MA3203 - Problem sheet 4 - 2006

Problem 1. Given $\Lambda = k\Gamma/\langle\rho\rangle$, where Γ is the quiver $1 \xrightarrow{\alpha} 2 \underset{\gamma}{\overset{\beta}{\rightrightarrows}} 3$ with relations $\rho = \{\beta\alpha\}$ and k is a field. Find the radicals and tops of representations of Λe_i for different possible values of i.

Problem 2. Given $\Lambda = k\Gamma/\langle\rho\rangle$, where Γ is the quiver $1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2$ with relations $\rho = \{\alpha\beta\}$ and k is a field. Let J be the ideal in $k\Gamma$ generated by the arrows.

(a) Show that there is some t such that $J^t \subset \langle\rho\rangle \subset J^2$. What is the dimension of Λ over k?
(b) Find the representations of Λe_i for different possible values of i and find their radicals and tops.
(c) Find the radical of Λ.

Problem 3. Let $\Lambda = k\Gamma$ for a field k, where Γ is the quiver:

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$$

Find the projective covers and the kernel of the projective covers of the following representations :

(1) $k \xrightarrow{0} k \xrightarrow{0} 0$.
(2) $k \xrightarrow{1} k \xrightarrow{0} k$.
(3) $k^2 \xrightarrow{(1\ 0)} k \xrightarrow{\binom{1}{1}} k^2$.

Problem 4. Given $\Lambda = k\Gamma/\langle\rho\rangle$ for a field k, where Γ is the quiver:

$$1 \xrightarrow{\alpha} 2 \underset{\gamma}{\overset{\beta}{\rightrightarrows}} 3$$

and the relations $\rho = \{\beta\alpha\}$. Find the projective covers and the kernel of the projective covers of the following representations:

(1) $k \xrightarrow{1} k \underset{\binom{0}{1}}{\overset{0}{\rightrightarrows}} k^2$.
(2) $k \xrightarrow{\binom{0}{1}} k^2 \underset{(1\ 1)}{\overset{(1\ 0)}{\rightrightarrows}} k$.
(3) $0 \xrightarrow{0} k^2 \underset{\binom{1\ 0}{1\ 0}}{\overset{\binom{1\ 0}{1\ 0}}{\rightrightarrows}} k$.

Problem 5. Let $f\colon A \to B$ and $g\colon B \to C$ be two essential epimorphisms of left Λ-modules. Show that gf is an essential epimorphism.

Problem 6. Let Λ be an artin algebra, M a finitely generated Λ-module and P an indecomposable projective Λ-module.

Show that $\mathrm{Hom}_\Lambda(P, M) \neq (0)$ if and only if $P/\mathfrak{r}P$ is a composition factor of M.

Problem 7. Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be an exact sequence of Λ-modules for some ring Λ. Show that the following are equivalent.

- There is a map $z\colon B \to A$ such that $zf = 1_A$
- There is a map $v\colon C \to B$ such that $gv = 1_C$

In case these hold, the sequence is called *split exact*. Show that $B \simeq A \oplus C$, when the sequence is split exact.