# Iterated Commutators, Lie's Reduction Method and Ordinary Differential Equations on Matrix Lie Groups

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**Abstract.** In the context of devising geometrical integrators that retain qualitative features of the underlying solution, we present a family of numerical methods (*the method of iterated commutators,* [5, 13]) to integrate ordinary differential equations that evolve on matrix Lie groups. The schemes apply to the problem of finding a numerical approximation to the solution of

$$Y' = A(t, Y)Y, \qquad Y(0) = Y_0,$$

whereby the exact solution $Y$ evolves in a matrix Lie group G and $A$ is a matrix function on the associated Lie algebra $\mathfrak{g}$. We show that the method of iterated commutators, in a linear setting, is intrinsically related to Lie's reduction method for finding the fundamental solution of the Lie-group equation $Y' = A(t)Y$.

## 1 Introduction

In many applications of practical (and theoretical) interest, a given problem is often reduced to the solution of a differential equation or a family of differential equations evolving on some prescribed manifold. In the framework of devising structural discretization methods for Ordinary Differential Equations (ODEs) on manifolds (cf. [6, 7] for a discussion on such matters), we describe *the method of iterated commutators* for those manifolds that posses an underlying Lie-group structure. Such manifolds are interesting *per sé,* as for instance the *orthogonal group,* the *unitary group* etc., but also because, once we have Lie-group invariant discretization methods, we can solve ODEs on *homogeneous spaces* whilst retaining the mathematical structure of the underlying solution [11]. For the sake of completeness, we mention that other Lie-group invariant schemes are also available. To our knowledge, the first such methods, essentially of a Runge–Kutta-type, were introduced by Crouch and Grossman [4] and a similar approach is currently followed also by Owren and Marthinsen [8], who have presented systematic analysis of order conditions. A slightly different approach is being currently pursued by Munthe-Kaas: his schemes are also of a Runge–Kutta-type, but the order analysis in made in the algebra instead of the group. Thus, his methods can be obtained as modification of classical schemes with the introduction of correction functions on the Lie algebra $\mathfrak{g}$ [9, 10].

## 2 The Method of Iterated Commutators

The method of iterated commutators was introduced by Iserles in [5] to solve linear systems with variable coefficients,

$$Y' = A(t)Y, \qquad Y(0) = Y_0, \tag{1}$$

where $Y \in \mathbb{R}^d$ and $A(t) \in \mathbb{R}^{d\times d}$ is an analytic function. Iserles proved that the solution $Y(t)$ of (1) can be obtained as the infinite product

$$Y(t) = \lim_{m\to\infty} \mathrm{e}^{B_0(t)} \cdots \mathrm{e}^{B_m(t)} Y_0, \tag{2}$$

where the matrix functions $B_i(t), i \in \mathbb{Z}^+$, are evaluated in the following iterative manner. First of all, we let $A_0(t) = A(t)$ and for $i = 0, 1, \ldots,$ we evaluate the integrals

$$B_i(t) = \int_0^t A_i(\tau)\,\mathrm{d}\tau. \tag{3}$$

Next we construct the sequence of commutators

$$C_{i,0}(t) = A_i(t), \qquad C_{i,k+1}(t) = [C_{i,k}(t), B_i(t)], \quad k \geq 0, \tag{4}$$

and, letting

$$A_{i+1}(t) = \sum_{k=1}^{\infty} \frac{k}{(k+1)!} C_{i,k}(t),$$

we set

$$B_{i+1}(t) = \int_0^t A_{i+1}(\tau)\,\mathrm{d}\tau.$$

The importance of (2) comes to light when we truncate the infinite product to a finite number $M \in \mathbb{Z}^+$ of terms,

$$Y(t) \approx \mathrm{e}^{B_0(t)} \cdots \mathrm{e}^{B_M(t)} Y_0. \tag{5}$$

If we denote by $\|\cdot\|$ the Frobenius norm on matrices, then the order of approximation of the scheme (5) is ensured by following result.

**Theorem 1 (Theorem 3, [5]).** *Let $Y^{[M]}(t) = \mathrm{e}^{B_0(t)} \cdots \mathrm{e}^{B_M(t)} Y_0$, where the matrices $B_i(t)$, $0 \leq i \leq M$, are evaluated as explained above. Then, if $A(t)$ is sufficiently smooth function of $t$, we have*

$$\|Y^{[M]}(t) - Y(t)\| = \mathcal{O}\left(t^{2^{M+2}-1}\right). \tag{6}$$

*Therefore, for every integer $M \geq 0$ the formula (5) is an approximation of order $p = 2^{M+2} - 2$.* □

The procedure that we have presented requires the construction of fairly complicated functions. In order to implement the schemes numerically, we replace the integrals $B_i(t)$ by quadrature formulas and truncate the infinite series $A_{i+1}(t)$ to a finite number of terms. As long as these quadrature formulas are of sufficiently high order and we have a sufficient number of terms in the series, the order analysis in Theorem 1 remains valid.

We will assume that the matrix exponentials are evaluated exactly. Thus, with a single exponential we can approximate up to order two, with two up to order six, etc. The order increases exponentially with the number of exponentials we use.

When $A$ commutes with its derivative, we have

$$B_1(t) = \cdots = B_M(t) = O$$

and the order of approximation depends only on the quadrature formula that we employ in approximating the integral $B_0(t)$.

The numerical implementation of the method of iterated commutators is described in [5, 13]. In [13], we prove that the method of iterated commutators for Lie-group equations (8) produces numerical approximations in the group G and that the algorithm can be implemented in a nonlinear setting.

## 3 Group Actions on Manifolds and Lie's Reduction Method

In this section we intend to show that Iserles's method of iterated commutators is in fact related to the well known method of reduction for *equations of Lie type* due to Sophus Lie. We refer the reader to [1] for theory and notation in the current section.

Let G be a Lie group and $\mathcal{M}$ a manifold. A *left action* of G on $\mathcal{M}$ is an operation $\lambda : \mathrm{G} \times \mathcal{M} \to \mathcal{M}$ such that

$$\begin{aligned} \lambda(e, m) &= m, \\ \lambda(g_1 g_2, m) &= \lambda(g_1, \lambda(g_2, m)), \qquad g_1, g_2 \in \mathrm{G}, \quad m \in \mathcal{M}, \end{aligned}$$

where $e$ denotes the identity element in G. When there is no danger of confusion, we will denote the left action by means of a dot, that is

$$\lambda(g, m) = g \cdot m, \qquad g \in \mathrm{G}, \quad m \in \mathcal{M}.$$

**Definition 2.** Let $\mathcal{M}$ be a manifold endowed with a left action $\lambda$, $\mathfrak{X}(\mathcal{M})$ the Lie algebra of vector fields on $\mathcal{M}$, $\mathfrak{g}$ the Lie algebra of the group G. If $\lambda_* : \mathfrak{g} \to \mathfrak{X}(\mathcal{M})$ is the Lie algebra homomorphism induced by the left action, then a curve $\gamma : t \in \mathbb{R} \to \mathcal{M}$ on the manifold $\mathcal{M}$ obeys the differential equation

$$\gamma'(t) = \lambda_*(A(t))(\gamma(t)) \tag{7}$$

where $A : \mathbb{R} \to \mathfrak{g}$ is a curve on the Lie algebra. The equation (7) is called an *equation of Lie type.*

All the solutions of (7) can be characterized in terms of the following theorem, due to Sophus Lie.

**Theorem 3 (Prop. 3, [1]).** *On any manifold $\mathcal{M}$ endowed with a left* G*-action, the solution of the equation of Lie type (7) with initial condition $\gamma(0) = m \in \mathcal{M}$, is given by*

$$\gamma(t) = S(t) \cdot m,$$

*where $S : \mathbb{R} \to \mathrm{G}$ is the fundamental solution of the Lie-group equation*

$$S'(t) = Y_{A(t)}(S(t)), \quad S(0) = e, \tag{8}$$

*and $Y_{A(t)}$ is the right-invariant vector field.*

In the matrix case, $Y_{A(t)}(S(t)) = A(t)S(t)$, and (8) reduces merely to the Lie-group equation

$$S'(t) = A(t)S(t), \quad S(0) = I, \tag{9}$$

where $I$ denotes the identity matrix.

Therefore, having integrators that solve differential equations on Lie groups means, at the same time, that we are able to construct $\mathcal{M}$-invariant integrators for all those manifolds which are endowed with a left G-action.

As an example, we consider the following group action of the special orthogonal group $\mathrm{SO}(d)$ on the manifold $\mathcal{M}_{[d]}$ of $d \times d$ real matrices. The action is given by

$$U \cdot L = ULU^T \qquad U \in \mathrm{SO}(d), \quad L \in \mathcal{M}_{[d]}.$$

The corresponding equation of Lie type is

$$L' = [B(L), L], \qquad L(0) = L_0, \tag{10}$$

where $B(L)$ maps $L$ into a skew-symmetric matrix. Recall that the Lie algebra of $\mathrm{SO}(d)$ is denoted by $\mathfrak{so}(d)$ and it consists of all $d \times d$ skew-symmetric matrices. The equation (10) is well known, since it characterizes *isospectral flows.* The solution $L(t)$ of (10) is similar to the initial condition $L_0$, as it can be evinced by the group action, hence both $L(t)$ and $L_0$ have the same spectrum. The numerical solution of these flows has been discussed with greater generality in [2], to which we refer the reader for further references.

The knowledge of the fundamental solution $U(t)$ that solves the Lie-group equation (8),

$$U' = B(L)U, \qquad U(0) = I, \tag{11}$$

allows us to obtain the solution to (10). We have

$$L(t) = U(t) \cdot L_0 = U(t)L_0U(t)^T.$$

In this light, the approach proposed in [2] to solve numerically (10) whilst retaining isospectrality, can be depicted as finding the fundamental solution to (11) by means of an $\mathrm{SO}(d)$-invariant numerical method. This allowed the authors to introduce *isospectral* numerical methods, whereas classical methods for ODEs fail to retain the isospectrality of (10).

### 3.1 Lie's Reduction Method

The main problem reduces to find the *fundamental solution* of (9) where $S$ evolves in a Lie group G. Assume that $S$ can be written in the form

$$S(t) = T_1(t)S_1(t), \tag{12}$$

whereby $T_1(t)$ is an arbitrary curve in G. Differentiation and substitution in (8) yields the following differential equation for $S_1$:

$$S_1' = \left(T_1^{-1}(AT_1 - T_1')\right) S_1, \tag{13}$$

together with the initial condition $S_1(0) = I$. The equation (13) is still in the Lie group G, but in particular $S_1$ evolves in a proper subgroup of G, see [1] for details.

The choice of Iserles for the arbitrary curve $T_1 \in \mathrm{G}$ corresponds to

$$T_1(t) = \exp\left(\int_0^t A(\tau)\,\mathrm{d}\tau\right) = \exp(B_0(t)), \tag{14}$$

accordingly to the notation introduced in Sect. 2. A long calculation[3] leads to the following formula for the derivative of $T_1$,

$$\frac{\mathrm{d}}{\mathrm{d}t}T_1(t) = \exp(B_0(t))\sum_{k=0}^{\infty}\frac{1}{(k+1)!}C_{0,k}(t),$$

and therefore, substitution in (13) yields

$$\begin{aligned}
S_1' &= T_1^{-1}\left(AT_1 - T_1'\right) S_1 \qquad (15)\\
&= \exp(-B_0(t))\left[A\exp(B_0(t)) - \exp(B_0(t))\sum_{k=0}^{\infty}\frac{1}{(k+1)!}C_{0,k}(t)\right] S_1\\
&= \left[\exp(-B_0(t))A\exp(B_0(t)) - \sum_{k=0}^{\infty}\frac{1}{(k+1)!}C_{0,k}(t)\right] S_1\\
&= \left[\sum_{k=0}^{\infty}\frac{1}{k!}C_{0,k}(t) - \sum_{k=0}^{\infty}\frac{1}{(k+1)!}C_{0,k}(t)\right] S_1\\
&= \left[\sum_{k=0}^{\infty}\frac{k}{(k+1)!}C_{0,k}(t)\right] S_1 .
\end{aligned}$$

Note that $\sum_{k=0}^{\infty}\frac{k}{(k+1)!}C_{0,k}(t)$ is just the function $A_1(t)$ introduced in Sect. 2. Next step consists in applying the same splitting to $S_1 = T_2S_2$, where $T_2 = \exp(B_1(t))$ and finding the differential equation for $S_2$, $S_2' = A_2(t)S_2$ and so on. The iterated procedure yields exactly the method of iterated commutators described in Sect. 2.

[3] A similar expression appears implicitly in [5]. The general formula for the derivation of the exponential can be found in [12] and in [10]

### 3.2 Another Interpretation of Lie's Reduction Method

We have seen that the main task for solving equations on Lie groups and equations on manifolds on which there is a group action is to find the fundamental solution of the system (8). For this construction we require that G is connected and simply connected.

Assume that we can find a homomorphism $\phi : \mathrm{G} \to \mathrm{H}$ to some (possibly simpler) group H with Lie algebra $\mathfrak{h}$ on which it is possible to solve the corresponding fundamental equation more easily. Then our fundamental solution can be written in the form

$$S = T_1 S_1, \tag{16}$$

where $T_1 \in \phi^{-1}(\phi(S))$ and $S_1 \in \ker \phi$. An optimal situation is for instance when H is Abelian, therefore all ordinary differential equations of H can be explicitly solved by quadrature.

In passing we mention that a similar approach was proposed in [3] whereby the authors proposed the *method of diffeomorphisms.* Such method consists of reducing, whenever possible, the integration on some difficult manifold $\mathcal{M}$ to the integration on simpler manifold $\mathcal{N}$ numerically tractable with some classical scheme for ODEs. However, in this Lie-group setting we do not need the map $\phi$ to be one-to-one, since we are able to incorporate the contribution of its kernel, $\ker \phi$.

How to find the homomorphism $\phi$ and a simpler group H? Given the algebra $\mathfrak{g}$, we construct the linear space $[\mathfrak{g}, \mathfrak{g}]$ of elements that are commutators of some elements of $\mathfrak{g}$. This linear space is an *ideal* of $\mathfrak{g}$, since $[[\mathfrak{g}, \mathfrak{g}], \mathfrak{g}] \subset [\mathfrak{g}, \mathfrak{g}]$, therefore it is possible to construct the quotient algebra $\mathfrak{t} = \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$. The elements of $\mathfrak{t}$ are equivalence classes $\langle u \rangle = u + [\mathfrak{g}, \mathfrak{g}]$, and each coset consists of those elements of $\mathfrak{g}$ of the form $u + [v, w]$ for some $v, w \in \mathfrak{g}$. If $\varphi_0 : \mathfrak{g} \to \mathfrak{t}$ denotes the canonical quotient mapping,

$$\varphi_0(v) = \langle v \rangle = v + [\mathfrak{g}, \mathfrak{g}],$$

then there exists a unique Lie-group homomorphism (cf. [1]), $\phi_0 : \mathrm{G} \to \mathfrak{t}$, such that the induced Lie algebra homomorphism coincides with $\varphi_0$. The relation between $\phi_0$ and $\varphi_0$ is that $\varphi_0 = \phi_0'(I)$. Thus, $\mathrm{H} = \mathfrak{t}$. In fact, $\mathfrak{t}$ might be regarded as a group, and, being a vector space, its Lie algebra can be identified with $\mathfrak{t}$ itself. The advantage of this intricate construction is that $\mathfrak{t}$ is Abelian! For simplicity, we denote $\bar{\mathfrak{g}} = [\mathfrak{g}, \mathfrak{g}]$ and let $\langle u \rangle, \langle v \rangle \in \mathfrak{t}$. Then $\langle u \rangle = u + \bar{\mathfrak{g}}$ and $\langle v \rangle = v + \bar{\mathfrak{g}}$. Thus,

$$\begin{aligned}[\langle u \rangle, \langle v \rangle] &= [u + \bar{\mathfrak{g}}, v + \bar{\mathfrak{g}}] = [u, v] + [\bar{\mathfrak{g}}, v] + [u, \bar{\mathfrak{g}}] + [\bar{\mathfrak{g}}, \bar{\mathfrak{g}}] = \bar{\mathfrak{g}} \\ &= 0 + \bar{\mathfrak{g}} = \langle 0 \rangle,\end{aligned}$$

since $\bar{\mathfrak{g}}$ is an ideal in $\mathfrak{g}$. If we denote $\gamma = \phi_0(S) \in \mathfrak{t}$, then $\gamma$ obeyes the ordinary differential equation

$$\gamma' = R_\gamma'(\varphi_0 \circ A),$$

whereby $R_\gamma'$ denotes the right multiplication induced on the algebra $\mathfrak{t}$. Abusing notation, we rewrite the above equation as

$$\gamma' = \langle A(t) \rangle \gamma,$$

whose fundamental solution, because of commutativity, is given by

$$\gamma = \exp\left(\int_0^t \langle A(\tau)\rangle \,\mathrm{d}\tau\right) = \left\langle \exp\left(\int_0^t A(\tau)\,\mathrm{d}\tau\right)\right\rangle .$$

We let $T_1 \in \phi_0^{-1}(\gamma)$ and we are allowed to choose

$$T_1 = \exp\left(\int_0^t A(\tau)\,\mathrm{d}\tau\right),$$

then we need to find the differential equation for $S_1$ in $\ker\phi_0$, which is given exactly by (16). Letting $\mathrm{G}_1 = \ker\phi_0$, its Lie algebra being denoted by $\mathfrak{g}_1$, the same procedure is repeated in an iterative fashion, to generate a sequence $\mathfrak{g} \equiv \mathfrak{g}_0, \mathfrak{g}_1, \mathfrak{g}_2 \ldots$, obeying the condition $\ldots \subset \mathfrak{g}_2 \subset \mathfrak{g}_1 \subset \mathfrak{g}_0$. If after some iteration $k$ we have $\mathfrak{g}_k = \{0\}$, then the procedure terminates, the solution of (8) can be found exactly in a finite number of steps and the group is said to be *solvable.* In such cases, the method of iterated commutators, if the numerical approximation is carried with sufficient accuracy, samples the exact solution and the matrices $B_k(t)$ from a certain index $k$ onwards are zero matrices.

## 4 Conclusions

We have evidenced that the method of iterated commutators for matrix ODEs can be viewed as a numerical implementation of the Lie's reduction method, which is genererally used to solve some special ODEs by quadrature. The method of iterated commutators, however, comes together with an order analysis, therefore it is possible to decide the order of accuracy for each scheme. Some numerical methods from this family are presented in [5] and in [13], where we consider application to a nonlinear case and establish that the order analysis is still valid, assuming sufficient regularity of the underlying problem. It is remarkable that Lie's procedure turns out to be a viable way to solve numerically ordinary differential equations.

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