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# MULTIRATE PARTITIONED RUNGE-KUTTA METHODS

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**Abstract.**

The coupling of subsystems in a hierarchical modelling approach leads to different time constants in the dynamical simulation of technical systems. Multirate schemes exploit the different time scales by using different time steps for the subsystems. The stiffness of the system or at least of some subsystems in chemical reaction kinetics or network analysis, for example, forbids the use of explicit integration schemes. To cope with stiff problems, we introduce multirate schemes based on ROW methods which avoid the coupling between active and latent components based on interpolating and extrapolating state variables. The most efficient implementation of such schemes turn out to be a multirate extension of Partitioned Runge-Kutta methods. Order conditions and test results for such a lower order MPRK method are presented.

*AMS subject classification:* 65L05.

*Key words:* Coupled systems, Stiff systems, Multirating, Partitioned Runge-Kutta schemes, Rosenbrock-Wanner methods, Partitioning, Latency, Inverter chain.

## 1 Introduction.

The dynamical behaviour of coupled systems is characterized by different time scales of the subsystems. In radio frequency applications, for example, one has to deal with coupled systems of analogue and digital networks. Whereas the analogue part runs in the microsecond scale, the digital part is measured in nanoseconds. This requires multirate integration schemes using the inherent step size of each subsystem.

In general, the dynamics of the whole system can be described by an initial value problem of ordinary differential equations

$$\dot{y} = f(t, y), \qquad y(t_0) = y_0, \qquad y \in \mathbb{R}^n. \tag{1.1}$$

Due to a hierarchical modeling process, the systen is often given partitioned into $p$ subsystems:

$$\begin{aligned} \dot{y}_1 &= f_1(t, y_1, \ldots, y_p) \\ &\vdots \\ \dot{y}_p &= f_p(t, y_1, \ldots, y_p) \end{aligned} \tag{1.2}$$

with initial value $y(t_0) = y_0$ for the partitioned solution vector $y = (y_1, \ldots, y_p)^\top$, $y_i \in \mathbb{R}^{n_i}$, $i = 1, \ldots, n$, $\sum_{i=1}^{p} n_i = n$.

For the numerical solution of (1.2) multirate schemes have been derived, where the different step sizes are introduced explicitly in the discretization. Basic work on backward differentiation formulas (BDF) can be found in Gear and Wells [3], Skelboe [10], and Skelboe and Anderson [9]. Multirate extrapolation [1] and Runge-Kutta [2] schemes are successfully used in stellar problems by Engstler and Lubich. In Günther and Rentrop [4, 5] multirate Rosenbrock-Wanner (MROW) methods are used for VLSI applications of electrical networks. One shortcoming of all these multirate methods derived so far is the coupling between active and latent components by interpolating and extrapolating state variables, which makes their implementation very difficult into existing simulation packages.

Recently, Kværnø and Rentrop [7] have derived a concise theory of explicit multirate Runge-Kutta methods which avoids the large overhead caused by coupling subsystems via interpolation and extrapolation. However, in many technical applications such as chemical reaction kinetics or electrical network simulation, the stiffness of the system or at least of some subsystems forbids the use of explicit schemes. To cope also with stiff problems, we generalize in this paper the former approach to ROW methods.

In section 2 generalized multirate ROW methods are introduced. Among different possible implementations a generalization of Partitioned Runge-Kutta (PRK) to Multirate PRK (MPRK) methods seems to be the most efficient one. Section 3 includes the order conditions with and without simplifying conditions, and a coefficient set for an embedded method of order (2)3 is derived. Finally, an inverter chain taken from electric circuit simulation, shows the performance of the approach.

## 2 Generalized Multirate ROW Methods.

In the following we concentrate our investigations on autonomous initial value problems, whose state vector $y \in \mathbb{R}^n$ is partitioned into only two parts: latent components $y_L \in \mathbb{R}^{n_L}$ and a small number of active components $y_A \in \mathbb{R}^{n_A}$ with $n_A + n_L = n$ and $n_A \ll n_L$,

$$\dot{y}_A = f_A(y_A, y_L), \qquad y_A(t_0) = y_{A0} \tag{2.1a}$$

$$\dot{y}_L = f_L(y_A, y_L), \qquad y_L(t_0) = y_{L0} \tag{2.1b}$$

The active components $y_A$ are integrated with a small step size $h$, the latent components $y_L$ with a large step size $H$. A synchronization of the micro steps

$h$ and the macro step $H$ is performed after each macrostep. In each macro step the micro steps are assumed to be constant: $H = mh$.

A quite natural generalization of the multirate RK methods derived in [7] is now given in a compact notation by

*Compound macro and first micro step:*

$$
\begin{aligned}
&\text{(2.2a)} & y_L^H &= y_{L0} + (b_L^\top \otimes I_L) k_L \\
&\text{(2.2b)} & y_{A,1}^h &= y_{A0} + (b_A^\top \otimes I_A) k_A^0 \\
& & k_L &= H f_L \Big( \mathbb{1}_A \otimes y_{A0} + m(\widetilde{D} \otimes I_A) k_A^0, \dots \\
&\text{(2.2c)} & & \qquad \mathbb{1}_L \otimes y_{L0} + (\widetilde{A} \otimes I_L) k_L \Big) + H(\widetilde{G} \otimes J_L) k_L \\
& & k_A^0 &= h f_A \left( \mathbb{1}_A \otimes y_{A0} + (A \otimes I_A) k_A^0, \dots \right. \\
&\text{(2.2d)} & & \qquad \left. \mathbb{1}_L \otimes y_{L0} + m^{-1}((D + \eta(0)) \otimes I_L) k_L \right) + h(G_0 \otimes J_A^0) k_A^0
\end{aligned}
$$

with the Jacobians

$$
J_L = \frac{\partial f_L}{\partial y_L}(y_{A0}, y_{L0}) \quad \text{and} \quad J_A^0 = \frac{\partial f_A}{\partial y_A}(y_{A0}, y_{L0}) \, .
$$

*Micro steps for $\lambda = 1, \dots, m-1$:*

$$
\begin{aligned}
& & y_{A,\lambda+1}^h &= y_{A\lambda}^h + (b_A^\top \otimes I_A) k_A^\lambda \\
& & k_A^\lambda &= h f_A \left( \mathbb{1}_A \otimes y_{A\lambda} + (A \otimes I_A) k_A^\lambda, \dots \right. \\
&\text{(2.2e)} & & \qquad \left. \mathbb{1}_L \otimes y_{L0} + m^{-1}((D + \eta(\lambda)) \otimes I_L) k_L \right) + h(G_\lambda \otimes J_A^\lambda) k_A^\lambda
\end{aligned}
$$

with the Jacobians

$$
J_A^\lambda = \frac{\partial f_A}{\partial y_A} \left( y_{A\lambda}, y_{L0} + m^{-1}(\eta(\lambda) \otimes I_L) k_L \right) .
$$

Here we have used the abbreviations $\mathbb{1}_A = (1, \dots, 1)^\top \in \mathbb{R}^{s_A \times 1}$ and $\mathbb{1}_L = (1, \dots, 1)^\top \in \mathbb{R}^{s_L \times 1}$. The identity matrices in $\mathbb{R}^{n_A \times n_A}$ and $\mathbb{R}^{n_L \times n_L}$ are denoted by $I_A$ and $I_L$, resp. The coefficient matrices

$$
A = (\alpha_{ij}) \in \mathbb{R}^{s_A \times s_A}, D = (\delta_{ij}) \in \mathbb{R}^{s_A \times s_L},
$$

$$
\widetilde{A} = (\widetilde{\alpha}_{ij}) \in \mathbb{R}^{s_L \times s_L}, \widetilde{D} = (\widetilde{\delta}_{ij}) \in \mathbb{R}^{s_L \times s_A},
$$

are lower triangular matrices with vanishing diagonal elements, and

$$
\begin{aligned}
\widetilde{G} &= (\widetilde{\gamma}_{ij}) + \operatorname{diag}(\widetilde{\gamma}, \dots, \widetilde{\gamma}) \in \mathbb{R}^{s_L \times s_L}, \\
G_0 &= (\gamma_{ij}) + \operatorname{diag}(\gamma, \dots, \gamma) \in \mathbb{R}^{s_A \times s_A}, \\
G_\lambda &= (\bar{\gamma}_{ij}) + \operatorname{diag}(\bar{\gamma}, \dots, \bar{\gamma}) \in \mathbb{R}^{s_A \times s_A}, \quad \lambda = 1, \dots, m-1,
\end{aligned}
$$

are diagonally lower triangular matrices. The matrix function

$$\eta : \mathbb{N} \longrightarrow \mathbb{R}^{s_A \times s_L}, \quad \eta(0) = 0,$$

has polynomials $\eta_{ij} = \eta_j$ in $\lambda$ as matrix entries.

The coupling of both step sizes in (2.2a-2.2d) is the crucial point. For each stage number, the increments $k^0_{A,i}$ and $k_{L,i}$ can be computed simultaneously. However, the increments of one part depend on increments of lower stages of the other part. The coupling of active to latent terms in (2.2c) relies on the available information in the first micro step. The latent terms are coupled to the active terms in (2.2d) by averaging the latent increments. We use a polynomial, which is constructed by the increments of the first micro step and which is extrapolated to the macro step interval $[t_0, t_0 + H]$. To get more degrees of freedom, the time history of the micro steps is included in the polynomials $\eta_j(\lambda)$ in (2.2e).

Some simplifications can be made to save computational costs in order to get a more efficient implementation of method (2.2). Three levels can be distinguished:

1. *W methods in the last $m-1$ micro steps*
   The Jacobian $J^\lambda_A$ in (2.2e) can be approximated by $\partial f_A/\partial y_A(y_{A\lambda}, y_{L0})$ or $\partial f_A/\partial y_A(y_{A0}, y_{L0})$, which is of $\mathcal{O}(H)$ in the first or both components, resp. This corresponds to the use of W-type methods for the last $m-1$ micro steps. Thus the order conditions for the second part (2.2e) can be derived analogous to W-methods [11].

   Stiffness is only expected in the latent part, whereas the active part is characterized by rapidly changing solution courses. This justifies the following simplifications.

2. *Explicit computation of active part in the last $m-1$ micro steps*
   Asigning $G_\lambda = 0$ for $\lambda = 1, \ldots, m$ in (2.2e) leads to an explicit RK method for the last $m-1$ micro steps. The corresponding order conditions for the second part can be found in [7].

3. *Explicit computation of active part in all steps*
   Using $G_\lambda = 0$ for $\lambda = 0, \ldots, m-1$ one gets an explicit computation of the active part, only coupled in the first micro step with the linearly implicit computation of the latent part. One notes that this third simplification corresponds to a Partitioned Runge-Kutta (PRK) method [8] if applied to the single rate case $m = 1$. Thus PRK methods are generalized to multirate schemes, or shortly: we get a Multirate Partitioned Runge-Kutta (MPRK) method.

For sufficiently smooth problems, the convergence theory for the MROW methods [4] can be transferred for all mentioned three variants of (2.2): *The order $p$ of the local truncation error implies global convergence of order $p$.*

The method is of order $p$, if

$$y^H_{L,1} = y_L(t_0 + H) + \mathcal{O}(H^{p+1}) \tag{2.3a}$$

$$y^h_{A,1} = y_A(t_0 + h) + \mathcal{O}(h^{p+1}) \tag{2.3b}$$

Table 3.1: Order conditions for compound macro and first micro step

| number | order | active | latent |
|---|---|---|---|
| 1 | 1 | $b_A^\top \mathbb{1}_A = 1$ | $b_L^\top \mathbb{1}_L = 1$ |
| 2 | 2 | $2b_A^\top A\mathbb{1}_A = 1$ | $2b_L^\top \widetilde{B}\mathbb{1}_L = 1$ |
| 3 | | $2b_A^\top D\mathbb{1}_L = 1$ | $2b_L^\top \widetilde{D}\mathbb{1}_A = 1$ |
| 4 | 3 | $6b_A^\top A^2\mathbb{1}_A = 1$ | $6b_L^\top \widetilde{B}^2\mathbb{1}_L = 1$ |
| 5 | | $6b_A^\top AD\mathbb{1}_L = 1$ | $6b_L^\top \widetilde{B}\widetilde{D}\mathbb{1}_A = 1$ |
| 6 | | $6mb_A^\top D\widetilde{D}\mathbb{1}_A = 1$ | $(6/m)b_L^\top \widetilde{D}D\mathbb{1}_L = 1$ |
| 7 | | $6mb_A^\top D\widetilde{B}\mathbb{1}_L = 1$ | $(6/m)b_L^\top \widetilde{D}A\mathbb{1}_A = 1$ |
| 8 | | $3b_A^\top \mathrm{diag}(A\mathbb{1}_A)A\mathbb{1}_A = 1$ | $3b_L^\top \mathrm{diag}(\widetilde{A}\mathbb{1}_L)\widetilde{A}\mathbb{1}_L = 1$ |
| 9 | | $3b_A^\top \mathrm{diag}(A\mathbb{1}_A)D\mathbb{1}_L = 1$ | $3b_L^\top \mathrm{diag}(\widetilde{A}\mathbb{1}_L)\widetilde{D}\mathbb{1}_A = 1$ |
| 10 | | $3b_A^\top \mathrm{diag}(D\mathbb{1}_L)D\mathbb{1}_L = 1$ | $3b_L^\top \mathrm{diag}(\widetilde{D}\mathbb{1}_A)\widetilde{D}\mathbb{1}_A = 1$ |

holds for the compound macro and first micro step, starting from the exact solution $y(t_0)$, and

$$y^h_{A,\lambda+1} = y_A(t_0 + \lambda h) + \mathcal{O}(h^{p+1}), \quad \lambda = 1, \ldots, m-1 \tag{2.3c}$$

as well for the last $m-1$ micro steps, if starting every time from the exact solution $y_A(t_0 + \lambda h)$.

In the following we restrict our investigations of generalized multirate ROW methods to the special case of MPRK schemes. We derive the order conditions and construct an embedded MPRK method of order (2)3.

## 3 MPRK methods.

Consider now MPRK schemes, i. e. method (2.2e) with

$$G_\lambda = 0 \quad \text{for} \quad \lambda = 0, \ldots, m-1.$$

This scheme fits into the P-series approach of Hairer [6]. The order conditions sufficient for (2.3a–2.3b) up to order 3, i. e. for local order $p$ of the compound step, are listed in Tab. 3.1, with the abbreviation $\widetilde{B} = \widetilde{A} + \widetilde{G}$.

Additionally, the order conditions given in Tab. 3.2 have to be fulfilled for (2.3c), i. e. for local order $p$ of the last $m-1$ macro step sizes. These order conditions can be found in [7], since only explicit RK steps are used.

*Simplifying conditions.* In the following, we demand the simplifying conditions

$$\eta(\lambda)\mathbb{1}_L = \lambda\mathbb{1}_L, \tag{3.1a}$$

$$A\mathbb{1}_A = D\mathbb{1}_L, \tag{3.1b}$$

$$\widetilde{A}\mathbb{1}_L = \widetilde{D}\mathbb{1}_A. \tag{3.1c}$$

Table 3.2: Order conditions for last $m-1$ micro steps: $\lambda = 1,\dots,m-1$

| number | order | condition |
|---|---|---|
| 1 | 2 | $2b_A^\top(D+\eta(\lambda))\mathbb{1}_L = (\lambda+1)^2-\lambda^2$ |
| 2 | 3 | $3b_A^\top\mathrm{diag}(\lambda\mathbb{1}_A + A\mathbb{1}_A)(D+\eta(\lambda))\mathbb{1}_L = (\lambda+1)^3-\lambda^3$ |
| 3 | | $3b_A^\top\mathrm{diag}((D+\eta(\lambda))\mathbb{1}_L))(D+\eta(\lambda))\mathbb{1}_L = (\lambda+1)^3-\lambda^3$ |
| 4 | | $3b_A^\top(\lambda^2\mathbb{1}_A + 2A(D+\eta(\lambda))\mathbb{1}_L) = (\lambda+1)^3-\lambda^3$ |
| 5 | | $6mb_A^\top(D+\eta(\lambda))\widetilde{D}\mathbb{1}_A = (\lambda+1)^3-\lambda^3$ |
| 6 | | $6mb_A^\top(D+\eta(\lambda))\widetilde{A}\mathbb{1}_L = (\lambda+1)^3-\lambda^3$ |

Table 3.3: Order conditions for MPRK methods with simplifying conditions.

| number | order | origin | order condition |
|---|---|---|---|
| 1 | 1 | PRK | $b^\top\mathbb{1} = 1$ |
| 2 | 2 | PRK | $2b^\top\widetilde{B}\mathbb{1} = 1$ |
| 3 | 2 | PRK | $2b^\top A\mathbb{1} = 1$ |
| 4 | 3 | PRK | $6b^\top\widetilde{B}\widetilde{B}\mathbb{1} = 1$ |
| 5 | 3 | PRK | $6b^\top\widetilde{B}A\mathbb{1} = 1$ |
| 6 | 3 | PRK | $6b^\top A^2\mathbb{1} = 1$ |
| 7 | 3 | PRK | $3b^\top\mathrm{diag}(A\mathbb{1}_A)A\mathbb{1}_A = 1$ |
| 8 | 3 | M | $(6/m)b^\top\widetilde{D}A\mathbb{1} = 1$ |
| 9 | 3 | M | $6mb^\top DA\mathbb{1} = 1$ |
| 10 | 3 | M | $6mb^\top D\widetilde{B}\mathbb{1} = 1$ |
| 11 | 3 | M | $2mb^\top\eta(\lambda)A\mathbb{1} = \lambda(\lambda+1)$ |

Now, the order condition $(2,3)$, $(4,5)$ and $(8,9,10)$ coincide for the active part, and $(6,7)$, $(8,9,10)$ for the latent part. Order condition (1–4) of Tab. 3.2 are automatically fulfilled, and $(5,6)$ coincide.

Choosing $s_A = s_L$ makes it very natural to demand

$$A = \widetilde{A}, \tag{3.1d}$$

$$b_A = b_L =: b. \tag{3.1e}$$

In this case, only the coupling coefficients $D$ and $\widetilde{D}$ depend on $m$. The resulting order conditions for a MPRK method up to order 3 are given in Tab. 3.3.

REMARK 3.1. *For $m = 1$ the order conditions for PRK schemes coincide with the order conditions of Tab. 3.3 marked with* PRK. *The order conditions for MROW methods form a subset of these order conditions. Only for order $p \geq 3$ additional coupling conditions have to be fulfilled for MPRK methods (marked* M*) in Tab. 3.3.*

Now we construct embedded MPRK methods of order (2)3 based on the simplifying assumptions (3.1).

LEMMA 3.1 (MPRK(2)3). *There exist MPRK methods of order $p = 3$ with stage number $s := s_L = s_A = 4$. For step size control a method of order $p = 2$ with stage number $\hat{s} := \hat{s}_L = \hat{s}_A = 3$ is embedded. The parameters $\widetilde{\gamma}, \widetilde{\beta}_{21}, \widetilde{\beta}_{43}, \widetilde{\alpha}_{21}, \widetilde{\alpha}_3, \widetilde{\alpha}_{41}, \widetilde{\alpha}_{42}, \widetilde{\alpha}_{43}, b_1, \delta_{42}, \widetilde{\delta}_{42}, \eta_3$ and $\eta_4$ are free. Here we have used the abbreviation $\alpha_i := e_i^\top A\mathbf{1}_A$.*

REMARK 3.2. *Without embedding technique an MPRK method of order 3 can be achieved with stages $s_L = s_A = 3$.*

PROOF. The order conditions for the method of order 3 are given by

$$
\begin{aligned}
&(3.2a) & b_1 + b_2 + b_3 + b_4 &= 1 \\
&(3.2b) & b_2\widetilde{\beta}_{21} + b_3(\widetilde{\beta}_{31} + \widetilde{\beta}_{32}) + b_4(\widetilde{\beta}_{41} + \widetilde{\beta}_{42} + \widetilde{\beta}_{43}) &= \frac{1}{2} - \widetilde{\gamma} =: p_1 \\
&(3.2c) & b_2\widetilde{\alpha}_2 + b_3\widetilde{\alpha}_3 + b_4\widetilde{\alpha}_4 &= \frac{1}{2} \\
&(3.2d) & b_3\widetilde{\beta}_{32}\widetilde{\beta}_{21} + b_4(\widetilde{\beta}_{42}\widetilde{\beta}_{21} + \widetilde{\beta}_{43}(\widetilde{\beta}_{32} + \widetilde{\beta}_{31})) &= \frac{1}{6} - \widetilde{\gamma} + \widetilde{\gamma}^2 =: p_3 \\
&(3.2e) & b_3\widetilde{\beta}_{32}\alpha_2 + b_4(\widetilde{\beta}_{42}\alpha_2 + \widetilde{\beta}_{43}\alpha_3) &= \frac{1}{6} - \frac{\widetilde{\gamma}}{2} =: p_2 \\
&(3.2f) & b_3\alpha_{32}\alpha_2 + b_4(\alpha_{42}\alpha_2 + \alpha_{43}\alpha_3) &= \frac{1}{6} \\
&(3.2g) & b_2\alpha_2^2 + b_3\alpha_3^2 + b_4\alpha_4^2 &= \frac{1}{3} \\
&(3.2h) & b_3\widetilde{\delta}_{32}\alpha_2 + b_4(\widetilde{\delta}_{42}\alpha_2 + \widetilde{\delta}_{43}\alpha_3) &= \frac{m}{6} \\
&(3.2i) & b_3\delta_{32}\alpha_2 + b_4(\delta_{42}\alpha_2 + \delta_{43}\alpha_3) &= \frac{1}{6m} \\
&(3.2j) & b_3\delta_{32}\widetilde{\beta}_2 + b_4(\delta_{42}\widetilde{\beta}_{21} + \delta_{43}(\widetilde{\beta}_{31} + \widetilde{\beta}_{32}) &= \frac{1}{6m} - \frac{\widetilde{\gamma}}{2} =: p_{2m} \\
&(3.2k) & \eta_2\widetilde{\alpha}_2 + \eta_3\widetilde{\alpha}_3 + \eta_4\widetilde{\alpha}_4 &= \frac{\lambda(\lambda+1)}{2m}
\end{aligned}
$$

The embedded method of order 2 has only to fulfill

$$
\begin{aligned}
&(3.2l) & \hat{b}_1 + \hat{b}_2 + \hat{b}_3 &= 1 \\
&(3.2m) & \hat{b}_2\widetilde{\beta}_{21} + \hat{b}_3(\widetilde{\beta}_{32} + \widetilde{\beta}_{31}) &= p_1 \\
&(3.2n) & \hat{b}_2\alpha_2 + \hat{b}_3\alpha_3 &= \frac{1}{2}
\end{aligned}
$$

Since $s_L > p$, the E-polynomial depends only on $\widetilde{\gamma}$, if

$$
(3.2o) \qquad b_{l4}\widetilde{\beta}_{43}\widetilde{\beta}_{32}\widetilde{\beta}_{21} = \frac{1}{24} - \frac{1}{2}\widetilde{\gamma} + \frac{3}{2}\widetilde{\gamma}^2 - \widetilde{\gamma}^3 := p_8
$$

The linear systems (3.2a,3.2c,3.2g) and (3.2l,3.2m,3.2n) define the weights $b_2, b_3, b_4$ and $\hat{b}_1, \hat{b}_2, \hat{b}_3$ uniquely, resp. Then $\widetilde{\beta}_{32}$, $\alpha_{32}$ and $\alpha_{31} = \alpha_3 - \alpha_{32}$, $\widetilde{\beta}_{42}$, $\widetilde{\beta}_{31}$ and $\widetilde{\beta}_{41}$ are fixed by equation (3.2o), (3.2f), (3.2e), (3.2d) and (3.2b). Now all coefficients of an PRK method are defined.

The coupling coefficients are derived as follows: equation (3.2h) defines $\widetilde{\delta}_{32}$. $\delta_{32}$ and $\delta_{43}$ are given by the linear system (3.2i,3.2j). The remaining $\delta$-coefficients are fixed by the simplifying assumptions (3.1b,3.1c). Finally, $\eta_1$ and $\eta_2$ are given by (3.2k) and (3.1a). □

Table 3.4: Coefficients for MPRK(2)3

| | |
|---|---|
| $\gamma = 0.395$ | |
| $b_1 = 0,$ | $b_2 = 2/3,$ |
| $b_3 = -1/3,$ | $b_4 = 2/3,$ |
| $\hat{b}_1 = \frac{3/10-\gamma}{15p_3-6p_2} - 1,$ | $\hat{b}_2 = 2(1 - \frac{3/10-\gamma}{15p_3-6p_2}),$ |
| $\hat{b}_3 = \frac{3/10-\gamma}{15p_3-6p_2}$ | |
| $\alpha_{21} = \alpha_{31} = \alpha_{32} = 1/4,$ | $\alpha_{41} = -15/80,$ |
| $\alpha_{42} = 3/4,$ | $\alpha_{43} = 15/80,$ |
| $\beta_{21} = \beta_{43} = 1/10,$ | $\beta_{31} = 15p_3 - 150p_8 - 6p_2 + 1/5,$ |
| $\beta_{32} = 150p_8,$ | $\beta_{41} = 3(p_1 - 6p_2 + 5p_3 - 50p_8 + 2/30)/2,$ |
| $\beta_{42} = 6p_2 + 75p_8 - 1/5$ | |
| $\delta_{21} = 1/4,$ | $\delta_{31} = 1/2 + \frac{2}{m}(1 - \frac{mp_{2m}-1/15}{5p_3-2p_2}),$ |
| $\delta_{32} = -\frac{2}{m}(1 - \frac{mp_{2m}-1/15}{5p_3-2p_2}),$ | $\delta_{41} = 3/4 - \frac{mp_{2m}-1/15}{m(10p_3-4p_2)}$ |
| $\delta_{42} = 0,$ | $\delta_{43} = \frac{mp_{2m}-1/15}{m(10p_3-4p_2)}$ |
| $\widetilde{\delta}_{21} = 1/4,$ | $\widetilde{\delta}_{31} = 1/2 + 2m,$ |
| $\widetilde{\delta}_{32} = -2m,$ | $\widetilde{\delta}_{41} = 3/4$ |
| $\widetilde{\delta}_{42} = \widetilde{\delta}_{43} = 0$ | |
| $\eta_1 = \lambda - 2\lambda(\lambda+1)/m,$ | $\eta_2 = 2\lambda(\lambda+1)/m,$ |
| $\eta_3 = \eta_4 = 0$ | |

A coefficient set for MPRK(2)3 is given in Tab. 3.4. The latent components are computed by an A-stable method, if $0.39434 \leq \widetilde{\gamma} \leq 1.28057$ holds.

REMARK 3.3. *An A-stable MPRK(2)3 method with stage number $s_L = 3$ is not possible, since A-stable MROW(2)3 schemes with only three stages do not exist [4]. Remember that the order conditions for MROW methods are a subset of the ones for PRK methods.*

REMARK 3.4. *As for PRK [8] and MROW [4] methods, the A-stability of MPRK methods has been proved for an MPRK1 scheme applied to the $2 \times 2$ test problem $\dot{y} = Ay, A \in \mathbb{R}^{2\times 2}$ with diagonal entries $a_{11} \ll 0, a_{22} > 0$ and a small coupling given by $|a_{12}|, |a_{21}| \ll 1$. The inherent extrapolation in (2.2) does not destroy the stability properties of the ROW scheme. The stiff parameter $a_{11}$ enters the approximative MPRK solution only via rational expressions, which leads to the correct asymptotical behavior of the approximative solution independent of the used step size.*

## 4 Test examples.

As a test example for MPRK(2)3 we used the inverter chain with models and parameters described in [7]. A MOSFET inverter transforms a given signal form the logical value 0 to 1 and vice versa. If the number of inverters is even, a series of coupled inverters, the inverter chain, delays the runtime of a signal. A window of active elements passes through the circuit in time. The inverter chain reflects very nicely the unidirectional signal flow that is typical for MOS structures. It has to be remarked that the coupling from the left to the right involves not only latent, but also active components.

Due to its regular behavior and easy scalability, the inverter chain serves as a suitable test set for multirate and partitioning strategies [4, 5]. The solution structure is very regular, so one can easily check the precision of the numerical simulation. The size of the circuit can be easily enlarged. The circuit contains a small time dependent active part of about 20 inverters, independent of the circuit size, and a large latent part.

In addition to the rather mildly stiff model used in [7], we enlarged the drain-source current by a factor of $K = 100$ which corresponds to increasing the width to length ratio of the MOS structure by a factor of 100. This change has two effects. First, the maximum clock rate is increased, and correspondingly the voltage courses become much stepper. The activity level of the active part is drastically increased. Second, the eigenvalues of the Jacobian corresponding to the latent part are shifted to the left. Hence the stiffness of the latent part is increased by a factor of order $K$. Summing up: In comparison with the case $K = 1$, the active part is much more active, and the system much more stiff.

The proposed method MPRK(2)3 was tested in a MATLAB5 environment and compared with the built-in ROW-type scheme `ode23s` of the same order. As test examples we used inverter chains with 100 and 500 inverters for both $K = 1$ and $K = 100$. In all cases, the integration interval covered the time the signal passed through the chain. The stiffness of our test problems with $K = 100$ is verified by results obtained with the built-in explicit Runge-Kutta type integrator `ode23`. This method failed after more than 20 000 steps with an out-of-memory message.

For MPRK(2)3 we used a partitioning depending on the activity level of each inverter. The maximum macro step size $h_{\max}$ is $50\,\mathrm{ps}/114.5\,\mathrm{ps}$ for $K = 100$ and $K = 1$, resp, according to the steepness of signal courses. This guarantees that any signal change is detected during integration.

The test results are listed in Tab. 4.1 with the following abbreviations used:

- NSTEP is the number of steps and macro steps, resp., for `ode23s` and MPRK(2)3, resp.
- NSTEPREJ is the number of rejected steps and rejected macro steps, resp., for `ode23s` and MPRK(2)3, resp.
- NSTEPMIC is the number of micro steps scaled with the ratio of active to all elements (only for MPRK(2)3).

Table 4.1: Test results on MPRK(2)3 and `ode23s` for the inverter chain with $K = 1/100$ and 100/500 inverters

| | $K=1$ | | | | $K=100$ | | | |
|---|---|---|---|---|---|---|---|---|
| | MPRK(2)3 | | `ode23s` | | MPRK(2)3 | | `ode23s` | |
| | 100 | 500 | 100 | 500 | 100 | 500 | 100 | 500 |
| NSTEP | 80 | 286 | 212 | 910 | 84 | 285 | 766 | 2667 |
| NSTEPREJ | 16 | 16 | 4 | 104 | 5 | 6 | 237 | 845 |
| NSTEPMIC | 148 | 151 | — | — | 1259 | 1244 | — | — |

The usual number of micro steps per macro step was $m = 8$ for $K = 1$, and $m = 124$ for $K = 100$, with 20 active inverters in each macrostep. Hence NSTEPMIC, measuring the computation costs for the active part, does not increase with the number of inverters $N$, and can be neglected for very large values of $N$.

The ratio between $h_{\max}$ in MPRK(2)3 and the average stepsize in `ode23s` is approx. 3.5/10.5 for $K = 1/100$. This defines the limit for the theoretical gain by MPRK(2)3. Neglecting rejected steps, the corresponding values in Tab.4.1 (2.7/3.2 for $K = 1$, and 9.1/9.4 for $K = 100$) show that MPRK(2)3 is reaching this limit.

The maximum error of solution components at the end of the integration intervals is lower than $10^{-5}$ for both methods. The results show that the partitioning used for MPRK(2)3 works very reliable. The small number of rejected macro steps is mainly caused by the unsteadiness of the initial phase. All macro steps can be executed with the maximum macro step size. The high rate of rejected steps in `ode23s` is caused by the discontinuities in the right-hand side introduced by the transistor switches. These discontinuities can only be detected in `ode23s` by a failure of the step size prediction.

Experiments with higher stiffness parameters $K$ show that even slightly active components show a stiff behavior which can cause stability problems for MPRK methods: if slightly active components are treated as active to maximize the possible macro step size for the latent part, stiffness is introduced into the active part. However, this may force a step size limitation due to stability reasons for the explicit method depending on its stability region. On the other hand, if these components are treated as latent too early in order to overcome this problem, some dynamic behavior may be introduced into the latent part. Hence the macro step size may be limited too much.

Summing up, MPRK methods are an option for simulating numerically coupled systems of ODEs, provided that the equations *active = non stiff* and *latent = stiff* are valid, and the partitioning approach can be based on that.

## 5 Conclusion.

Multirate Partitioned Runge-Kutta methods are a natural generalization of both partitioned Runge-Kutta [8] and multirate Rosenbrock-Wanner methods [4] designed for the efficient numerical integration of coupled systems. Based on ideas developed in [7], internal stages are used for coupling the integrations schemes for the subsystems. No information derived by extrapolation/interpolation has to be used with all its overhead if implemented in existing simulation packages. The numerical results for the inverter chain have shown the effectiveness of MPRK methods, provided that the stiffness of the overall problem is not extremely large. In this case, one may overcome this problem by using an implicit scheme also for the active part at the price of increasing computational costs.

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