Exercise set 3 (29. September)
MA3402 Fall, 2003

Projects for the two *special weeks* 29 Sept. - 10 Oct.

You should work carefully with these exercises and try to understand the concepts which may be described in the text below.

Altso write down the solutions you find !!!

Sometimes you may have to check definitions from the book, mostly from Chap.7.

Project 1: **Investigate and re-interpret the classical vector calculus operations** gradient**,** curl **and** divergence **in terms of the exterior derivative operator d for differential forms**, using the following scheme:

Consider the underlying manifold to be some open subset U of $\mathbb{R}^3$ (and $\mathbb{R}^2$), and use coordinates x, y, z. Let

$$\begin{aligned} \Lambda^k(U) &= \text{the vector space of k-forms on } U \\ \mathcal{V}(U) &= \text{the vector space of vector fields on } U \end{aligned}$$

and consider the two horizontal sequences

$$\begin{array}{ccccccc} \Lambda^0(U) & \stackrel{d}{\to} & \Lambda^1(U) & \stackrel{d}{\to} & \Lambda^2(U & \stackrel{d}{\to} & \Lambda^3(U) \\ \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow \\ C^\infty(U) & \stackrel{grad}{\to} & \mathcal{V}(U) & \stackrel{curl}{\to} & \mathcal{V}(U) & \stackrel{div}{\to} & C^\infty(U) \end{array}$$

(**a)** For appropriately chosen vertical correspondences (which are linear isomorphisms) the above diagram will be commutative. Explain how to choose these vertical maps and also verify the commutativity of the diagram. What are the "classical theorems" in vector calculus corresponding to the fact that $d \circ d = 0$?

(b) Define the de Rham cohomology groups (actually finite dimensional vector spaces) $H^k(U)$ by introducing the notion of closed or exact k-forms. These groups depend on the topology of U. Comment on why these groups may be useful to answer why the "inverse" of the classical theorems will not hold in general.

(c) Check the text book or whatever (you dont need to know the proofs in the book) and try to answer which of the following cohomology groups $H^k(U)$, $k \geq 0$, are zero :

(1) : $U = \mathbb{R}$; (2) : $U = \mathbb{R} - \{0\}$; (3) : $U =$ open disk of $\mathbb{R}^2$; (4) : $U =$ open disk of $\mathbb{R}^3$

(5) : $U = \mathbb{R}^3 - \{0\}$; (6) : $U =$ open annulus (region between two concentric circles in the plane)

(6) : $U =$ interior of a torus (imbedded in $\mathbb{R}^3$)

(d) Consider the following 1-form defined on $U = \mathbb{R}^2 - \{0\}$:

$$\omega = -\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$$

Try to justify or explain what we mean by the following statement :

$$\omega \text{ represents a cohomology class } [\omega] \in H^1(U)$$

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Then, use ω to show that $H^1(U). \neq 0$. Hint: Use classical vector calculus, where we calculate the (usual) line integral of 1-forms over oriented paths. Then figure out what we can say if the form is exact and the path is closed.

(e) Let $f : M \to N$ be a smooth map. (Here M, N can be any smooth manifolds). Using the fact that the pullback map $f^* : \Lambda^k(N) \to \Lambda^k(M)$ commutes with the exterior differentiation d, explain why there is induced a linear map (we use the same notation for this map, for simplicity)

$$f^* : H^k(N) \to H^k(M)$$

This is acually functorial, in the sense that

$$\begin{aligned} (i) \quad &: \quad \text{If } f = Id : M \to M, \text{ then } f^* = Id \\ (ii) \quad &: \quad \text{If } \ M \xrightarrow{f} N \xrightarrow{g} P \ \text{ then } (g \circ f)^* = f^* \circ g^* \end{aligned}$$

Why must diffeomorphic manifolds have have the same cohomology groups for all k ?

(f) We claim that

$$H^1(S^1) \neq 0$$

where S^1 is the circle. You can verify this by considering the restriction of the above 1-form ω on $\mathbb{R}^2 - \{0\}$ to the unit circle. How ?

Use this result to prove that there is no (smooth) map

$$\varphi : \mathbb{R}^2 \to S^1 \text{ (unit circle)}$$

such that $\varphi(p) = p$ for each $p \in S^1$.

Project 2: **Volume forms etc.**

A volume form on a manifold M of dimension n is an n-form ω such that $\omega(p) \neq 0$ at each point $p \in M$.

(a) Consider $\mathbb{R}^3$ (or an open set here) with coordinate system (x, y, z) with respect to the (usual right-handed orthonormal frame). Then the volume form is defined to be

$$\omega = dx \wedge dy \wedge dz$$

and hence any 3-form can be written as $\eta = f(x, y, z)\omega$ for some function f. The integral of η (if it exists) is defined to be

$$\int_{\mathbb{R}^3} \eta = \int f(x, y, z) dx dy dz$$

where the last expression is the usual Riemann integral. For example, the integral of the 3-form $f dy \wedge dx \wedge dz = -f dx \wedge dy \wedge dz$ is the negative of the above integral etc.

Express ω in terms of

(i) sylinder coordinates (r, θ, z)
(ii) spherical coordinates (ρ, θ, φ)

(b) Consider the unit sphere

$$S^2 : x^2 + y^2 + z^2 = 1$$

in $\mathbb{R}^3$. The basic 1-forms dx, dy, dz are no longer independent when restricted to S^2 (that is, when pulled back to S^2 via the inclusion map). Explain why, and find a relation of type

$$P dx + Q dy + R dz = 0$$

which holds on S^2 for suitable functions P, Q, R .

(c) The position vector $\mathbf{n} = (x, y, z)$ of a point on S^2 is also a normal vector to the tangent plane, and we can orient S^2 by saying that $\mathbf{n}$ points in the positive direction, which is outward. We will find a volume (or call it area) form Ω for S^2. To do this, observe that any 2-form on $\mathbb{R}^3$ may be written

$$Cdx \wedge dy + Ady \wedge dz + Bdz \wedge dx \tag{1}$$

where A, B, C are functions, so we seek such an expression whose restriction to S^2 is an area form for S^2. Determine such functions A, B, C.

Hint: Regard the tangent space of S^2 at $\mathbf{n}$ as a subspace of $\mathbb{R}^3$ spanned by two vectors $\mathbf{u}$, $\mathbf{v}$ such that $\mathbf{n} \cdot \mathbf{u} = \mathbf{n} \cdot \mathbf{v} = 0$. Let Θ denote the form (1) and calculate

$$\Theta(\mathbf{n})(\mathbf{u}, \mathbf{v})$$

Does the answer look like the calculation of a determinant? Remember that Θ will be an area form iff $\Theta(\mathbf{n}) \neq 0$ at each point $\mathbf{n}$. A, B, C are not unique, so try to choose them so that the form is as simple as possible.

(**d**) Finally, express also your area form for S^2 in terms of spherical coordinates (θ, φ). (Note that there will be no term with $d\rho$ since $\rho = 1$ on S^2 and hence $d\rho = 0$).

Project 3 **Lie algebra of vector fields**

We consider smooth vector fields $X, Y, ...$on a manifold M. Regard these as 1.order differential operators on the space $C^\infty(M)$ of functions (acting as a derivation: $f \to Xf$ in the usual way), and define the Lie product (bracket) by

$$[X, Y] = XY - YX$$

where XY means composition of operators, and hence it is a 2.order diff. operator , in fact. (The point is that the difference is actually a 1. order operator).

(a) Verify that $[X, Y]$ is, indeed, a derivation, and hence is a vector field.

(**b**) Show that the product rule $[X, Y]$ is bilinear (over $\mathbb{R}$), skew-symmetric, and satisfies

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0 \qquad \text{(Jacobi identity)}$$

(The properties in (a),(b) makes the space of vector fields to a Lie algebra)

(c) Investigate if the product rule is also linear over the functions, i.e., does $[fX, Y] = f[X, Y]$ hold ? Is the map

$$(X, Y) \to [X, Y] \text{ a tensor field ?}$$

Explain your answer.

Project 4

Do (carefully) exercise 21 in Spivak, page 235.