MA3203 - Problem sheet 5 - 2006

Problem 1. Given a ring Λ, let $F, G\colon \operatorname{mod}\Lambda \longrightarrow Ab$ be two functors, F covariant and G contravariant.

(a) If $0 \to F(A) \to F(B) \to F(C)$ ($G(C) \to G(B) \to G(A) \to 0$) is an exact sequence whenever $0 \to A \to B \to C$ is, we say that F is *left exact* functor (G is *right exact* functor).

(b) If $F(A) \to F(B) \to F(C) \to 0$ ($0 \to G(C) \to G(B) \to G(A) \to 0$) is an exact sequence whenever $0 \to A \to B \to C$ is, we say that F is *right exact* functor (G is *left exact* functor).

(c) If $0 \to F(A) \to F(B) \to F(C) \to 0$ ($0 \to G(C) \to G(B) \to G(A) \to 0$) is an exact sequence whenever $0 \to A \to B \to C \to 0$ is, we say that F is *exact* functor (G is *exact* functor).

Let $X \in \operatorname{mod}\Lambda$.

(1) Show that the functor $F = \operatorname{Hom}_\Lambda(X, \ )$ is a left exact (covariant) functor. Show also that $F = \operatorname{Hom}_\Lambda(X, \ )$ is an exact functor if and only if X is a projective Λ-module.

(2) Show that the functor $F = \operatorname{Hom}_\Lambda(\ , X)$ is a left exact (contravariant) functor. Show also that $F = \operatorname{Hom}_\Lambda(\ , X)$ is an exact functor if and only if X is an injective Λ-module.

Problem 2. Let (Γ, ρ) be a quiver with relations such that $J^t \subseteq \langle\rho\rangle \subseteq J^2$ for some $t \geq 2$. Let $\Lambda = k\Gamma/\langle\rho\rangle$. Let $F\colon \operatorname{mod}\Lambda \to \operatorname{Rep}(\Gamma, \rho)$ be the equivalence defined in the lectures. Show that M in $\operatorname{mod}\Lambda$ is indecomposable if and only if $F(M)$ is.

Problem 3. Consider the following commutative diagram with exact rows in $\operatorname{Mod}\Lambda$:

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C \\ & & & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' \end{array}$$

Show that there exists an $f\colon A \to A'$ such that the diagram is commutative. Also show that if g and h are isomorphisms, then f is also an isomorphism.

Problem 4. Let k be a field, and let $\operatorname{Vec}(k)$ be the category of finite dimensional vectorspaces over k. Let $D = \operatorname{Hom}_k(\ , k)\colon \operatorname{Vec}(k) \to \operatorname{Vec}(k)$ be the contravariant functor, and let $\phi_V\colon V \to DD(V)$ given by $\phi_V(v)(f) = f(v)$ for f in $D(V) = \operatorname{Hom}_k(V, k)$.

(a) Show that ϕ_V is a monomorhism, and hence an isomorphism

(b) Show that we get an induced isomorphism of functors.

Now let Λ be a finite dimensional k-algebra, and X an object in $\operatorname{mod}\Lambda$.

(c) Show that $D(X)$ is a left Λ^{op}-module via $(\lambda \cdot f)(x) = f(\lambda x)$ for f in $D(X)$, x in X and λ in Λ.

(d) For $f\colon X \to Y$, a map in $\operatorname{mod}\Lambda$, show that $D(f)\colon D(Y) \to D(X)$ given by $g \mapsto gf$, is a Λ^{op}-homomorphism.