S_+ and left shift S_- operators

$$S_+ f(x) = f(x+h), \qquad S_- f(x) = f(x-h) \tag{2.13}$$

as

$$\underset{+h}{D} = \frac{S_+ - 1}{h}; \qquad \underset{-h}{D} = \frac{1 - S_-}{h} \tag{2.14}$$

We will consider the space of discrete derivatives $\underset{h}{u}^{(n)} = (u, \underset{h}{u}_1, \underset{h}{u}_2, ..., \underset{h}{u}_n)$ and corresponding functionals of the form

$$\underset{h}{\mathcal{P}} = \sum_{\Omega} \underset{h}{P}(t, x, h, \underset{h}{u}^{(n)}) h, \tag{2.15}$$

where the summation is taken over all space points of the mesh Ω. We denote the space of such functionals as $\underset{h}{\mathcal{F}}$.

We assume that the Hamiltonian operator $\mathcal{D}$ can be approximated by an operator $\underset{h}{\mathcal{D}}$ such that the discrete bracket

$$\{\underset{h}{\mathcal{P}}, \underset{h}{\mathcal{L}}\}_h = \sum_{\Omega} \frac{\delta \underset{h}{\mathcal{P}}}{\delta u} \underset{h}{\mathcal{D}} \left(\frac{\delta \underset{h}{\mathcal{L}}}{\delta u} \right) h \tag{2.16}$$

defines a Poisson bracket for functionals from $\underset{h}{\mathcal{F}}$. This means that $\{\cdot, \cdot\}_h$ is skew–symmetric and satisfies the Jacobi identity. We can choose some approximation $\underset{h}{\mathcal{H}}$ of the Hamiltonian functional $\mathcal{H}$ and get a set of Hamiltonian semidiscrete equations

$$\dot{u} = \underset{h}{\mathcal{D}} \left(\frac{\delta \underset{h}{\mathcal{H}}}{\delta u} \right), \tag{2.17}$$

which approximate the equation (2.1) on the mesh (2.12).

Using the bracket notation, we can express a total time derivative of a functional $\underset{h}{\mathcal{P}} \in \underset{h}{\mathcal{F}}$ as

$$\frac{d}{dt} \underset{h}{\mathcal{P}} = \frac{\partial}{\partial t} \underset{h}{\mathcal{P}} + \{\underset{h}{\mathcal{P}}, \underset{h}{\mathcal{H}}\}_h. \tag{2.18}$$

Definition For a given Hamiltonian operator $\underset{h}{\mathcal{D}}$ a distinguished functional is a functional $\underset{h}{\mathcal{C}}(x,h,\underset{h}{u}^{(n)})$ such that

$$\underset{h}{\mathcal{D}}\left(\frac{\delta\,\underset{h}{\mathcal{C}}}{\delta u}\right)=0$$

It follows that a functional is distinguished if and only if its Poisson bracket with every other functional is trivial:

$$\{\underset{h}{\mathcal{C}},\underset{h}{\mathcal{H}}\}_h=0 \quad \text{for any} \quad \underset{h}{\mathcal{H}}\in\underset{h}{\mathcal{F}}.$$

Example 2.3 Let us consider the bracket based on the central difference derivative operator

$$\underset{h}{D}{}_0=\frac{S_+-S_-}{2h}, \tag{2.19}$$

defined with the help of the right and left shift operators (2.13). It is easy to check that the bracket is skew–symmetric and satisfies the Jacobi identity. This bracket has two linearly independent distinguished functionals

$$\underset{h}{\mathcal{C}}{}_1=\sum_{i=-\infty}^{\infty}u_ih \quad \text{and} \quad \underset{h}{\mathcal{C}}{}_2=\sum_{i=-\infty}^{\infty}(-1)^iu_ih. \tag{2.20}$$

In the continuous limit the first distinguished functional gives the only distinguished functional of the continuous bracket (2.3) generated by the total differentiation operator D_x (the continuous counterpart of $\underset{h}{D}{}_0$):

$$\underset{h}{\mathcal{C}}{}_1\to\int udx \quad \text{as} \quad h\to0.$$

The second distinguished functional disappears in the continuous limit: $\underset{h}{\mathcal{C}}{}_2\to0$ as $h\to0$.

Alternatively, we can present the distinguished functionals as

$$\underset{h}{\tilde{\mathcal{C}}}{}_1=\sum_{i=-\infty}^{\infty}u_{2i}h \quad \text{and} \quad \underset{h}{\tilde{\mathcal{C}}}{}_2=\sum_{i=-\infty}^{\infty}u_{2i+1}h.$$

These expressions indicate that conservations of mass for even and odd points of the mesh hold independently.

$\Diamond$

Example 2.4 We can choose a higher order approximation of the operator D_x (up to $O(h^4)$):

$$\underset{h}{\hat{D}}=\frac{1}{3}\left(4\underset{h}{\mathcal{D}}{}_0-\underset{2h}{\mathcal{D}}{}_0\right)=\frac{-S_+^2+8S_+-8S_-+S_-^2}{12h}. \tag{2.21}$$

The discrete bracket (2.16) based on $\underset{h}{\hat{D}}$ has four linearly independent distinguished functionals:

$$\begin{aligned}\underset{h}{\hat{\mathcal{C}}}{}_1&=\sum_{i=-\infty}^{\infty}u_ih; \qquad \underset{h}{\hat{\mathcal{C}}}{}_2=\sum_{i=-\infty}^{\infty}(-1)^iu_ih;\\ \underset{h}{\hat{\mathcal{C}}}{}_3&=\sum_{i=-\infty}^{\infty}(4-\sqrt{15})^iu_ih; \qquad \underset{h}{\hat{\mathcal{C}}}{}_4=\sum_{i=-\infty}^{\infty}(4+\sqrt{15})^iu_ih.\end{aligned} \tag{2.22}$$

Note that the functionals $\underset{h}{\hat{\mathcal{C}}}{}_1$ and $\underset{h}{\hat{\mathcal{C}}}{}_2$ are the same as in the previous example.

$\Diamond$

II.3 Discrete Euler operator

It is useful to have some practical formula to compute the variation derivatives of the discrete functionals. A discrete analog of the Euler operator (2.6) was introduced in order to study variations of discrete Lagrangian problems [5],[6],[7]. For Hamiltonian functionals of the form

$$\underset{h}{\mathcal{H}} = \sum_{\Omega} H(x, h, u, \underset{h}{u}_1, \underset{h}{u}_2, ..., \underset{h}{u}_n) h \tag{2.23}$$

the variational derivative can be found with the help of the discrete Euler operator $\underset{h}{E}$:

$$\frac{\delta \underset{h}{\mathcal{H}}}{\delta u} = \underset{h}{E} H, \tag{2.24}$$

where

$$\underset{h}{E} = \frac{\partial \cdot}{\partial u} - \underset{-h}{D} \left(\frac{\partial \cdot}{\partial u_1} \right) + \underset{+h}{D} \underset{-h}{D} \left(\frac{\partial \cdot}{\partial u_2} \right) + \cdots + (-1)^n (\underset{-h}{D})^{(n \ mod \ 2)} (\underset{+h}{D} \underset{-h}{D})^{\lfloor n/2 \rfloor} \left(\frac{\partial \cdot}{\partial u_n} \right). \tag{2.25}$$

The function $\lfloor \cdot \rfloor$ denotes the integer part and $(\cdot \ mod \ 2)$ the remainder of the division by 2.

Another approach can be used for the Hamiltonian functionals of the form

$$\underset{h}{\mathcal{H}} = \sum_{i=-\infty}^{\infty} H[u_i] h = \sum_{i=-\infty}^{\infty} H(x_i, h, u_{i-m}, ..., u_i, ..., u_{i+n}) h \tag{2.26}$$

for some $m, n \in \mathbb{N}$. The variation derivative of the functional with respect to u_i is

$$\begin{gathered} \underset{h}{E} H = \frac{\partial H[u_{i-n}]}{\partial u_i} + \cdots + \frac{\partial H[u_i]}{\partial u_i} + \cdots + \frac{\partial H[u_{i+m}]}{\partial u_i} \\ = \left(\frac{\partial H[u_i]}{\partial u_{i+n}} \right)_{-n} + \cdots + \left(\frac{\partial H[u_i]}{\partial u_i} \right) + \cdots + \left(\frac{\partial H[u_i]}{\partial u_{i-m}} \right)_{m} = \sum_{k=-m}^{n} \left(\frac{\partial H[u_i]}{\partial u_{i+k}} \right)_{-k}, \end{gathered} \tag{2.27}$$

where $(\cdot)_k$ denotes that the quantity inside bracket is shifted on k steps. Thus in this case we get the following presentation of the discrete Euler operator:

$$\underset{h}{E} = \sum_{k=-m}^{n} \left(\frac{\partial \cdot}{\partial u_{i+k}} \right)_{-k} \tag{2.28}$$

III Symmetries and conservation laws

III.1 Invariance of semidiscrete equations

Let $\underset{h}{Z}$ be the space of sequences of variables $(t, x, h, u, \underset{h}{u}_1, \underset{h}{u}_2, ...)$ and $\underset{h}{\mathcal{A}}$ be the space of analytic functions of a finite number of variables z from $\underset{h}{Z}$.

Let us consider the infinite set of semidiscrete equations

$$\dot{u} = F(z), \quad F \in \underset{h}{\mathcal{A}} \tag{3.1}$$

defined at points of some two–dimensional mesh Ω. This equation is written on the finite number of points (stencil) of the difference mesh

$$\Omega(h_+, z) = 0, \qquad h_+ = x_+ - x = (S_+ - 1)x, \tag{3.2}$$

which may be uniform or nonuniform. Then we say that the semidiscrete equation (3.1) on mesh (3.2) admits the group G if

$$\begin{cases} \dot{u} = F(z) \\ \\ \Omega(h_+, z) = 0 \end{cases} \tag{3.3}$$

is an invariant manifold of the group G.

Symmetries of the system of equation (3.3) are transformations generated by vector fields of the form

$$X = \xi^t(z)\frac{\partial}{\partial t} + \xi^x(z)\frac{\partial}{\partial x} + \eta(z)\frac{\partial}{\partial u}, \tag{3.4}$$

where ξ^t, ξ^x, $\eta \in \underset{h}{\mathcal{A}}$, which leave the system invariant. The infinitesimal criterion for invariance of (3.3) under transformation generated by (3.4) is given by the following conditions:

$$\mathbf{pr}\ X(\dot{u} - F(z))|_{\dot{u}=F,\Omega=0} = 0; \tag{3.5}$$

$$\mathbf{pr}\ X(\Omega)|_{\dot{u}=F,\Omega=0} = 0; \tag{3.6}$$

where the operator X is prolonged on all variables appearing in the system of equation (3.3)

$$\mathbf{pr}\ X = X + \zeta_1^t\frac{\partial}{\partial \dot{u}} + \underset{h}{\zeta}{}_1^x\frac{\partial}{\partial \underset{h}{u}_1} + \underset{h}{\zeta}{}_2^x\frac{\partial}{\partial \underset{h}{u}_2} + \cdots \tag{3.7}$$

On the space–uniform grid (2.12) the coefficients of the prolonged operator are calculated through the prolongation formulas

$$\begin{gathered} \zeta_1^t = D_t(\eta) - \dot{u}D_t(\xi^t) - u_1 D_t(\xi^x), \quad \underset{h}{\zeta}{}_1^x = \underset{+h}{D}(\eta) - S_+(\dot{u})\underset{+h}{D}(\xi^t) - \underset{h}{u}_1\underset{+h}{D}(\xi^x), \\ \underset{h}{\zeta}{}_2^x = \underset{-h}{D}\underset{+h}{D}(\eta) - 2\underset{h}{u}_2\underset{+h}{D}(\xi^x) - \tfrac{1}{h}S_+(\dot{u})\underset{+h}{D}(\xi^t) + \tfrac{1}{h}S_-(\dot{u})\underset{-h}{D}(\xi^t), \quad \cdots \end{gathered} \tag{3.8}$$

These formulas are continuous for the time derivative $\dot{u}$ [1],[18],[19],[20] and discrete for the space derivatives [6]. Note that $u_1 = D_x(u)$ is the "continuous" derivative. It needs to be in some discrete representation, for example, $\underset{h}{\tilde{D}}{}_0(u)$, which will be introduced below.

To check out the invariance one has to act by the prolonged operator $\mathbf{pr}\ X$ on the equation of the system (3.3) and verify that the obtained expressions are annihilated on the equations (3.3) and its differential consequences.

The conditions (3.5) and (3.6) require the invariance of the equations and of the mesh, correspondingly. For the mesh (2.12) the last condition can be split into two:

$$\underset{-h}{D}\underset{+h}{D}(\xi^x)|_{\dot{u}=F,\Omega=0} = 0 \tag{3.9}$$

and

$$D_t(\xi^x)|_{\dot{u}=F,\Omega=0} = 0. \tag{3.10}$$

The bracket $\{\cdot,\cdot\}_h$ requires a space–uniform mesh that implies condition (3.9), which means the conservation of the space mesh uniformity under the group transformation [21],[6]. Condition (3.10) has a clear geometrical meaning, it enforces that a vertical line of the time–space plane $x_i = const$, passing through a mesh point x_i, will remain vertical under the transformation. Let us note that in general the group transformation corresponding to the operator (3.4) destroys grid geometry.

Operators of the form (3.4) are called Lie–Bäcklund symmetries (they are also called *generalized* symmetries). An important subclass of the Lie–Bäcklund symmetries (3.4) is constituted by Lie point symmetries. They are given by the vector fields with coefficients ξ^t, ξ^x and η depending only on dependent and independent variables

$$X = \xi^t(t,x,u)\frac{\partial}{\partial t} + \xi^x(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u}. \tag{3.11}$$

Such symmetries are given by the same vector fields in the continuous and discrete cases.

Example 3.1 Groups satisfying invariance conditions.

If we require conditions (3.9) and (3.10) to be true for Lie point symmetries (3.11) independently of the considered equations, we find, as an example, that the transformations with generators

$$X = (ax+b)\frac{\partial}{\partial x} + \xi^t(t,x,u)\frac{\partial}{\partial t} + \eta(t,x,u)\frac{\partial}{\partial u}, \tag{3.12}$$

where a and b are constant, keep the mesh invariant. $\Diamond$

Remark In example 3.1 we require the mesh to be strongly invariant. For weak invariance of the semidiscrete equations we need the conditions (3.5), (3.9) and (3.10) to be satisfied on the solutions of the equations. Therefore a set of operators which keep equations invariant is broader than the set of operators which keep the mesh invariant independently on the considered equations.

Example 3.2 A group not satisfying invariance conditions.

Galilean transformation is central in models of Newton mechanics and is admitted by many equations. For example, the transfer equation (also known as nonviscous Burgers equation)

$$u_t = uu_x \tag{3.13}$$

and Korteweg-de Vries equation (2.8) are invariant with respect to the Galilean transformation

$$\begin{aligned} x^* &= x + t\alpha, \\ t^* &= t, \\ u^* &= u - \alpha, \end{aligned} \tag{3.14}$$

where α is a transformation parameter. Galilean transformation corresponds to the operator

$$X = t\frac{\partial}{\partial x} - \frac{\partial}{\partial u}.$$

The transformation does not keep vertical lines of the mesh (condition (3.10)). Figure 2 shows that the vertical lines of the two–dimensional mesh (2.12) (see Figure 1) become inclined under the transformation.

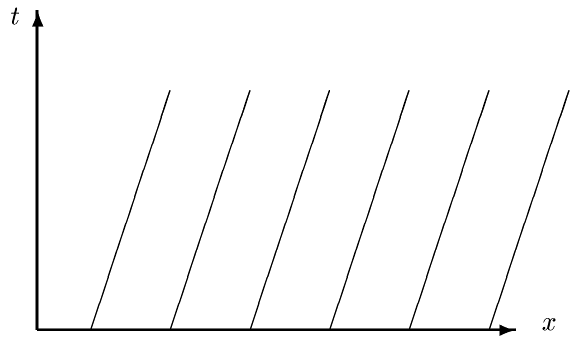

Fig. 2. Mesh deformation under the action of Galilean transformation.

$\Diamond$

III.2 Factorization of the operators

In this point we overview factorization of operators both in continuous and discrete cases. The factorizations will be used to transfer the admitted symmetries into the canonical form. For simplicity we consider the space (x, u), where u is dependent variable and x is independent one.

a) It is known that Taylor group generated by the total differentiation operator D_x (2.7) provides a transition to the quotient algebra of the Lie–Bäcklund operators [20]. The operators $\xi(z)D_x$, where $\xi(z)$ is an analytic function of a finite number of variables in the space $(x, u, u_1, u_2, ...)$, form the ideal in the Lie algebra of operators

$$X = \xi(z)\frac{\partial}{\partial x} + \eta(z)\frac{\partial}{\partial u} + \cdots , \tag{3.15}$$

Thus instead of the operator (3.15) one can consider the operator $\bar{X}$ in the canonical form

$$\bar{X} = X - \xi D_x = \bar{\eta}\frac{\partial}{\partial u} + \cdots , \qquad \bar{\eta} = \eta - \xi u_x \tag{3.16}$$

The representatives of this quotient algebra have independent variables as invariants and the extension formulas for them have simple and convenient form.

b) Similar factorization can be done in the discrete case. Following [6],[22] we consider the Lie–Bäcklund operator in the space $\underset{h}{Z} = (x, h, u, \underset{h}{u}_1, \underset{h}{u}_2, ..., \underset{h}{u}_n, ...)$:

$$X = \xi(z)\frac{\partial}{\partial x} + \eta(z)\frac{\partial}{\partial u} + \underset{h}{\zeta}_1\frac{\partial}{\partial \underset{h}{u}_1} + \underset{h}{\zeta}_2\frac{\partial}{\partial \underset{h}{u}_2} + \cdots + h\underset{+h}{D}(\xi)\frac{\partial}{\partial h}, \tag{3.17}$$

where

$$\begin{gathered}\zeta_1 = \underset{+h}{D}(\eta) - \underset{h}{u}_1\underset{+h}{D}(\xi), \quad \zeta_2 = \underset{-h}{D}(\zeta_1) - \underset{h}{u}_2\underset{-h}{D}(\xi), \quad \cdots \\ \zeta_{2k+1} = \underset{+h}{D}(\zeta_{2k}) - \underset{h}{u}_{2k+1}\underset{+h}{D}(\xi), \quad \zeta_{2k+2} = \underset{-h}{D}(\zeta_{2k+1}) - \underset{h}{u}_{2k+2}\underset{-h}{D}(\xi), \quad \cdots\end{gathered} \tag{3.18}$$

Proposition 3.1 *(Dorodnitsyn [6],[22]) The set of Lie–Bäcklund operators defined on the same uniform mesh forms a Lie algebra with multiplication*

$$[X_1, X_2] = X_1X_2 - X_2X_1.$$

Let us consider the special operation of left multiplication of a Lie-Bäcklund operator by an analytic function $\tilde{\xi}(z) \in \underset{h}{\mathcal{A}}$:

$$\tilde{\xi} * X = \tilde{\xi}\xi\frac{\partial}{\partial x} + \tilde{\xi}\eta\frac{\partial}{\partial u} + \left(\underset{+h}{D}(\tilde{\xi}\eta) - \underset{h}{u}_1\underset{+h}{D}(\tilde{\xi}\xi)\right)\frac{\partial}{\partial \underset{h}{u}_1} + \cdots + \underset{+h}{D}(\tilde{\xi}\xi)\frac{\partial}{\partial h}. \tag{3.19}$$

The first coordinates in the operator $\tilde{\xi} * X$ are multiplied by $\tilde{\xi}(z)$ and the remaining coordinates are constructed according to the prolongation formulas (3.18).

Two representations for the operator of the Taylor group in the mesh space $\underset{h}{Z}$ were proposed in [6],[22]. They are the exact representations of the operator of total differentiation D_x in the space of difference variables

$$\begin{gathered}D^+ = \frac{\partial}{\partial x} + \underset{+h}{\tilde{D}}(u)\frac{\partial}{\partial u} + \ldots , \qquad \underset{+h}{\tilde{D}} = \sum_{n=1}^{\infty}\frac{(-h)^{n-1}}{n}\underset{+h}{D}^n \\ D^- = \frac{\partial}{\partial x} + \underset{-h}{\tilde{D}}(u)\frac{\partial}{\partial u} + \ldots , \qquad \underset{-h}{\tilde{D}} = \sum_{n=1}^{\infty}\frac{h^{n-1}}{n}\underset{-h}{D}^n,\end{gathered} \tag{3.20}$$

where $\underset{+h}{D}$ and $\underset{-h}{D}$ are right and left difference operators on a uniform mesh.

It was shown in [6],[22] that in the discrete case one can use ideals $\tilde{\xi}(z) * D^{+}$ and $\tilde{\xi}(z) * D^{-}$ for factorization of the operators (3.17). It is important to notice that every difference equation on a regular mesh admits the operators D^{+} and D^{-}, which do not change the mesh, and that the representation (3.20) is true only for uniform meshes.

Another exact representation of the Taylor group operator in the mesh space $\underset{h}{Z}$ can be found with the help the central difference derivative (2.19). It is known that the finite transformations of a continuous groups have a one–to–one relation with infinitesimal transformations. This relation is expressed by a finite system of Lie equations for pointwise groups and by the unique recurrent sequence of coefficients of formal series for Lie–Bäcklund groups [20]. For both group types the solution of the system can be presented in the form of an exponential map. A finite transformation of any coordinate $z^i \in \underset{h}{Z}$ generated by a vector field X

$$(z^i)^* = \exp(aX)(z^i) = \sum_{k=0}^{\infty} \frac{a^k}{k!} X^k(z^i).$$

Let us now invert the series, finding an infinitesimal transformation $aX(z^i)$ from the finite transformation $\exp(aX)(z^i)$:

$$aX(z^i) = \ln(\exp(aX))(z^i).$$

Taking the parameter value $a = h$, we can reconstruct the tangent field from the finite transformation of the Taylor group

$$\exp\left(a \underset{h}{\tilde{D}}_0 \right)\bigg|_{a=h} = S_+$$

using the presentation of the right shift operator in terms of powers of $\underset{h}{D}_0$:

$$S_+ = h \underset{h}{D}_0 + \sqrt{1 + h^2 \underset{h}{D}_0^2} = h \underset{h}{D}_0 + \sum_{k=0}^{\infty} C_k^{1/2} h^{2k} \underset{h}{D}_0^{2k},$$

found from the relation

$$\underset{h}{D}_0 = \frac{S_+ - (S_+)^{-1}}{2h}.$$

We get

$$\underset{h}{\tilde{D}}_0 = \frac{1}{h} \sinh^{-1}(h \underset{h}{D}_0) = \frac{1}{h} \ln\left(h \underset{h}{D}_0 + \sqrt{1 + h^2 \underset{h}{D}_0^2} \right).$$

Using

$$\sinh^{-1}(x) = \ln(x + \sqrt{1 + x^2}) = \sum_{k=0}^{\infty} \alpha_{2k+1} x^{2k+1}$$

with the expansion coefficients

$$\alpha_{2k+1} = (-1)^k \frac{1}{2}\frac{3}{4} \cdots \frac{2k-1}{2k} \frac{1}{2k+1} = \frac{(-1)^k}{2k+1} \frac{(2k-1)!!}{2^k k!},$$

where $(2k-1)!! = 1 \cdot 3 \cdot 5 \cdots (2k-1)$, we obtain the discrete representation of the operator D_x:

$$\underset{h}{\tilde{D}}_0 = \sum_{k=0}^{\infty} \alpha_{2k+1} h^{2k} \underset{h}{D}_0^{2k+1}. \tag{3.21}$$

Finally, the infinitesimal operator of the Taylor group on $\underset{h}{Z}$ can be written as follows:

$$D^0 = \frac{\partial}{\partial x} + \underset{h}{\tilde{D}}_0(u) \frac{\partial}{\partial u} + \underset{h}{\tilde{D}}_0(\underset{h}{u}_1) \frac{\partial}{\partial \underset{h}{u}_1} + \cdots + \underset{h}{\tilde{D}}_0(\underset{h}{u}_n) \frac{\partial}{\partial \underset{h}{u}_n} + \cdots, \tag{3.22}$$

where we used commutativity of the operators $\underset{+h}{D}$, $\underset{-h}{D}$ and $\underset{h}{D}{}_0$ on the uniform grid. One can also see that the operator D^0 satisfies the conditions (3.18), i.e, the group generated by this operator is a nontrivial Lie–Bäcklund group.

Remark The operators D^+ and D^- given in (3.20) use Newton series interpolation to the right and to the left, correspondingly. In the opposite directions they represent extrapolation. The operator D^0 represents interpolation on the whole real line.

Lemma 3.2 *For any Lie–Bäcklund operators X and $\tilde{\xi} * D^0$, $\tilde{\xi} \in \underset{h}{A}$ defined on the same uniform mesh it is true that*

$$[\tilde{\xi} * D^0, X] = [\tilde{\xi} * D^0(\xi) - X(\tilde{\xi})] * D^0 \tag{3.23}$$

The proof of this Lemma is identical to the proof of the same result for the operators D^+ and D^- [22]. The Lemma provides us the following result:

Theorem 3.3 *The set of operators of the form*

$$X_* = \tilde{\xi}(z) * D^0 = \tilde{\xi}(z)\frac{\partial}{\partial x} + \tilde{\xi}(z)\,\underset{h}{\tilde{D}}{}_0(u)\frac{\partial}{\partial u} + \cdots \tag{3.24}$$

with arbitrary coefficients $\tilde{\xi}(z)$ forms an ideal in the Lie algebra of all Lie–Bäcklund operators $\{X\}$ on the uniform grid.

Consequently, instead of the Lie algebra of operators (3.17) we could consider the quotient algebra by the ideal $\tilde{\xi}(z)*D^0$. Although we can also use ideal $\tilde{\xi}(z)*D^+$ or $\tilde{\xi}(z)*D^-$ for factorization of the operators in the mesh space, the presence of the operator $\underset{h}{D}{}_0$ in the bracket suggests us to choose the ideal $\tilde{\xi}(z) * D^0$. As we will later see this choice is motivated by applications.

As representatives of the quotient algebra we shall consider the operators with the coordinate $\xi = 0$:

$$\bar{X} = \bar{\eta}\frac{\partial}{\partial u} + \cdots, \qquad \bar{\eta} = \eta - \xi\,\underset{h}{\tilde{D}}{}_0(u). \tag{3.25}$$

Let us note that the independent variable is invariant for the canonical operator (3.25). Furthermore it follows that the step of the mesh h is also invariant.

III.3 Conservation laws

For the system of Hamiltonian equations (2.17) considered on the grid (2.12) we have the following types of the conservation laws:

1. If there are any distinguished functionals for the bracket (2.16), then they are conserved quantities. The distinguished functionals can be viewed as functionals corresponding to the symmetry $X_0 \equiv 0$.

2. Hamiltonian form of Noether's theorem.

For the bracket based on the central difference derivative $\underset{h}{D}{}_0$ we can factorize the symmetries

$$X = \xi^t\frac{\partial}{\partial t} + \xi^x\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial u} \tag{3.26}$$

by a combination of the discrete factorization (3.25) in space and continuous factorization (3.16) in time

$$\bar{X} = \bar{\eta}\frac{\partial}{\partial u}, \qquad \bar{\eta} = \eta - \xi^t\,\underset{h}{\mathcal{D}}\left(\frac{\delta\,\underset{h}{\mathcal{H}}}{\delta u}\right) - \xi^x\,\underset{h}{\tilde{D}}{}_0(u), \tag{3.27}$$

where we substituted the expression for $\dot{u}$ from the equations (2.17).

Definition The Hamiltonian vector field associated with a functional $\underset{h}{\mathcal{P}}$ is the unique smooth vector field X_P satisfying

$$X_P(\underset{h}{\mathcal{F}}) = \{\underset{h}{\mathcal{F}}, \underset{h}{\mathcal{P}}\}_h \tag{3.28}$$

In coordinate form it can be presented as the operator

$$X_P = \underset{h}{\mathcal{D}} \left(\frac{\delta \underset{h}{\mathcal{P}}}{\delta u} \right) \frac{\partial}{\partial u} \tag{3.29}$$

Some symmetries (3.4) are given as Hamiltonian vector fields or are equivalent to Hamiltonian vector fields under factorization (3.27).

The Hamiltonian formulation of Noether's theorem provides the connection between symmetries and conservation laws.

Theorem 3.4 *For a Hamiltonian system of semidiscrete evolution equations (2.17) a Hamiltonian vector field X_P determines a generalized symmetry of the system if and only if there is an equivalent functional $\underset{h}{\tilde{\mathcal{P}}} = \underset{h}{\mathcal{P}} - \underset{h}{\mathcal{C}}$, differing only from $\underset{h}{\mathcal{P}}$ by a time–dependent distinguished functional $\underset{h}{\mathcal{C}}(t, x, h, u^{(n)})$, such that $\underset{h}{\tilde{\mathcal{P}}}$ determines a conservation law.*

The proof of this theorem is identical to that in the continuous case [1] and we will not reproduce it here.

Let us note that if the Hamiltonian functional and the Hamiltonian symmetry are time independent, then the corresponding conservation law is linear in time.

Remark The total derivative of the Hamiltonian equals to its partial derivative

$$\frac{d}{dt} \underset{h}{\mathcal{H}} = \frac{\partial}{\partial t} \underset{h}{\mathcal{H}} + \{\underset{h}{\mathcal{H}}, \underset{h}{\mathcal{H}}\}_h = \frac{\partial}{\partial t} \underset{h}{\mathcal{H}} \tag{3.30}$$

because $\{\underset{h}{\mathcal{H}}, \underset{h}{\mathcal{H}}\}_h = 0$. Thus $\underset{h}{\mathcal{H}}$ is a conservation law if it is time independent. It is interesting to note that conservation of the time–independent Hamiltonian functional can be viewed as invariance of the equations with respect to time translation. Equations generated by time–independent Hamiltonian functionals are also time–independent, that is they are invariant with respect to the operator

$$X_t = \frac{\partial}{\partial t}.$$

In the canonical form this operator can be presented as

$$\bar{X}_t = \dot{u} \frac{\partial}{\partial u} = \underset{h}{\mathcal{D}} \left(\frac{\delta \underset{h}{\mathcal{H}}}{\delta u} \right) \frac{\partial}{\partial u}$$

which shows that the Hamiltonian functional generates a symmetry and according the Theorem 3.4 it is a conservation law.

IV Examples of semidiscrete equations with conservation laws

In this section we present several examples which demonstrate how Hamiltonian symmetries can be used to find conservation laws of the semidiscrete equations.

Example 4.1 Linear semidiscrete equations with infinitely many conservation laws.

Let us consider the Hamiltonian functional

$$\underset{h}{\mathcal{H}} = \sum_{\Omega} H[u]h, \quad H[u] = \frac{1}{2}(c_0 u^2 + c_1 \underset{h}{u}{}_1^2 + c_2 \underset{h}{u}{}_2^2 + \dots + c_n \underset{h}{u}{}_n^2) \tag{4.1}$$

and the Poisson bracket based on the central difference derivative (2.19). They provide us the infinite set of ODEs

$$\dot{u}_i = c_0 \frac{u_{i+1} - u_{i-1}}{2h} - c_1 \frac{u_{i+2} - 2u_{i+1} + 2u_{i-1} - u_{i-2}}{2h^3} + \dots + (-1)^n c_n \underset{h}{D}{}_0(\underset{h}{u}{}_{2n}) \tag{4.2}$$

with $i = 0, \pm 1, \pm 2, ...$ In the continuous limit these equations correspond to the PDE

$$u_t = c_0 u_1 - c_1 u_3 + \dots + (-1)^n c_n u_{2n+1}$$

for a function $u(t,x)$, $x \in \mathbb{R}$.

From the first glance we see that these equations conserve the distinguished functionals $\underset{h}{\mathcal{C}}{}_1$ and $\underset{h}{\mathcal{C}}{}_2$ given in (2.20) and the Hamiltonian functional (4.1). In addition to these evident conserved functionals we can construct conservation laws corresponding to the following symmetries:

1. The admitted Hamiltonian operator

$$X_1 = \frac{\partial}{\partial u} \tag{4.3}$$

corresponds to the functional

$$\underset{h}{\mathcal{P}}{}_1 = \sum_{i=-\infty}^{\infty} x_i u_i h$$

Because

$$\frac{d}{dt} \underset{h}{\mathcal{P}}{}_1 = -c_0 \sum_{i=-\infty}^{\infty} u_i h = -c_0 \underset{h}{\mathcal{C}}{}_1$$

we obtain the conservation law

$$\underset{h}{\mathcal{P}}{}_1 + tc_0 \underset{h}{\mathcal{C}}{}_1 = \sum_{i=-\infty}^{\infty} x_i u_i h + tc_0 \sum_{i=-\infty}^{\infty} u_i h. \tag{4.4}$$

In the continuous limit we have

$$\underset{h}{\mathcal{P}}{}_1 + tc_0 \underset{h}{\mathcal{C}}{}_1 \to \int xu dx + tc_0 \int u dx.$$

2. The symmetry

$$X_2 = x \frac{\partial}{\partial u}$$

is admitted if $c_0 = 0$. It provides us the conservation law

$$\underset{h}{\mathcal{P}}{}_2 = \frac{1}{2} \sum_{i=-\infty}^{\infty} x_i^2 u_i h, \tag{4.5}$$

which goes into $\frac{1}{2} \int x^2 u dx$ in the continuous limit.

3. The symmetry

$$X_3 = x^2 \frac{\partial}{\partial u} \tag{4.6}$$

is also admitted in the case $c_0 = 0$. Although it is not a Hamiltonian operator, the linear combination

$$X_3 + \frac{h^2}{3} X_1 = \left(x^2 + \frac{h^2}{3} \right) \frac{\partial}{\partial u}$$

is. It is generated by the functional

$$\underset{h}{\mathcal{P}}_3 = \frac{1}{3} \sum_{i=-\infty}^{\infty} x_i^3 u_i h \tag{4.7}$$

which lets us find the conservation law $\underset{h}{\mathcal{P}}_3 + 2tc_1 \underset{h}{\mathcal{C}}_1$ which in the limit $h \to 0$ becomes $\frac{1}{3}\int x^3 u dx + 2tc_1 \int u dx$.

4. Another Lie point symmetry admitted by the set of equations (4.2) is the translation in space presented by the operator

$$X_4 = \frac{\partial}{\partial x}. \tag{4.8}$$

This operator is not canonical. Applying the factorization (3.27), we obtain the canonical symmetry

$$\bar{X}_4 = \underset{h}{\tilde{D}}_0(u) \frac{\partial}{\partial u},$$

which is generated by the functional $\underset{h}{\mathcal{P}}_4$ satisfying

$$\underset{h}{D}_0 \left(\frac{\delta \underset{h}{\mathcal{P}}_4}{\delta u} \right) = \underset{h}{\tilde{D}}_0(u).$$

Further, with the help of the expression (3.21) we get

$$\left. \frac{\delta \underset{h}{\mathcal{P}}_4}{\delta u} \right|_i = \sum_{k=0}^{\infty} \alpha_{2k+1} h^{2k} \underset{h}{D}_0^{2k} u_i = \sum_{k=0}^{\infty} \alpha_{2k+1} \frac{(S_+ - S_-)^{2k}}{2^{2k}} u_i. \tag{4.9}$$

Applying

$$(S_+ - S_-)^{2k} = \sum_{j=0}^{2k} (-1)^j \frac{(2k)!}{j!(2k-j)!} S_{2(k-j)} = \sum_{p=-k}^{k} (-1)^{k-p} \frac{(2k)!}{(k-p)!(k+p)!} S_{2p},$$

where $S_i = S_{\text{sign}(i)}^{|i|}$ represents a shift of indexes operator, to (4.9) and changing the order in the double sum, we arrive at

$$\left. \frac{\delta \underset{h}{\mathcal{P}}_4}{\delta u} \right|_i = \sum_{p=-\infty}^{\infty} b_{2p} S_{2p}(u_i) = \sum_{p=-\infty}^{\infty} b_{2p} u_{i+2p}, \tag{4.10}$$

where

$$b_{2p} = \sum_{k=|p|}^{\infty} \alpha_{2k+1} \frac{(-1)^{k-p}}{2^{2k}} \frac{(2k)!}{(k-p)!(k+p)!} = (-1)^p \sum_{k=|p|}^{\infty} \frac{1}{2k+1} \frac{1}{2^{2k}} \frac{((2k-1)!!)^2}{(k-p)!(k+p)!} \tag{4.11}$$

Let us note that $b_{-2p} = b_{2p}$.

Lemma 4.1 *The series (4.11) defining coefficients b_{2p} for $p \in \mathbb{Z}$ are convergent.*

Proof: Substituting $(k-p)!(k+p)! \geq (k!)^2$ into (4.11), we get

$$|b_{2p}| \leq \sum_{k=p}^{\infty} \frac{1}{2k+1} \frac{1}{2^{2k}} \left(\frac{(2k-1)!!}{k!} \right)^2 = \sum_{k=p}^{\infty} \frac{1}{2k+1} \left(\frac{(2k)!}{(2^k (k)!)^2} \right)^2 .$$

We use the bounds on the factorial provided by the Stirling's expansion

$$n^n \exp(-n)\sqrt{2\pi n} < n! < n^n \sqrt{2\pi n} \exp\left(-n + \frac{1}{12n}\right) \quad \text{for} \quad n \in \mathbb{N}.$$

to obtain the inequality

$$\frac{1}{2k+1} \left(\frac{(2k)!}{(2^k (k)!)^2} \right)^2 < \frac{1}{\pi k(2k+1)} \exp\left(\frac{1}{12k}\right) \quad \text{for} \quad k \in \mathbb{N}$$

that insures us that the series defining b_{2p} converges, moreover it can be shown that $|b_{2p}| \sim 1/|p|$ as $p \to \infty$.

□

A number of the first coefficients b_{2p} is given in the Table 1.

Table 1. Some coefficients b_{2p}.

b_0	$b_{\pm 2}$	$b_{\pm 4}$	$b_{\pm 6}$	$b_{\pm 8}$	$b_{\pm 10}$	$b_{\pm 12}$	$b_{\pm 14}$
1.1662	−0.1070	0.0344	−0.0164	0.0095	−0.0061	0.0043	−0.0031

At this point we can construct the functional corresponding to the Hamiltonian symmetry $\bar{X}_4$. It is the conservation law of the equations (4.2)

$$\underset{h}{\mathcal{P}}_4 = \frac{1}{2} \sum_{i=-\infty}^{\infty} b_0 u_i^2 h + \sum_{p=1}^{\infty} b_{2p} \sum_{i=-\infty}^{\infty} u_i u_{i+2p} h = \frac{1}{2} \sum_{i=-\infty}^{\infty} u_i \sum_{p=-\infty}^{\infty} b_{2p} u_{i+2p} h. \qquad (4.12)$$

For convergence of (4.12) it is essential that $u_i \to 0$ as $i \to \pm\infty$ fast enough.

Lemma 4.2 *It is true that*

$$\underset{h}{\mathcal{P}}_4 \;\to\; \frac{1}{2} \int u^2 dx + O(h^2) \quad \textit{as} \quad h \to 0.$$

Proof: Let us compare the functional $\underset{h}{\mathcal{P}}_4$ with the functional $\frac{1}{2} \sum_{i=-\infty}^{\infty} u_i^2 h$:

$$\underset{h}{\mathcal{P}}_4 - \frac{1}{2} \sum_{i=-\infty}^{\infty} u_i^2 h = \frac{1}{2} \sum_{i=-\infty}^{\infty} u_i \left(\sum_{p=-\infty}^{\infty} b_{2p} u_{i+2p} - u_i \right) h$$

$$= \frac{1}{2} \sum_{i=-\infty}^{\infty} u_i \left(\sum_{k=0}^{\infty} \alpha_{2k+1} h^{2k} \underset{h}{D}{}_0^{2k} - 1 \right) u_i h = \frac{1}{2} \sum_{i=-\infty}^{\infty} u_i \sum_{k=1}^{\infty} \alpha_{2k+1} h^{2k} \underset{h}{D}{}_0^{2k} u_i h$$

$$= \frac{1}{2} \sum_{i=-\infty}^{\infty} u_i (\alpha_3 h^2 u''(x_i) + O(h^4)) h = \frac{h^2}{2} \sum_{i=-\infty}^{\infty} \alpha_3 u_i u''(x_i) h + O(h^4) = -\frac{h^2}{12} \int u u'' dx + O(h^4)$$

On the other hand we have

$$\frac{1}{2} \sum_{i=-\infty}^{\infty} u_i^2 h = \frac{1}{2} \int u^2 dx + O(h^2)$$

which combined with the previous result gives the statement of the lemma.

□

Let us note that in physical applications the integral (4.12) is referred as linear momentum.

5. Equations (4.2) also admit the operators

$$Y_k = \underset{h}{D}{}_0^{2k+1}(u)\frac{\partial}{\partial u}, \quad k = 0, 1, 2, ... \tag{4.13}$$

These operators are true Lie–Bäcklund symmetries, i.e., symmetries which are not equivalent to Lie point symmetries (3.11). All these operators are Hamiltonian with generating functionals

$$\underset{h}{\mathcal{R}}{}_k = \frac{1}{2}\sum_{i=-\infty}^{\infty} u_i \underset{h}{D}{}_0^{2k}(u_i)h = \frac{(-1)^k}{2}\sum_{i=-\infty}^{\infty} (\underset{h}{D}{}_0^k u_i)^2 h, \quad k = 0, 1, 2, ... \tag{4.14}$$

The functionals $\underset{h}{\mathcal{R}}{}_k$ are the conservation laws of (4.2), which in the continuous limit correspond to

$$\mathcal{R}_k = \frac{(-1)^k}{2}\int u_k^2 dx, \qquad k = 0, 1, 2, ...$$

It is interesting to note that the operators Y_k form the canonical operator $\bar{X}_4$ and that functionals $\underset{h}{\mathcal{R}}{}_k$ form the momentum functional (4.12) as the infinite series:

$$\bar{X}_4 = \sum_{k=0}^{\infty} \alpha_{2k+1} h^{2k} Y_k, \qquad \underset{h}{\mathcal{P}}{}_4 = \sum_{k=0}^{\infty} \alpha_{2k+1} h^{2k} \underset{h}{\mathcal{R}}{}_k.$$

◊

Example 4.2 In Examples 2.1 and 2.2 we saw that the Korteweg-de Vries equation and the transfer equation (2.11) can be rewritten as Hamiltonian PDEs. Let us now provide Hamiltonian semidiscretizations of these. We can take the operator $\underset{h}{D}{}_0$ for the discrete bracket and approximate the Hamiltonian functionals as

$$\underset{h}{\mathcal{H}}{}_1 = \sum_{i=-\infty}^{\infty} \left(\frac{u_i^3}{6} - \frac{1}{2}\left(\frac{u_{i+1} - u_i}{h}\right)^2 \right) h \tag{4.15}$$

and

$$\underset{h}{\mathcal{H}}{}_2 = \sum_{i=-\infty}^{\infty} h(u_i)h, \tag{4.16}$$

leading to the semidiscrete equations

$$\dot{u}_i = \frac{u_{i+1}^2 - u_{i-1}^2}{4h} + \frac{u_{i+2} - 2u_{i+1} + 2u_{i-1} - u_{i-2}}{2h^3} \tag{4.17}$$

and

$$\dot{u}_i = \frac{h'(u_{i+1}) - h'(u_{i-1})}{2h}, \tag{4.18}$$

which approximate the original equations to order $O(h^2)$.

Some of the symmetries admitted by the original continuous equations [18] carry over to the semidiscretized equations. The invariance of the equations (4.17) and (4.18) on a uniform mesh with respect to translation in space provides us the conservation law (4.12). The equations also preserve the distinguished functionals (2.20) and their own Hamiltonian functionals.

◊

That each of the conservation laws found above is conserved can, of course, be proved directly. The advantage of the approach we used is that it is constructive and gives not only conservation laws, which we can expect (distinguished and Hamiltonian functionals), but also conservation laws of the form we would probably not try to find by ad hoc methods (see discrete momentum (4.12)). We sum up the considered symmetries and their generating functionals in the Table 2.

V Conclusions

In the paper we presented Hamiltonian form of Noether's theorem for semidiscrete equations. The approach was illustrated on discretizations of Hamiltonian PDEs which preserve the Poisson bracket and some conservation laws of the original continuous equations. Using the connection between symmetries and conservation laws, we can point two reasons why some conservation laws get lost after Poisson structure preserving space discretization. First, the symmetry can be lost because it is not admitted by the semidiscrete system on a uniform mesh (see Example 3.2). Second, there is a difference between continuous and discrete differentiation and some relations, which hold in the limit $h \to 0$, may not be true for finite steps h. One can easily see on the discrete Leibniz rule for the central discrete derivative

$$\underset{h}{D}_0(FG) = \frac{G_+ + G_-}{2} \underset{h}{D}_0(F) + \frac{F_+ + F_-}{2} \underset{h}{D}_0(G)$$

or

$$\underset{h}{D}_0(FG) = G \underset{h}{D}_0(F) + F \underset{h}{D}_0(G) + \frac{h^2}{2}(\underset{-h}{D}\,\underset{+h}{D} G) \underset{h}{D}_0(F) + \frac{h^2}{2}(\underset{-h}{D}\,\underset{+h}{D} F) \underset{h}{D}_0(G)$$

why some symmetries can be Hamiltonian in the continuous case, but lose this property under space discretization.

Although the necessity to have a discrete bracket satisfying skew–symmetry and Jacobi identity imposes limitations on the applications of the presented approach, extensions are possible in several ways. The results of the paper can be extended on the case of several dependent variables or several space variables. Work is in progress on conservation laws of the semidiscretized Schrödinger and wave equations, which can be presented as Hamiltonian equations with the help of the canonical bracket [23]. It is interesting to examine semidiscrete equations with periodic solutions, where the set of semidiscrete equations is finite. For such equations one can find first integrals preserving time discretizations using first integral preserving discretization methods for Hamiltonian ODEs [24],[25],[26],[27]. Note that, in general, it is impossible to preserve symplectic structure and all first integrals simultaneously [28].

The present approach might also be applied to equations which have Lagrangian formulation. The consideration of semidiscrete equations with the Lagrangian formulation can take an intermediate position between well–known continuous case and fully discrete case [5],[6],[7]. It can probably explain whether space or time discretization is the reason why particular conservation laws get lost under a discretization.

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Appendix

Table 2. Some symmetries and their generating functionals for the bracket given by the central difference derivative. The functionals (up to some distinguished functional) are conservation laws of the Hamiltonian semidiscrete equations constructed with the help of this bracket.

Operator	Operator in canonical form	Functional	Continuous limit of functional
	$0\dfrac{\partial}{\partial u}$	$\sum\limits_{i=-\infty}^{\infty} u_i h, \quad \sum\limits_{i=-\infty}^{\infty} (-1)^i u_i h$	$\int u dx, \quad 0$
$\dfrac{\partial}{\partial t}$	$\underset{h}{D}{}_0 \left(\dfrac{\delta \underset{h}{\mathcal{H}}}{\delta u} \right) \dfrac{\partial}{\partial u}$	$\underset{h}{\mathcal{H}}$	$\mathcal{H}$
	$\dfrac{\partial}{\partial u}$	$\sum\limits_{i=-\infty}^{\infty} x_i u_i h$	$\int x u dx$
	$x\dfrac{\partial}{\partial u}$	$\dfrac{1}{2} \sum\limits_{i=-\infty}^{\infty} x_i^2 u_i h$	$\dfrac{1}{2} \int x^2 u dx$
	$\left(x^2 + \dfrac{h^2}{3} \right) \dfrac{\partial}{\partial u}$	$\dfrac{1}{3} \sum\limits_{i=-\infty}^{\infty} x_i^3 u_i h$	$\dfrac{1}{3} \int x^3 u dx$
$\dfrac{\partial}{\partial x}$	$\sum\limits_{k=0}^{\infty} \alpha_{2k+1} h^{2k} \underset{h}{D}{}_0^{2k+1}(u) \dfrac{\partial}{\partial u}$	$\dfrac{1}{2} \sum\limits_{i=-\infty}^{\infty} u_i \sum\limits_{p=-\infty}^{\infty} b_{2p} u_{i+2p} h$	$\dfrac{1}{2} \int u^2 dx$
	$\underset{h}{D}{}_0^{2k+1}(u) \dfrac{\partial}{\partial u}, \quad k = 0, 1, \ldots$	$\dfrac{(-1)^k}{2} \sum\limits_{i=-\infty}^{\infty} (\underset{h}{D}{}_0^k u_i)^2 h$	$\dfrac{(-1)^k}{2} \int u_k^2 dx$