MA3402
Exercise 2, September 2005

Problem 1: Consider the differential d , acting on scalar functions on M, as an $\mathbb{R}$-linear map from the vector space of smooth functions to the vector space of covector fields, that is, sections of the corangent bundle

$$d : C^\infty(M) \to \Gamma(T^*(M)), \quad \varphi \to d\varphi$$

Given a smooth map $f : M \to N$, recall that we defined the pull back of functions on N as an $\mathbb{R}$-linear map, namely

$$f^* : C^\infty(N) \to C^\infty(M), \quad f^*(\psi) = \psi \circ f$$

Verify the formula

$$f^*(d\psi) = d(f^*(\psi)) \ \text{ (i.e., } f^* \text{ and } d \text{ commutes)}$$

Moreover, suppose we have local coord. $(x_1, .., x_m)$ in some nbd of M and $(y_1, .., y_n)$ in N respectively. Calculate the covector fields $f^*(dy_i)$ in the local coordinates of M (where defined).

Problem 2: Show Leibniz rule

$$d(\varphi\psi) = \varphi d\psi + \psi d\varphi$$

Here $d\varphi$ is regarded as a covector field on M, ie. a section of the cotangent bundle. Observe that sections of vector bundles can be multiplied by scalar functions on the base space, so that, for example, $\Gamma(T^*(M))$ becomes a module over the ring (or $\mathbb{R}$-algebra) $C^\infty(M)$. Therefore the above identity makes sense.

Note See Spivak's exercise #2, p.127. What is wrong?

Problem 3: Cf. Spivak, Ex. 1 (a), (b), (c) (and also (d) a little later).

Here we are given a smooth map $f : M \to N$, which induces linear maps between tangent spaces and cotangent spaces (and more generally, between corresponding fibers of the various tensor bundles):

$$f_{*p} : M_p \to N_{f(p)}, \ \text{ and its dual } f_p^* : N_{f(p)}^* \to M_p^*$$

Describe these linear maps in terms of local coordinates. Namely, you must choose local coord. systems at p and $f(p)$, corresponding bases for the vector spaces involved, and describe the coefficient matrix of the linear maps (in terms of the local coordinates and f).