STABILITY OF MULTIRATE RUNGE-KUTTA SCHEMES

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Abstract: Some systems of ordinary differential equations can be split into subsystems, working on different timescales. Multirate methods use different stepsizes for each of those subsystems. In this paper, two multirate strategies based on Runge-Kutta schemes are presented, and a procedure for doing linear stability analysis of such methods is outlined.
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1. Introduction

Some systems of ordinary differential equations (ODEs) may be partitioned into rapidly and slowly varying subsystems. Multirate methods try to reduce integration time by integrating the slow subsystem with a larger steplength than the fast subsystem.

We consider a system of ODEs, for which the solution $y(t)$ may be split into active (or fast) components, $y_A(t)$, and latent (or slow) components $y_L(t)$, thus

$$y(t) = \begin{pmatrix} y_A(t) \\ y_L(t) \end{pmatrix}, \quad y_A(t) \in \mathbb{R}^{n_A}, \quad y_L(t) \in \mathbb{R}^{n_L}, \quad n_A + n_L = n.$$

The ODE under consideration can then be written as a split system

$$\begin{aligned} \dot{y}_A(t) &= f_A(y_A(t), y_L(t)), \qquad y_A(t_0) = y_{A,0}, \\ \dot{y}_L(t) &= f_L(y_A(t), y_L(t)), \qquad y_L(t_0) = y_{L,0}, \end{aligned} \tag{1}$$

which for simplicity, but without loss of generality, is assumed autonomous. Our purpose in partitioning is to use as large stepsize as possible for each group of components, to gain efficiency. Andrus [1, 2] describes multirate algorithms based on Runge-Kutta formulas. Basic work for multistep methods can be found in Gear and Wells [4] and Skelboe and Andersen [10, 11]. In Günther and Rentrop [5, 6] a multirate Rosenbrock-Wanner method is discussed, where the coupling between fast and slow components is realized by interpolating and extrapolating state variables. Kværnø and Rentrop [7] have recently presented a technique avoiding extrapolation of state variables, thus creating a real one-step method.

Stability properties of various multirate schemes have been discussed in [3, 4, 5, 9, 10, 11]. However, most of these discussions are not very detailed, nor very conclusive. In particular, the coupling between the systems is assumed "sufficiently weak". In section 3, a strategy for doing linear stability analysis of one-step multirate schemes is presented. Emphasis has been laid on the coupling between the systems. The analysis is applied to a rather simple scheme, but can easily be adapted to more complicated schemes as well.

2. Multirate Runge-Kutta Methods.

Consider the split system (1). The active/fast components y_A are integrated with the small stepsize h, the latent/slow components y_L are integrated with the large stepsize H. The methods used for the integration can, but do not have to, be the same. A synchronization of the *microsteps* h and the *macrosteps* H is performed at the end of each macrostep. The number of microsteps within a macrostep is m, thus $H = mh$. The definition of a multirate Runge-Kutta method — MRK method — for the numerical solution of (1) becomes:

- The active components y_A are defined by

$$k_{A,i}^{\lambda} = f_A(y_{A,\lambda} + h\sum_{j=1}^{s} a_{ij}\, k_{A,j}^{\lambda},\ \tilde{Y}_{L,i}^{\lambda}), \quad i = 1, 2, \dots, s, \tag{2.a}$$

$$y_{A,\lambda+1} = y_{A,\lambda} + h\sum_{i=1}^{s} b_i\, k_{A,i}^{\lambda} \ \approx\ y(t_0 + (\lambda+1)h), \tag{2.b}$$

  for $\lambda = 0, 1, 2, \dots, m-1$. Here, $c_i = \sum_{j=1}^{s} a_{ij}$ and $\tilde{Y}_{L,i}^{\lambda} \approx y_L(t_0+(\lambda+c_i)h)$.

- The latent components y_L are defined by

$$k_{L,i} = f_L(\tilde{Y}_{A,i},\ y_{L,0} + H\sum_{j=1}^{\bar{s}} \bar{a}_{ij}\, k_{L,j}), \qquad j = 1, 2, \dots, \bar{s}, \tag{2.c}$$

$$y_{L,1} = y_{L,0} + H\sum_{i=1}^{\bar{s}} \bar{b}_i\, k_{L,i} \ \approx\ y_L(t_0 + H), \tag{2.d}$$

where $\bar{c}_i = \sum_{j=1}^{\bar{s}} \bar{a}_{ij}$ and $\tilde{Y}_{A,i} \approx y_A(t_0 + \bar{c}_i\, H)$.

The coefficients $(a_{ij})_{i,j=1}^{s}$, $(b_i)_{i=1}^{s}$, $(\bar{a}_{ij})_{i,j=1}^{\bar{s}}$ and $(\bar{b}_i)_{i=1}^{\bar{s}}$ are coefficients defining the basic methods for the active and the latent methods respectively. The coupling from the latent to the active subsystem and vice versa are performed by the intermediate stage values $\tilde{Y}_{A,i}$ and $\tilde{Y}_{L,i}^{\lambda}$. Traditionally, these are computed by extrapolation/interpolation techniques, based on information from the preceding steps. Although a natural choice, this inevitably turns the Runge-Kutta method into a two-step method. Kværnø and Rentrop [7] constructed a genuine one (macro) step method by computing the coupling stage values by

$$\tilde{Y}_{L,i}^{\lambda} = y_{L,0} + h \sum_{j=1}^{\bar{s}} (\gamma_{ij} + \eta_j(\lambda))\, k_{L,j}, \quad i = 1, 2, \ldots s, \quad \lambda = 0, 1, \ldots, m-1, \tag{2.e}$$

$$\tilde{Y}_{A,i} = y_{A,0} + H \sum_{j=1}^{s} \bar{\gamma}_{ij}\, k_{A,j}^{0}, \quad i = 1, 2 \ldots, \bar{s}. \tag{2.f}$$

Strategy (2.a-f), hereafter called MRKI, is cheap in the sense that no extra computational costs are involved. However, the $\tilde{Y}_{A,i}$-values are obtained by extrapolation from the first micro-step, and the strategy is therefore reliable only if y_A do not vary much within a macrostep, or the coupling from the active to the latent part of the system is weak. A more robust technique, although not that cost-efficient, can be obtained by computing the $\tilde{Y}_{A,i}$-values by some cheap, low order method. This strategy will in the following be denoted by MRKII. If, as we will assume, the active part of the system is nonstiff, a forward Euler method can be used. For methods of order 1 and 2 this is quite straightforward, for higher order methods some modifications are required. For details on both methods, including consistency analysis, see [7].

3. Stability of Multirate Methods

The absolute stability properties of an integration method are usually studied by applying the method to the scalar test equation $y' = \alpha y$, with $\alpha \in \mathbb{C}^{-}$. Doing so for a standard Runge-Kutta method, the solution after one step h is given by

$$y_1 = R(h\alpha) y_0$$

where the stability function $R(h\alpha)$ is some rational function. The method is stable if and only if $|R(h\alpha)| < 1$. However, multirate formulas require at least two equations. An appropriate test problem might be

$$\begin{pmatrix} \dot{y}_A \\ \dot{y}_L \end{pmatrix} = \mathcal{A} \begin{pmatrix} y_A \\ y_L \end{pmatrix}, \qquad \mathcal{A} = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \in \mathbb{R}^{2\times 2} \tag{3}$$

The assumptions

$$\alpha_{11}, \alpha_{22} < 0 \quad \text{and} \quad \gamma = \frac{\alpha_{12}\alpha_{21}}{\alpha_{11}\alpha_{22}} < 1 \tag{4}$$

ensure that both eigenvalues of $\mathcal{A}$ have negative real parts. The parameter γ can be viewed as a measure for the coupling between the equations. We also introduce $\kappa = \alpha_{22}/\alpha_{11} > 1$ as a measure for the stiffness of the system.

A compound step of one of the multirate methods suggested in the preceding section applied to (3) can be expressed by

$$\begin{pmatrix} y_{A,m} \\ y_{L,1} \end{pmatrix} = \mathcal{K} \begin{pmatrix} y_{A,0} \\ y_{L,0} \end{pmatrix}. \tag{5}$$

where the matrix $\mathcal{K}$ depends on the stepsizes H and h, and the coefficients of the method. Obviously, the method is asymptotically stable if and only if the eigenvalues of $\mathcal{K}$ are both within the unit disk.

The scenario we are going to study is as follows: Two methods have been chosen, one for the active and one for the latent part. Stepsizes H and h are chosen to ensure that the uncoupled system ($\alpha_{12} = \alpha_{21} = 0$) is stable, thus the stability functions $R_A(h\alpha_{11})$ and $R_L(H\alpha_{22})$ of the active and the latent part respectively are both of absolute value less than one. One interesting question is how the coupling between the two systems affects the stability. It is straightforward to show that as long as either α_{12} or α_{21} is zero, the eigenvalues of $\mathcal{K}$ becomes R_L and R_A^m. It seems convenient for our purpose to express the matrix $\mathcal{K}$ in terms of R_A and R_L rather than the stepsizes H and h. Further, R_A and R_L are related through m, the number of microsteps within the macrostep. Thus, we can study the stability region, that is, the values of R_A and γ for which eigenvalues of $\mathcal{K}$ is within the unit disk, and how this region varies with increasing m and stiffness ratio κ. A similar approach has been used by Owren [8] to show stability in modular integration techniques.

To demonstrate the idea, we will use a simple scheme composed by a forward Euler method for the active part, and a backward Euler method for the latent part. For the uncoupled systems, that is

$$\begin{aligned} y_{A,\lambda+1} &= y_{A,\lambda} + h\alpha_{11}y_{A,\lambda}, \qquad\qquad \lambda = 0, 1, \ldots, m-1 \\ y_{L,1} &= y_{L,0} + H\alpha_{22}y_{L,1}. \end{aligned}$$

we have $R_A = 1 + h\alpha_{11}$ and $R_L = 1/(1 - H\alpha_{22})$. Now, $|R_A| < 1$ if and only if $-2 < \alpha_{11}h < 0$, while $0 < R_L < 1$ since $\alpha_{22} < 0$. R_A and R_L is related through $H = m \cdot h$, giving

$$R_L = \frac{1}{1 + m(1 - R_A)\kappa}, \qquad \kappa = \frac{\alpha_{22}}{\alpha_{11}}$$

Applying the forward Euler method to the active part, and the backward Euler methods to the latent part gives

$$k_A^\lambda = \alpha_{11} y_{A,\lambda} + \alpha_{12}\tilde{Y}_L^\lambda, \qquad y_{A,\lambda+1} = y_{A,\lambda} + hk_A^\lambda, \quad \lambda = 0,1,\dots,m-1,$$
$$k_L = \alpha_{21}\tilde{Y}_A + \alpha_{22}(y_{L,0} + Hk_L), \qquad y_{L,1} = y_{L,0} + Hk_L.$$

Strictly speaking, for an order one method we could use $\tilde{y}_L^\lambda = y_{L,0}$ and $\tilde{Y}_A = y_{A,0}$. This simple choice is however not very representative for the methods outlined in section 2. Thus we rather choose $\tilde{Y}_L^\lambda = y_{L,0} + h\lambda k_L$, and then one macro-step can be expressed by

$$\begin{aligned} y_{A,m} &= R_A^m y_{A,0} + \big(-(1-R_A^m)\frac{\alpha_{12}}{\alpha_{11}} + C_1 R_L \frac{\alpha_{12}\alpha_{22}}{\alpha_{11}^2}\big) y_{L,0} + C_1 R_L \frac{\alpha_{12}\alpha_{21}}{\alpha_{11}^2}\tilde{Y}_A, \\ y_{L,1} &= R_L y_{L,0} - (1-R_L)\frac{\alpha_{21}}{\alpha_{22}}\tilde{Y}_A, \end{aligned} \tag{6}$$

where $C_1 = m(1-R_A) - (1-R_A^m)$ and $\tilde{Y}_A$ depends on the strategy, as explained below.

MRKI: In this case, we use $\tilde{Y}_A = y_{A,0} + Hk_{A,0}$, which gives

$$\begin{aligned} \tilde{Y}_A &= y_{A,0} + H(\alpha_{11} y_{A,0} + \alpha_{12} y_{L,0}) \\ &= \big(1 - \frac{1-R_L}{R_L}\frac{\alpha_{11}}{\alpha_{22}}\big) y_{A,0} - \frac{1-R_L}{R_L}\frac{\alpha_{12}}{\alpha_{22}} y_{L,0} \end{aligned}$$

Inserting this into (6) gives the elements of the matrix $\mathcal{K}$:

$$\begin{aligned} K_{11} &= R_A^m + C_1\frac{\alpha_{12}\alpha_{21}}{\alpha_{11}^2}\big(R_L - (1-R_L)\frac{\alpha_{11}}{\alpha_{22}}\big) \\ K_{12} &= -(1-R_A^m)\frac{\alpha_{12}}{\alpha_{11}} + C_1\frac{\alpha_{12}\alpha_{22}}{\alpha_{11}^2}\big(R_L - (1-R_L)\frac{\alpha_{21}\alpha_{12}}{\alpha_{22}^2}\big) \\ K_{21} &= -(1-R_L)\frac{\alpha_{21}}{\alpha_{22}}\big(1 - \frac{1-R_L}{R_L}\frac{\alpha_{11}}{\alpha_{22}}\big) \\ K_{22} &= R_L + \frac{(1-R_L)^2}{R_L}\frac{\alpha_{21}\alpha_{12}}{\alpha_{22}^2} \end{aligned}$$

Next, we consider the conditions for which the spectral radius of the matrix $\mathcal{K}$ is less than one. To this end, we compute the characteristic polynomial of the matrix, and then apply the Routh-Hurwitz criterions. The eigenvalues of $\mathcal{K}$ turns out to be less than one if and only if the conditions

$$\begin{aligned} p_0 &= R_L(1-R_L)(1-R_A^m)(1-\gamma) > 0 \\ p_1 &= \kappa R_L(1-R_A^m R_L) \\ &\quad + \gamma\big(-C_1 R_L^2\kappa^2 + mR_L(1-R_A)(1-R_L)\kappa - (1-R_L)^2\big) > 0 \\ p_2 &= \kappa R_L(1+R_L)(1+R_A^m) \\ &\quad + \gamma\big(2C_1 R_L^2\kappa^2 - R_L(1-R_L)(1-R_A^m+2C_1)\kappa + 2(1-R_L)^2\big) > 0 \end{aligned}$$

are satisfied.

MRKII: In this case, $\tilde{Y}_A$ is computed by taking m forward Euler steps, thus

$$\begin{aligned} u_0 &= y_{A,0} \\ u_{\lambda+1} &= u_\lambda + h(\alpha_{11}u_\lambda + \alpha_{12}(y_{L,0} + h\lambda k_{L,0})), \quad \lambda = 0,1,\ldots,m-1 \\ \tilde{Y}_A &= u_m \end{aligned}$$

using $k_{L,0} = \alpha_{21}y_{A,0} + \alpha_{22}y_{L,0}$. This will eventually lead us to

$$\tilde{Y}_A = \left(R_A^m + C_1\frac{\alpha_{12}\alpha_{21}}{\alpha_{11}^2}\right)y_{A,0} + \left(-(1-R_A^m)\frac{\alpha_{12}}{\alpha_{11}} + C_1\frac{\alpha_{12}\alpha_{22}}{\alpha_{11}^2}\right)y_{L,0}$$

Inserting this into (6) gives the elements of $\mathcal{K}$:

$$\begin{aligned} K_{11} &= R_A^m + C_1\frac{\alpha_{12}\alpha_{21}}{\alpha_{11}^2}R_l\left(R_A^m + C_1\frac{\alpha_{12}\alpha_{21}}{\alpha_{11}^2}\right) \\ K_{12} &= -(1-R_A^m)\frac{\alpha_{12}}{\alpha_{11}}\left(1 + C_1R_L\frac{\alpha_{12}\alpha_{21}}{\alpha_{11}^2}\right) + C_1R_L\frac{\alpha_{12}\alpha_{22}}{\alpha_{11}^2}\left(1 + C_1\frac{\alpha_{12}\alpha_{21}}{\alpha_{11}^2}\right) \\ K_{21} &= -(1-R_L)\frac{\alpha_{21}}{\alpha_{22}}\left(R_A^m + C_1\frac{\alpha_{12}\alpha_{21}}{\alpha_{11}^2}\right) \\ K_{22} &= R_L - (1-R_L)\frac{\alpha_{21}\alpha_{12}}{\alpha_{11}\alpha_{22}}\left((1-R_A^m) + C_1\frac{\alpha_{22}}{\alpha_{11}}\right) \end{aligned}$$

and the eigenvalues of $\mathcal{K}$ is within the unit disk if and only if the following conditions are satisfied:

$$\begin{aligned} p_0 &= (1-R_L)(1-R_A^m)(1-\gamma)(1+C_1\gamma\kappa) > 0 \\ p_1 &= 1 - R_A^mR_l + \gamma\kappa R_A^mC_1(1-2R_L) \\ &\quad + \gamma^2\kappa C_1\left(-\kappa R_LC_1 + (1-R_A^m)(1-R_L)\right) > 0 \\ p_2 &= (1+R_A^m)(1+R_L) \\ &\quad - \gamma\left((1-R_A^m)(1-R_L) + \kappa C_1\left((1+R_A^m)(1-R_L) - 2R_A^mR_L\right)\right) \\ &\quad + \gamma^2\kappa C_1\left(2\kappa R_LC_1 - (1-R_A^m)(1-R_L)\right) > 0 \end{aligned}$$

The stability regions for different values of m and κ is given in Figure 1. Somewhat surprisingly, MRKI has a larger stability region than MRKII. Moreover, MRKI is stable for almost all $\gamma > -1/3$, independent of the stiffness ratio κ and R_A, which means that a rather strong coupling is feasible. To obtain a stable scheme for MRKII, either the coupling has to be quite weak, or the microsteps have to be small, so that R_A is sufficiently close to 1. The stability domain becomes smaller with increasing m and increasing κ. Thus, MRKI are able to handle quite stiff, strongly coupled systems, where MRKII might have stability problems. However, from these examples, it is too early to conclude that a general MRKI (2.a-f) is more stable than a general MRKII. But we do believe that the ideas presented here can be used to illuminate stability properties of more realistic schemes.

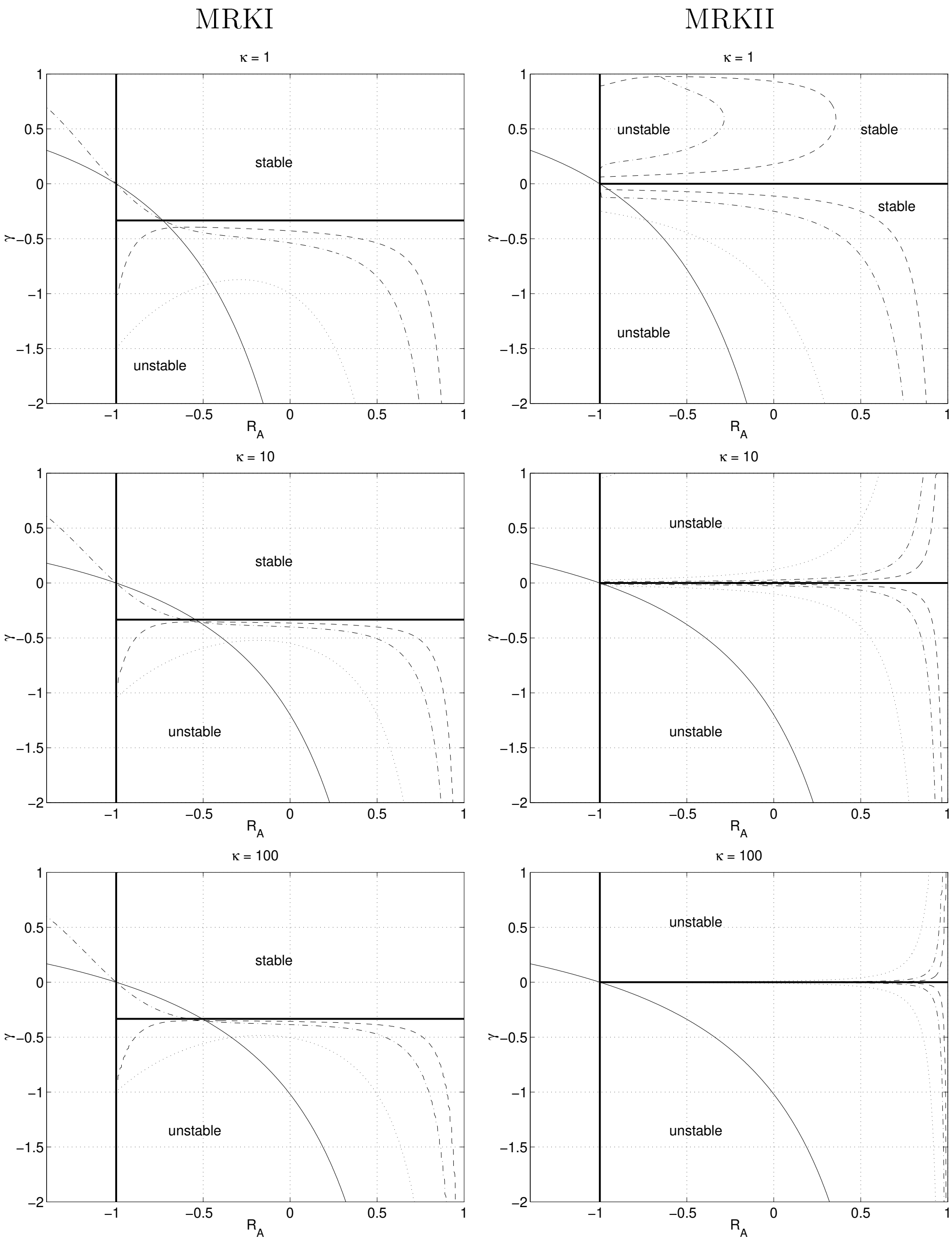

Figure 1: Boundaries of the stability regions for $m = 1$ (—), $m = 2$ ($\cdots$), $m = 5$ ($- \cdot - p\cdot$), $m = 10$ ($--$) and $m \to \infty$ (**—**). The methods are stable to the right of the boundaries, unstable to the left.

References

[1] J. F. Andrus. Numerical solution of systems of ordinary differential equations separated into subsystems. *SIAM J. Numer. Anal.*, **16** (1979), 605–611.

[2] J. F. Andrus. A Runge-Kutta method with stepsize control for separated systems of first-order ODEs. *Appl. Math. Comput.*, **59** (1993), 193–214.

[3] J. F. Andrus. Stability of a multi-rate method for numerical integration of ODE's. *Computers Math. Applic.*, **25** (1993), 3–14.

[4] C. W. Gear and D. R. Wells. Multirate linear multistep methods. *BIT*, **24** (1984), 484–502.

[5] M. Günther and P. Rentrop. Multirate ROW methods and latency of electric circuits. *Appl. Numer. Math*, **13** (1993), 83–102.

[6] M. Günther and P. Rentrop. Partitioning and multirate strategies in latent electric circuits. *International series of Numerical Mathematics*, **117** (1994), 33–60.

[7] A. Kværnø and P. Rentrop. Low order multirate Runge-Kutta methods in electric circuit simulation. Submitted to BIT.

[8] B. Owren. Stability of Runge–Kutta methods used in modular integraion. *BIT*, **62** (1995), 80–101.

[9] J. Sand and S. Skelboe. Stability of backward Euler multirate methods and convergence of waveform relaxation. *BIT*, **32** (1992), 350 – 366.

[10] S. Skelboe. Stability properties of backward differentiation multirate formulas. *Appl. Numer. Math.*, **5** (1989), 151–160.

[11] S. Skelboe and P.U. Andersen. Stability properties of backward Euler multirate formulas. *SIAM J. Sci. Stat. Comput.*, **10** (1989), 1000–1009.