Advisor : Eldar Straume
Room 1250, SB II
e-mail : eldars@math.ntnu.no
Tlf. : 73 59 66 83

PROJECT TOPICS

In general, I can offer projects for students in several directions, depending on the background and interest of the student. The underlying mathematical setting for these topics belong roughly to one of the following main areas, or some combination of them .

- 1. geometry/topology : By topology I mean algebraic topology or differential topology, not "general topology" which provides a basic common language for many mathematical disciplines (including algebra and analysis). Riemannian and symplectic geometry play a major role in modern mathematics and physics. When the ideas from the usual "flat" Euclidean geometry (length, angles, inner product etc,) are applied to more general "curved" spaces, you will be led to Riemannian geometry. But what is symplectic geometry? This behaves very differently from our daily "Euclidean" experiences, but it is very useful and provides, for example, the mathematical theory underlying Hamiltonian mechanics.Thus it also provides the key to quantum mechanics, quantization questions and so on.
  2. Lie groups and Lie algebras Here I have in mind simple matrix groups such as $SO(3)$ or $SU(2)$), their realization as transformation groups, and questions related to symmetry. In fact, Lie groups date back to the Norwegian mathematician Sophus Lie (around 1875), who showed that symmetry properties of differential equations can be expressed by these groups, in a way similar to the role of the Galois group for algebraic equations. Combined with modern computer algebra (Maple, Matlab, Mathematica etc) this yields the method called "Lie symmetry analysis" used to explore or solve differential equations.
  3. mathematical physics: topics within classical mechanics, especially celestial mechanics (e.g. 3-body problem, n-body problem), or topics from relativity theory making use of semi-Riemannian geometry or the Lorentz Lie group. Another direction is to quantum mechanical topics and applications of some (Lie) group theory. A basic understanding of (ordinary) differential equations may be necessary for some topics chosen from the areas listed above. One proposal is to work out a geometric reduction technique for the equations describing some classical mechanical system..

Within these topics I can offer a course, smaller projects or master thesis projects for 2007/2008. But your selection of sceduled courses for the spring

semester 2007 is also important. I will suggest that you take at least two courses from the following list, to be taken during 2007 (unless you already have taken the course) :

- TMA 4165 Diff. equations/dyn. systems (spring 2007)
- TMA 4190 Manifolds (spring 2007)
- MA 3002 General topology (spring 2007)
- MA 3402 Analysis on manifolds (fall 2007)
- MA 3403 Algebraic topology I (fall 2007)

You are welcome to discuss a possible project topic with me, and perhaps you have some ideas or preferences already?. But below I will list four topics more specifically, with a working title which you may refer to :

1. The Kepler problem In classical mechanics this is essentially the 2-body problem, and in many respects it is well understood. But there are also many surprises and not so well known facts, and books are still published about this topic. It also serves as a preparation for further studies, say, in the realm of the n-body problem for n >2. For literature, see e.g.

   a) John Milnor, "On the Geometry of the Kepler Problem", The American Mathematical Monthly, Vol. 90, No. 6 (1983)

   b) V. Guillemin and S. Sternberg , "Variations on a theme by Kepler.", Amer. Math. Society Colloquium Publications , vol 42 (1990)

   c) E.Straume, "Frå Kepler-problemet til tre-lekamsproblemet og dei trojanske asteroidar", Normat, hefte 3, 2000.

2. The Hopf fibration As a purely mathematical idea the Hopf fibration has been around since 1931. It is a type of map, called a principal fibration, between two spheres, and there are just three such maps. Usually we refer to the simplest case of $S^3 \to S^2$, which is a very special map from the 3-sphere to the 2-sphere. Hopf used it to calculate the third homotopy group of S^2, and Dirac applied it in his description of the magnetic monopole.Today it appears at least in 7 different situations in theoretical physics, disguised in various ways.(Use Google to get an impression of its many facets and applications)

   The project here is to describe and explain some of the geometric properties of the Hopf fibration, and also consider some of its applications to geometry or physics. The construction can also be described in terms of groups, namely as a quotient map

   $$S^3 \simeq SU(2) \to SU(2)/U(1) \simeq S^2$$

The fibers are circles $S^1 \simeq U(1)$ which are pairwise "linked" , that is, they cannot be continuously "liberated" from each other within the 3-sphere. Usage of the quaternions $\mathbb{H} \simeq \mathbb{R}^4$ also provides a nice construction of this Hopf fibration. The other two Hopf fibrations are in higher dimensions, namely

$$S^7 \to S^4, \qquad S^{15} \to S^8$$

To begin with, you may as well use the internet to find various papers involving Hopf fibrations, to get a first impression. Here are just three examples :

a) M. A. Porter, *"The Hopf Fibration and its Applications"* (ask the advisor)

b) http:// mathworld. wolfram.com/HopfMap.html

c) http://www.mathematik.uni-bielefeld.de/birep/top/hopf/hopf-fib.htm

3. Contact geometry The theory of contact transformations originates primarily from Sophus Lie and his studies in the 1870's. The Legendre transformation, well known in classical mechanics, is just a special case of these. Today contact geometry is a well established geometric discipline, which applies to odd dimensional spaces, in contrast to symplectic geometry which is only possible in even dimensions.

   The project is to investigate contact geometry in 3-space $\mathbb{R}^3$, to show how S. Lie used them to transform and simplify differential equations, and to consider some of the modern developments in contact geometry.

   Some very good books on the topic are a), b) below, while c) is an elementary introduction :

   a) S. Lie and G. Scheffers, "Geometrie der Berührungstransformationen", first published 1896 in Leipzig , see Chelsea Publ. Company, 1977.

   b) A. Arnold, "Mathematical methods of classical mechanics", Appendix 4; Second edition, Graduate Texts in Math., Springer-Verlag 1989.

   c) E.Straume, "Sophus Lie og differensialligninger", Normat, hefte 4, 1992..

4. The n-body problem This classical problem studies the motion of a system with n point masses under the action of the Newtonian gravitational potential. As opposed to the Kepler problem, the n-body problem is not integrable for n $>$ 2. Still, many prominent mathematicians and astronomers have been working on the problem during the last three centuries and up to present time. So what is the problem today, and what can we do about it?

   During the last 6 years I have given several student projects on this topic, mostly for $n = 3$. There are certainly many topics to work with, for a long time to come. One direction is to investigate, from a numerical viewpoint and with simulation, some of the differential equations which

arise in the study of the kinematics (e.g. Euler equations) or the dynamics (a reduction of Newton's equations).

For some historical background information, I refer to the recent book of Saari, listed below. While it provides an introduction to aspects of the N-body problem and describes several unsolved problems, many of the results are new and will be of interest also to experts. Reference b) is somewhat technical and sets the stage for a Riemannian geometric approach to the 3-body problem.

a) Donald Saari, "Collisions, Rings, and other Newtonian N-body Problems", American Math. Society, 2005. .

b) Wu-Yi Hsiang and E.Straume, "Kinematic geometry of triangles and the study of the three-body problem", NTNU 2005, : arXiv: math-ph/0608060,

c) Wu-Yi Hsiang and E.Straume, "Global geometry of 3-body motions with vanishing angular momentum, I", Berkeley 2006, arXiv: math-ph/0609076.