ON LIE GROUP CLASSIFICATION OF SECOND–ORDER ORDINARY DIFFERENCE EQUATIONS

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A group classification of invariant difference models, i.e. difference equations and meshes, is discussed. In the continuous limit the results go over into Lie's classification of second order ordinary differential equations. The discrete model is a three point one and we show that it can be invariant under Lie groups of dimension $0 \leq n \leq 6$.

1 Introduction

Lie group theory started out as a theory of transformations of solutions of sets of differential equations [1,2,3]. Over the years it has developed into a powerful tool for classifying differential equations and for solving them.

Applications of Lie group theory to difference equations are much more recent. Essentially there are two different points of view that have been adopted when studying continuous symmetries of equations involving discrete or discretely varying independent variables (see [9] for literature overview).

One point of view is that a difference equation is a priori given on some fixed lattice and the task is to determine a group of transformations, leaving the solution set invariant. Different approaches differ in their treatment of independent variables, and in the assumed form of the global and infinitesimal transformations considered. In any case, the distinction between point symmetries and generalized symmetries becomes blurred. In order to obtain symmetries that go into dilations, rotations, or Lorentz transformations in the continuous limit, it is necessary to significantly modify the Lie techniques used in the continuous case.

An alternative point of view is to pose the question: How does one discretize a differential equation while preserving all of its Lie point symmetries? Here one starts from a differential equation and finds its Lie point symmetries, using well known techniques [4,5,6]. Thus a symmetry group G and its Lie algebra are a priori given. One then looks for a difference scheme, i.e. a difference equation and a mesh that have the same symmetry group and the same symmetry algebra L. In particular the Lie algebra L is realized by the same vector fields in the continuous and in the discrete case.

In this paper we consider the second point of view. We start out from Lie's classification of second order ordinary differential equations (ODEs) according to their point symmetries. Our aim is to provide a similar classification of second order difference schemes.

Thus, Lie considered equations of the form

$$E(x, y, y', y'') = 0, \qquad \frac{\partial E}{\partial y''} \neq 0, \tag{1}$$

where E is an arbitrary sufficiently smooth function. The Lie algebra L of the symmetry group G of Eq. (1) is realized by vector fields of the form

$$X_\alpha = \xi_\alpha(x, y)\frac{\partial}{\partial x} + \eta_\alpha(x, y)\frac{\partial}{\partial y}, \qquad \alpha = 1, ..., n. \tag{2}$$

Lie showed [1,2] that the Lie algebra of Eq. (1) can be of dimension $n = \dim L = 0, 1, 2, 3, \text{or } 8$. Moreover, he classified equations with point symmetries into equivalence classes under the action of the infinite–dimensional group $\mathrm{Diff}(2, \mathbb{C})$ of all local diffeomorphisms of a complex plane $\mathbb{C}^2$ (his analysis was over the field $\mathbb{C}$ of complex numbers). For each equivalence class he chose a simple representative equation and gave the corresponding realization of L. He showed that all equations with $\dim \mathrm{L} \geq 2$ can be integrated in quadratures.

We provide a similar classification of discrete models of second order ODEs. We restrict ourselves to a three point stencil, as shown in Fig.1.

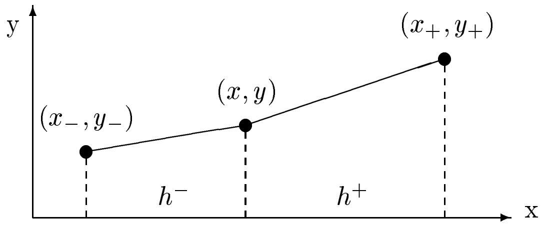

Fig.1. Elementary stencil for 3 point difference equation.

Thus, we are considering a six–dimensional subspace $(x, x_-, x_+, y, y_-, y_+)$ of the space of independent and dependent variables. The discrete model under consideration can be presented in terms of a pair of difference equations

$$\begin{cases} F(x, x_-, x_+, y, y_-, y_+) = 0, \\ \Omega(x, x_-, x_+, y, y_-, y_+) = 0, \end{cases} \tag{3}$$

such that

$$\det \frac{\partial(F, \Omega)}{\partial(x_+, y_+)} \neq 0, \quad \det \frac{\partial(F, \Omega)}{\partial(x_-, y_-)} \neq 0. \tag{4}$$

The conditions on the model are

1. In the continuous limit $h_+ = x_+ - x \to 0$ and $h_- = x - x_- \to 0$ the first equation should go into a second order ODE of the form (1). The second equation gives the lattice on which the first equation is considered. In the continuous limit the second equation becomes an identity $(0 = 0)$.

2. The difference system (3)–(4) is invariant under the same group as the ODE (1). That is, the difference scheme (3)–(4) and the ODE (1) are annihilated by the appropriate prolongations (see below) of the same vector fields (2).

Lie's classification of ODEs was performed over the field of complex numbers $\mathbb{C}$. He made use of his classification of all finite–dimensional Lie algebras that can be realized in terms of vector fields of the form (2). Thus, he classified all finite–dimensional subalgebras $L \subset \mathrm{diff}(2, \mathbb{C})$ [3].

Our classification of the difference models is over $\mathbb{R}$; that over $\mathbb{C}$ is obtained as a byproduct. We make use of a much more recent classification of finite–dimensional subalgebras of $\mathrm{diff}(2, \mathbb{R})$ [7].

2 Construction of invariant difference schemes

As stated in the introduction, we wish to construct all three–point difference schemes invariant under some transformation group. The Lie algebra of this group is realized by vector fields of the form (2), i.e. the Lie algebra, and the group action is the same as for differential equations.

An essential tool for studying Lie symmetries is prolongation theory. For a second order ODE we must prolong the action of a vector field (2) from the space (x, y) of independent and dependent variables to a four–dimensional space (x, y, y', y''). The prolongation formula for vector fields is [4,5,6]:

$$\mathbf{pr}^{(2)} X_\alpha = X_\alpha + \eta^1_\alpha(x, y, y') \frac{\partial}{\partial y'} + \eta^2_\alpha(x, y, y', y'') \frac{\partial}{\partial y''} \tag{5}$$

with

$$\eta_\alpha^1 = D_x(\eta_\alpha(x,y)) - y' D_x(\xi_\alpha(x,y)), \quad \eta_\alpha^2 = D_x(\eta_\alpha^1) - y'' D_x(\xi_\alpha(x,y)), \quad (6)$$

where D_x is the total differentiation operator.

In the discrete case we prolong the operators X_α to a six–dimensional space $(x, x_-, x_+, y, y_-, y_+)$. The prolongation formula is

$$\begin{aligned} \mathbf{pr}^{(2)} X_\alpha = X_\alpha + \xi_\alpha(x_-, y_-)\frac{\partial}{\partial x_-} + \xi_\alpha(x_+, y_+)\frac{\partial}{\partial x_+} + \\ + \eta_\alpha(x_-, y_-)\frac{\partial}{\partial y_-} + \eta_\alpha(x_+, y_+)\frac{\partial}{\partial y_+}. \end{aligned} \quad (7)$$

Let us assume that a Lie group G is given and that its Lie algebra is realized by vector fields of the form (2). If we wish to construct a second order ODE that is invariant under G, we proceed as follows. We choose a basis of L, namely $\{X_\alpha, \ \alpha = 1, ..., n\}$, and impose the equations

$$\mathbf{pr}^{(2)} X_\alpha \Phi(x, y, y', y'') = 0, \qquad \alpha = 1, ..., n, \quad (8)$$

with $\mathbf{pr}^{(2)} X_\alpha$ as in Eq. (5). Using the method of characteristics, we obtain a set of elementary invariants $I_1, ..., I_k$. Their number is

$$k = \mathrm{dim} M - (\mathrm{dim} G - \mathrm{dim} G_0), \quad (9)$$

where M is the manifold that G acts on and G_0 is the stabilizer of a generic point on M. In our case we have $M \sim \{x, y, y', y''\}$ and hence $\mathrm{dim} M = 4$. An equivalent, but more practical formula for the number of invariants is

$$k = \mathrm{dim} M - \mathrm{rank} Z, \qquad k \geq 0,, \quad (10)$$

where Z is the matrix

$$Z = \begin{pmatrix} \xi_1 & \eta_1 & \eta_1^1 & \eta_1^2 \\ \vdots & & & \\ \xi_n & \eta_n & \eta_n^1 & \eta_n^2 \end{pmatrix}. \quad (11)$$

The rank of Z is calculated at a generic point of M. The invariant equation is written as

$$E(I_1, ..., I_k) = 0, \quad (12)$$

where E must satisfy the condition from (1). Equation (12) obtained in this manner is "strongly invariant", i.e. $\mathbf{pr}^{(2)} X_\alpha E = 0$ is satisfied identically [8].

Further invariant equations are obtained if the rank of Z is less than maximum on some manifold described by the equation $E(x, y, y', y'') = 0$, itself satisfying the condition

$$\mathbf{pr^{(2)}} X_\alpha E \Big|_{E=0} = 0, \qquad \alpha = 1, ..., n. \tag{13}$$

We then obtain a "weakly invariant" equation of the form (1), i.e. Eq. (13) is satisfied on the solution set of the equation $E = 0$.

The procedure for obtaining an invariant second order difference model for a given group G is quite analogous. Instead of Eq. (8) we write

$$\mathbf{pr^{(2)}} X_\alpha \Phi(x, y, x_-, y_-, x_+, y_+) = 0, \qquad \alpha = 1, ..., n, \tag{14}$$

with $\mathbf{pr^{(2)}} X_\alpha$ as in Eq. (7). We use the method of characteristics to obtain the elementary invariants $I_1, ..., I_k$ in the space $M \sim \{x, y, x_-, y_-, x_+, y_+\}$. The number of invariants k satisfies Eq. (9) and (10). However, in this case we have $\dim M = 6$ and the matrix Z is

$$Z = \begin{pmatrix} \xi_1 & \eta_1 & \xi_{1,-} & \eta_{1,-} & \xi_{1,+} & \eta_{1,+} \\ \vdots & & & & & \\ \xi_n & \eta_n & \xi_{n,-} & \eta_{n,-} & \xi_{n,+} & \eta_{n,+} \end{pmatrix}, \tag{15}$$

where

$$\xi_{j,\pm} = \xi_j(x_\pm, y_\pm), \qquad \eta_{j,\pm} = \eta_j(x_\pm, y_\pm), \qquad 1 \le j \le n. \tag{16}$$

The "strongly invariant" difference scheme is then given by the equations

$$F(I_1, ..., I_k) = 0, \qquad \Omega(I_1, ..., I_k) = 0, \tag{17}$$

satisfying condition (4).

"Weakly invariant" difference schemes are obtained by finding invariant manifolds in M, i.e. finding surfaces $S(x, y, x_-, y_-, x_+, y_+) = 0$ on which the rank of Z is less than maximal. The system (17) represents both the difference equation and a mesh. In general, for the same group G we expect to have more difference invariants, than differential ones (since we have $\dim M = 6$ in the first case and $\dim M = 4$ in the second). We need two equations for a difference scheme, just one for a differential equation. That still leaves us with one more degree of freedom in the discrete case.

3 Examples of invariant difference models

In [9] we run through subalgebras of diff$(2,\mathbb{R})$ by dimension. For each representative subalgebra we construct the most general difference scheme. Then we specialize it by specific choices of the arbitrary functions involved, so as to obtain a scheme that has the appropriate ODE as its continuous limit. Let us present some examples of invariant difference schemes.

Example 1. Let us consider a three–dimensional Lie algebra which can be presented by vector fields

$$X_1 = \frac{\partial}{\partial x}; \qquad X_2 = \frac{\partial}{\partial y}; \qquad X_3 = x\frac{\partial}{\partial x} + (x+y)\frac{\partial}{\partial y}. \tag{18}$$

The corresponding invariant ODE is

$$y'' = \exp(-y'). \tag{19}$$

A basis for the difference invariants is

$$\left\{ \frac{h_+}{h_-}, \quad h_+ \exp\left(-y_x\right), \quad h_- \exp\left(-y_{\bar{x}}\right) \right\},$$

where we use notations

$$h_+ = x_+ - x, \quad h_- = x - x_-, \quad y_x = \frac{y_+ - y}{h_+}, \quad y_{\bar{x}} = \frac{y - y_-}{h_-}.$$

The general invariant difference scheme can be written as

$$\begin{cases} y_{x\bar{x}} = \exp\left(-\dfrac{y_x + y_{\bar{x}}}{2}\right) f\left(\sqrt{h_- h_+}\exp\left(-\dfrac{y_x + y_{\bar{x}}}{2}\right)\right); \\[2ex] h_+ = h_- g\left(\sqrt{h_- h_+}\exp\left(-\dfrac{y_x + y_{\bar{x}}}{2}\right)\right), \end{cases} \tag{20}$$

where $y_{x\bar{x}} = \frac{2}{h_+ + h_-}\left(\frac{y_+ - y}{h_+} - \frac{y - y_-}{h_-}\right)$. An invariant approximation of the ODE (19) is obtained by setting $f = g = 1$:

$$y_{x\bar{x}} = \exp\left(-\frac{y_x + y_{\bar{x}}}{2}\right); \qquad h_+ = h_-. \tag{21}$$

Example 2. The six–dimensional general affine Lie algebra $\mathfrak{gaff}(2,\mathbb{R})$ can be presented by vector fields

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x}; \qquad X_2 = \frac{\partial}{\partial y}; \qquad X_3 = x\frac{\partial}{\partial x}; \\ X_4 &= y\frac{\partial}{\partial x}; \qquad X_5 = x\frac{\partial}{\partial y}; \qquad X_6 = y\frac{\partial}{\partial y}. \end{aligned} \tag{22}$$

The equation $y'' = 0$ is an invariant manifold in the continuous case and the difference model

$$\begin{cases} y_{x\bar{x}} = 0; \\ \\ h_+ = Ch_-; \end{cases} \tag{23}$$

where C is an arbitrary constant, provides a weakly invariant difference scheme.

In the continuous case the symmetry group of a second order ODE can be at most 8–dimensional. Moreover, it is 8–dimensional only if the equation can be transformed into $y'' = 0$ by a point transformation. The symmetry group in that case is $SL(3, \mathbb{R})$. Any second order ODE invariant under a Lie group of dimension 4, 5, or 6 is also invariant under $SL(3, \mathbb{R})$. No ODE invariant under a Lie group of dimension 7 exists. The algebra $\mathfrak{sl}(3, \mathbb{R})$ can be presented by vector fields

$$\begin{gathered} X_1 = \frac{\partial}{\partial x}; \quad X_2 = \frac{\partial}{\partial y}; \quad X_3 = x\frac{\partial}{\partial x}; \quad X_4 = y\frac{\partial}{\partial x}; \quad X_5 = x\frac{\partial}{\partial y}; \\ X_6 = y\frac{\partial}{\partial y}; \quad X_7 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}; \quad X_8 = xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}. \end{gathered} \tag{24}$$

One can easily check that there is only one invariant manifold $y_{x\bar{x}} = 0$ for this algebra. Thus, we can not write a grid equation and, consequently, there is no invariant difference model for the group $SL(3, \mathbb{R})$ on a three point stencil.

All Lie groups admitted by invariant difference models were presented in Ref. [9]. We give the number of obtained models in Table 1.

Table 1. Number of invariant difference models and differential equations for finite–dimensional groups.

group dimension	1	2	3	4	5	6	8
number of difference models	1	4	14	8	2	1	–
number of differential equations	1	2	7	–	–	–	1

4 Conclusions

Let us compare Lie's classification of second order ordinary differential equations with the obtained classification of three point difference schemes.

1. For every ODE invariant under a Lie group G of dimension n, $1 \leq n \leq 3$, there exist a family of different schemes invariant under the same group G. In particular, for $n = 3$, the invariant ODE is specified up to at most a

constant. The invariant difference scheme in general involves two arbitrary functions.

2. All ODEs invariant under a Lie group G of dimension $n = 4$, 5, or 6 are also invariant under $SL(3, \mathbb{R})$ and we have $G \subset SL(3, \mathbb{R})$. Moreover, the ODE can be transformed into $y'' = 0$. Three point difference schemes invariant under groups of dimensions 4, 5, and 6 exist. If a continuous limit exists, it is $y'' = 0$. However, other invariant schemes exist that do not have continuous limits.

3. The "discrete free particle equation" $y_{x\bar{x}} = 0$ has a different symmetry behavior from its continuous limit. First of all, it is invariant at most under a six–dimensional Lie group, namely the group of all linear transformations of the space $\mathbb{R}^2$, or $\mathbb{C}^2$, i.e. $\{x, y\}$. This invariance occurs on the mesh $h_+ = Ch_-$, where $C > 0$ is a constant. The equation $y_{x\bar{x}} = 0$ is invariant under groups of dimension $1 \leq n \leq 4$ for more general meshes.

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