TENSOR CALCULUS
(A brief overview)

E. Straume (NTNU)

August 16, 2004

1 A brief introduction to differential forms

Chaper 6 in J.R. Munkres' book (Analysis on Manifolds), or M. Spivak's book (A Comprehensive Introduction to Differential Geometry, Vol.I, Chap.4) explains the basics of multilinear algebra which is needed to define tensors, tensor products, and general tensor fields in a precise way. In particular, *alternating covariant tensor fields*, called differential forms, are discussed more carefully. In this short note we will only briefly go into the details of tensor algebra and tensor calculus, and rather try to explain the nature of *differential forms* (and touch upon other tensor fields a little bit). Differential forms exist on any differentiable manifold M, they are just a special kind of tensor field. The simplest tensor fields are in fact the vector fields and covector fields (which are differential 1-forms). These are the "building blocks" for the more general tensor fields, but in order to construct differential k-forms we only need the 1-forms and the socalled wedge product of these. This will be explained below.

Why investigate differential forms? The reason is that they constitute an effective tool in differential and algebraic topology and they can also be very useful when we perform traditional vector calculus. Let us mention that integration theory on manifolds works best for differential forms. In fact, one cannot integrate a function on an (abstract) manifold M without having some metric (or Riemannian) structure, as is the case when M is imbedded in a Euclidean space (cf. Munkres). However, on an n-dimensional orientable manifold M we can integrate n-forms, and k-forms can be integrated along a k-dimensional submanifold.

1.1 What is a tensor field ?

To begin with, let us briefly look at tensor fields in general. One can explain and describe tensor fields in a coordinate free manner (just as we can describe a vector field without having to introduce local coordinates). This is the modern way and perhaps the most elegant way. However, the classical approach goes via coordinates. In fact, in the good old days they did not have manifolds, but

they had n-space with its coordinates $(x_1, x_2, ..., x_n)$, although for a long time only $n \leq 3$ made sense in their mind. Thus, in the 19th century and early 20th century a tensor field was a horrifying "animal "given by an expression of type

$$\Phi = \sum_{I,J} f_{I,J}(\frac{\partial}{\partial x_{i_1}} \otimes \frac{\partial}{\partial x_{i_2}} \otimes .... \otimes \frac{\partial}{\partial x_{i_k}}) \otimes (dx_{j_1} \otimes dx_{j_2} ... \otimes dx_{j_l}) \qquad (1)$$

where $I = (i_1, .., i_k)$ and $J = (j_1, ..., j_l)$ are multi-indices with each $i_1, ., , j_l$ in the range $\{1, 2, ..., n\}$, and each coefficient $f_{I,J}$ is a differentiable function. The tensor field is said to have **degree** (k, l), it is contravariant of degree k and covariant of degree l according to the traditional terminology. A tensor field of degree (0,0) is simply defined to be a scalar function. So, you see that Φ is specified by freely choosing a lot of coefficient functions $f_{I,J}$. How many? This is simple combinatorics; there are n^k choices of I and n^l choices of J, so altogether n^{k+l} choices.

Why call it a field ? Well, consider first the case $(k, l) = (1, 0)$, where we simply have a vector field

$$\Phi = \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i}$$

This is certainly a vector valued function, since $\Phi(p) = \Phi_p = \sum_{i=1}^{n} f_i(p) \frac{\partial}{\partial x_i}|_p$ is a tangent vector and hence belongs to the tangent space T_pM. Namely, the vector field is a section of the tangent bundle TM. Of course, here we have $M = \mathbb{R}^n$ so all the spaces T_pM may be naturally identified with $\mathbb{R}^n$ itself and hence the vector field may be viewed as a function $M \to \mathbb{R}^n$, sending p to the vector $f_1(p)\mathbf{e}_1 + ... + f_n(p)\mathbf{e}_n$.

A general tensor field is also a vector valued function, namely Φ_p belongs to a certain vector space $T_p^{(k,l)}M$. These vector spaces are the fibers of a certain vector bundle $T^{(k,l)}M$ over M, called a tensor bundle. The fibers $T_p^{(k,l)}M$ are constructed by taking the tensor product of k copies of T_pM and l copies of the dual space T_p^*M. Again, these bundles are trivial when $M = \mathbb{R}^n$ (and whenever TM is trivial), and the fiber dimension is n^{k+l}. At each point $p \in M$ the value of the tensor field in (1) is the following vector (called a tensor)

$$\Phi_p = \sum_{I,J} f_{I,J}(p)(\frac{\partial}{\partial x_{i_1}}|_p \otimes \frac{\partial}{\partial x_{i_2}}|_p \otimes .... \otimes \frac{\partial}{\partial x_{i_k}}|_p) \otimes (dx_{j_1}|_p \otimes dx_{j_2}|_p ... \otimes dx_{j_l}|_p) \qquad (2)$$

where we have expanded Φ_p as a linear combination of the "standard" basis of $T_p^{(k,l)}(M)$ associated with the coordinate system $(x_1, x_2, ..., x_n)$.

Thus, we call Φ a field on M because it is a section of some vector bundle over M, we call it a tensor field because the bundle can be described as the following tensor product of bundles

$$T^{(k,l)}M = (TM \otimes .. \otimes TM) \otimes (T^*M \otimes ... \otimes T^*M)$$

where there are k factors of TM and l factors of the dual (or cotangent) bundle T^*M.

In the particular case $k = l = 0$ we define $T^{(0,0)}M = M \times \mathbb{R}$, that is, the trivial 1-dimensional bundle, so that a section of this bundle is naturally identified with a function $f \in C^\infty(M)$. We will return later to the cotangent bundle T^*M, which is the dual of TM. We only mention that to each vector bundle E there is a dual bundle E^*, and $E^{**} \simeq E$.

Since you are, may be, not so familiar with tensor product constructions, it is still possible to introduce tensors and tensor fields without using vector bundles. Then, let us return to the tensor fields expressed in a chosen coordinate system such as in (1). For the moment, don't worry about what these "animals" really mean. However, we must understand their **transformation rules**, namely how do we calculate the same tensor (field) in another coordinate system We illustrate this with a (perhaps) true story.

Three brothers and mathematics students, Per, Pål and Espen Askeladd, were investigating a complicated problem and they decided to use tensor calculus. Per proposed to use a tensor Φ, say, the expression in (1), of some degree (k,l) and coefficient functions $f_{I,J}$ which he determined. Pål was, however, working in his favorite coordinate system $(y_1, ..., y_n)$ and arrived at a similar expression

$$\Psi = \sum_{I,J} g_{I,J}(\frac{\partial}{\partial y_{i_1}} \otimes \frac{\partial}{\partial y_{i_2}} \otimes .... \otimes \frac{\partial}{\partial y_{i_k}}) \otimes (dy_{j_1} \otimes dy_{j_2} ... \otimes dy_{j_l} \qquad (3)$$

Then came Espen, the true nerd, wondering about what his friends had worked out. Did they really have the same tensor field ? After some mysterious calculations Espen found out that they actually had constructed the same tensor. How could he decide this? What did he mean when he claimed that $\Phi = \Psi$?

This question is essential in understanding tensors. You should imagine some underlying space (manifold) on which we consider tensor fields, and that we are considering two different coordinate systems on a certain chart neighborhood. Espen simply used the transformation rules, when going from one coordinate system to another

$$\varphi : (x_1, ..., x_n) \to (y_1, ..., y_n)$$

So, when Espen looked at the coordinate transformation φ (using the necessary information about φ which he got from Per and Pål) he knew how to express each $\frac{\partial}{\partial y_i}$ in terms of $\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}$ and each dy_j in terms of $dx_1, ..., dx_n$. Then he plugged the expressions into (3), used linearity of the tensor product to expand, collected terms of the same kind and thus calculated the resulting coefficients $h_{I,J}$ in the final expression

$$\Psi = \sum_{I,J} h_{I,J}(\frac{\partial}{\partial x_{i_1}} \otimes \frac{\partial}{\partial x_{i_2}} \otimes .... \otimes \frac{\partial}{\partial x_{i_k}}) \otimes (dx_{j_1} \otimes dx_{j_2} ... \otimes dx_{j_l}) \qquad (4)$$

Finally, Espen knew that $\Phi = \Psi$ means precisely that each $h_{I,J}$ in (4) must equal $f_{I,J}$ in (1).

Here is what Espen used when he performed his impressing calculations. He first calculated the Jacobi matrix of the coordinate transformation

$$D(\varphi) = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & .... & \frac{\partial y_1}{\partial x_n} \\ ... & ... & ... \\ \frac{\partial y_n}{\partial x_1} & ... & \frac{\partial y_n}{\partial x_n} \end{bmatrix}, \quad D(\varphi^{-1}) = D(\varphi)^{-1} = \begin{bmatrix} \frac{\partial x_1}{\partial y_1} & .... & \frac{\partial x_1}{\partial y_n} \\ ... & ... & ... \\ \frac{\partial x_n}{\partial y_1} & ... & \frac{\partial x_n}{\partial y_n} \end{bmatrix}$$

and then used the chain rule to write up the relations

$$\frac{\partial \phi}{\partial x_j} = \sum_i \frac{\partial \phi}{\partial y_i}\frac{\partial y_i}{\partial x_j} \quad \text{and} \quad \frac{\partial \phi}{\partial y_j} = \sum_i \frac{\partial \phi}{\partial x_i}\frac{\partial x_i}{\partial y_j} \tag{5}$$

for any (scalar) function ϕ on n-space. Dropping the symbol ϕ in the expression (5) he got the relations

$$\frac{\partial}{\partial x_j} = \sum_i \frac{\partial y_i}{\partial x_j}\frac{\partial}{\partial y_i} \quad , \quad \frac{\partial}{\partial y_j} = \sum_i \frac{\partial x_i}{\partial y_j}\frac{\partial}{\partial x_i}$$

Concerning the basic "small animals" dx_i (which are differential 1-forms) he used the classical way of calculating the differential of a function ϕ, namely

$$d\phi = \sum_i \frac{\partial \phi}{\partial x_i} dx_i = \sum_j \frac{\partial \phi}{\partial y_j} dy_j \tag{6}$$

You may choose to interpret dx_i or dy_j as " infinitesimals", but are you satisfied with such a diffuse 'physical" explanation? Ignoring this for the moment, now Espen chose ϕ in (6) to be the coordinate function x_i or y_j and deduced the transformation formula

$$dy_j = \sum_i \frac{\partial y_j}{\partial x_i} dx_i, \quad dx_i = \sum_j \frac{\partial x_i}{\partial y_j} dy_j \tag{7}$$

1.2 Mathematical descriptions of tensors and tensor fields

Due to the above transformation rules for the vector fields $\frac{\partial}{\partial x_i}$ and 1-forms dx_i, when we change to another coordinate system, we already know how to switch over to another overlapping coordinate chart neighborhood. Therefore, for simplicity we still assume $M = \mathbb{R}^n$ and choose some (global) coordinate system $(x_1, ..., x_n)$. In particular, each tangent space $T_pM = T_p\mathbb{R}^n$ is naturally identified with $\mathbb{R}^n$ itself, and we regard the vector field

$$\frac{\partial}{\partial x_i} : \mathbb{R}^n \to T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n \to^{\pi_2} \mathbb{R}^n$$

as the constant vector field given by the standard basis vector $\mathbf{e}_i = (0, 0, .., 1, 0, .., 0)$. A vector field on $\mathbb{R}^n$ is of type $\mathbf{X} = \sum f_i \frac{\partial}{\partial x_i}$ for suitable coefficient functions f_i. These are the tensor fields of degree (1,0).

The covector fields, or 1-forms, are the tensor fields of degree (0,1). A general 1-form is expressed as

$$\omega = \sum g_i dx_i$$

for suitable coefficient functions g_i. The field ω is a section of the cotangent bundle $T^{(0,1)}M = T^*M$, which is the bundle **dual** to TM. In fact, the dual tangent space at p is the vector space

$$T_p^*M = \{\alpha : T_pM \to \mathbb{R} \ , \ \alpha \text{ a linear map } \}$$

consisting of all linear functionals on T_pM. The vectors

$$\frac{\partial}{\partial x_1}|_p, \frac{\partial}{\partial x_2}|_p, .., \frac{\partial}{\partial x_n}|_p \tag{8}$$

constitute a basis for T_pM, and now we simply define $dx_1|_p, ..., dx_n|_p$ to be the basis of T_p^*M which is dual to (8), in the sense that

$$dx_i|_p(\frac{\partial}{\partial x_j}|_p) = \delta_{ij} = \begin{cases} 0, \text{ if } i \neq j \\ 1, \text{ if } i = j \end{cases}$$

This defines each dx_i in a precise manner, and hence each 1-form becomes a section $\omega : M \to T^*M$. In this way, there is a bilinear pairing $\langle \omega, \mathbf{X} \rangle$ between vector fields $\mathbf{X}$ and 1-forms ω which produces a scalar function, namely

$$\begin{aligned} \langle \boldsymbol{\omega}, \mathbf{X} \rangle &= \left\langle \sum g_j dx_j, \sum f_i \frac{\partial}{\partial x_i} \right\rangle = \sum_{i,j} g_j f_i \left\langle dx_j, \frac{\partial}{\partial x_i} \right\rangle \\ &= \sum_{i,j} g_j f_i \delta_{ij} = f_1 g_1 + f_2 g_2 + ... + f_n g_n \in C^\infty(M) \end{aligned}$$

We could also write $\langle \omega, \mathbf{X} \rangle = \mathbf{X}(\omega) = \omega(\mathbf{X})$ just to indicate that $\mathbf{X}$ acts on covector fields ω, and from another viewpoint ω acts on vector fields $\mathbf{X}$.The bilinear pairing also satisfies

$$\langle f\mathbf{X}, \omega \rangle = \langle \mathbf{X}, f\omega \rangle = f \ \langle \mathbf{X}, \omega \rangle \quad \text{ for all } f \in C^\infty(M)$$

More generally, we can now give an alternative mathematical description of what is a tensor field Φ of degree (k,l) (without referring to the bundle $T^{(k,l)}M$), namely Φ is a multilinear map, acting on l vector fields $\mathbf{Y}_i$ and k covector fields η_j,

$$(\eta_1, ..., \eta_k; \mathbf{Y}_1, ...\mathbf{Y}_l) \to \Phi(\eta_1, ..., \eta_k; \mathbf{Y}_1, ..., \mathbf{Y}_l) \in C^\infty(M)$$

and such that Φ is also linear over $C^\infty(M)$, that is,

$$\begin{aligned} \Phi(\eta_1, ..., \eta_k; \mathbf{Y}_1, .., f\mathbf{Y}_q, .., \mathbf{Y}_l) &= \Phi(\eta_1, .., f\eta_p, .., \eta_k; \mathbf{Y}_1, .., \mathbf{Y}_l) \\ &= f \ \Phi(\eta_1, .., \eta_k; \mathbf{Y}_1, .., \mathbf{Y}_l) \end{aligned} \tag{9}$$

for any function f and indices p, q. The tensor product description of Φ, such as in (1), is in fact closely related to the above, and there is a basic theorem saying that all tensor fields of order (k,l) are just linear combinations (with functions as coefficients) of tensor fields of type

$$\Phi = \mathbf{X}_1 \otimes \mathbf{X}_2 \otimes .... \otimes \mathbf{X}_k) \otimes (\omega_1 \otimes \omega_2 \otimes .. \otimes \omega_l) \tag{10}$$

where the $\mathbf{X}_i$ are vector fields and ω_i are covector fields (1-forms). Note that (1) is of this type.

Finally, it remains to explain how the tensor field Φ in (10) acts as an operator on l vector fields $\mathbf{Y}_i$ and k covector fields η_j, cf. (9). To this end we simply use the above pairing between vector fields and covector fields and pair them together (in the obvious order) into a "big product" consisting of kl terms :

$$\mathbf{X}_1 \otimes .. \otimes \mathbf{X}_k) \otimes (\omega_1 \otimes .. \otimes \omega_l) : (\eta_1, ..., \eta_k; \mathbf{Y}_1, ...\mathbf{Y}_l) \rightarrow \prod_{i=1}^{k} \langle \eta_{\ i}, \mathbf{X}_i \rangle \prod_{j=1}^{l} \langle \omega_j, \mathbf{Y}_j \rangle$$

1.3 Differential k-forms are alternating tensor fields of type (0,k)

Henceforth, we will specialize to tensor fields of type (0,k), namely in (1) we have

$$\Phi = \sum_{j_1 < j_2 .. < j_k} f_J \ dx_{j_1} \otimes dx_{j_2} ... \otimes dx_{j_k} \tag{11}$$

Furthermore, we will also assume Φ is **alternating** (or skew-symmetric), that is, as an operator on k vector fields,

$$\Phi(\mathbf{Y}_1, \mathbf{Y}_2 .., \mathbf{Y}_k)$$

changes sign whenever we interchange $\mathbf{Y}_i$ and $\mathbf{Y}_j$ for $i \neq j$. Then we say Φ is a **k-form**. How will the sum in (11) look like in this case ?

At this point we introduce the **wedge product** of 1-forms as a method to produce k-forms. So, if $\omega_1, ...\omega_k$ are 1-forms we define a k-form (called their wedge product)

$$\omega_1 \wedge \omega_2 \wedge ... \wedge \omega_k(\mathbf{Y}_1, ..., \mathbf{Y}_k) = \det \begin{bmatrix} \omega_1(\mathbf{Y}_1) & ... & \omega_1(\mathbf{Y}_k) \\ .... & .... & .... \\ \omega_k(\mathbf{Y}_1) & ... & \omega_k(\mathbf{Y}_k) \end{bmatrix} \tag{12}$$

Clearly if we interchange two of the $\mathbf{Y}_i's$, then the determinant also changes sign. We have constructed a k-form since it is also clear that the operation on vector fields is multilinear (over scalar functions as well). It is a nice result that each k-form on n-space can be uniquely expressed as a sum

$$\Phi = \sum_{j_1 < j_2 .. < j_k} g_J \ dx_{j_1} \wedge dx_{j_2} ... \wedge dx_{j_k}$$

There is no nonzero m-form on $\mathbb{R}^n$ when $n < m$, since any product $\omega_1 \wedge \omega_2 \wedge ... \wedge \omega_m$ must vanish. Indeed, the above determinant is zero at each point p where the covectors $\{\omega_i(p)\}$ are linearly dependent, and the covectors lie in the n-dimensional cotangent space $T_p^*M \simeq \mathbb{R}^n$. Let us give some examples:

- $k = 2$: It is easy to check that the wedge product $dx_i \wedge dx_j$, regarded as a (0,2)-tensor field, is equal to

  $$dx_i \wedge dx_j = dx_i \otimes dx_j - dx_j \otimes dx_i$$

  In $\mathbb{R}^2$ with coordinates x, y, any 2-form looks like

  $$f(x, y)dx \wedge dy$$

  where $f \in C^\infty(\mathbb{R}^2)$. We have $dx \wedge dy = -dy \wedge dx$ and $dx \wedge dx = 0$.

  In $\mathbb{R}^3$ with coordinates x, y, z, any 2-form may be written

  $$f_1(x, y, z)dx \wedge dy + f_2(x, y, z)dy \wedge dz + f_3(x, y, z)dx \wedge dz$$

- $k = 3$: On $\mathbb{R}^2$ there is no 3-form $\neq 0$, and any 3-form on $\mathbb{R}^3$ can be written as

  $$f(x, y, z)dx \wedge dy \wedge dz$$

  Try to verify the formula

  $$\begin{aligned} dx \wedge dy \wedge dz \quad = \quad & dx \otimes dy \otimes dz + dy \otimes dz \otimes dx + dz \otimes dx \otimes dy \\ & -dx \otimes dz \otimes dy - dy \otimes dx \otimes dz - dz \otimes dy \otimes dx \end{aligned}$$

In general, let us denote the set of k-forms on M by $\Omega^k(M)$. It is a vector space, and in fact a module over $C^\infty(M) = \Omega^0(M)$, namely we can multiply k-forms by a function and get another k-form in the obvious way. Since any k-form can be written as a linear combination of wedge products of 1-forms, we can in fact construct a wedge product between any k-form and any l-form and obtain a (k+l)-form :

$$\Omega^k(M) \times \Omega^l(M) \to \Omega^{k+l}(M), \qquad (\Phi, \Psi) \to \Phi \wedge \Psi$$

Here we simply expand Φ and Ψ as a linear combination of wedge product of 1-forms (an operation we have already defined by the determinant construction in (12)) and then multiply out using linearity, putting together the wedge products of k and l 1-forms into a long wedge product of $k + l$ 1-forms:

$$(\omega_1 \wedge .. \wedge .\omega_k) \wedge (\eta_1 \wedge ... \wedge \eta_l) = \omega_1 \wedge .. \wedge \omega_k \wedge \eta_1 \wedge ... \wedge \eta_l$$

This product operation is not always commutative, in fact

$$\Phi^k \wedge \Psi^l = (-1)^{kl}\Psi^l \wedge \Phi^k$$

where the indices indicates their degree. We have also associativity

$$(\Phi \wedge \Psi) \wedge \Xi = \Phi \wedge (\Psi \wedge \Xi)$$

and the usual distributive law between addition and product in algebra.

1.4 How are differential forms transformed ?

Suppose we have a diffeomorphism

$$\varphi : (x_1, .., x_n) \to (y_1, .., y_n)$$

or you may call it a change of variables. Espen Askeladd has already shown us the basic step in (7), from which we can easily calculate how a 1-form in the x-coordinate system

$$\omega = f_1 dx_1 + ... + f_n dx_n$$

is transformed to the y-coordinate system. You simply substitute each $dx_i = \sum_j \frac{\partial x_i}{\partial y_j} dy_j$ into the above expression for ω, and of course re-express each f_i as a function of $y_1, .., y_n$ by replacing it with the composition $f_i \circ \varphi^{-1}$.

On the other hand, any k-form is built up as a linear combination of wedge products of 1-forms, so we simply transform the k-form by substituting the transformed 1-forms into the wedge products and multiply out using linearity of the wedge product. Of particular interest is the case $k = n$. Then any n-form in x-coordinates looks like

$$\omega = f(x) dx_1 \wedge ... \wedge dx_n$$

and the only problem is to find out how $dx_1 \wedge ... \wedge dx_n$ is transformed. Well, substitute the expression for each dx_i into the wedge product and expand. What do you get? Try first the cases $n = 2, 3, ..$ to see the pattern. The result is the nice formula

$$dx_1 \wedge ... \wedge dx_n = \frac{\partial(x_1, ..., x_n)}{\partial(y_1, ..., y_n)} dy_1 \wedge ... \wedge dy_n \tag{13}$$

where the coefficient is the determinant of the Jacobi matrix of φ^{-1}, that is,

$$\frac{\partial(x_1, ..., x_n)}{\partial(y_1, ..., y_n)} = \det(D(\varphi^{-1}))$$

1.5 Exterior derivative of differential forms

There is another operation on differential forms which we don't have for general tensor fields, namely exterior differentiation d, an $\mathbb{R}-$linear operator which to each k-form gives a (k+1)-form :

$$\Omega^0(M) \to^d \Omega^1(M) \to^d \Omega^2(M) \to^d .... \to^d .\Omega^{n-1}(M) \to^d \Omega^n(M) \to^d 0$$

For a function $f \in \Omega^0(M) = C^\infty(M)$ its differential is just the classical definition, already used by Espen Askeladd when he wrote up

$$df = \sum \frac{\partial f}{\partial x_i} dx_i$$

expressing the differential of a function as a linear combination of the differentials of the coordinate functions x_i. Since higher forms are constructed from 1-forms via the wedge product, we can now define d inductively by using the following "derivation" rule:

$$d(\Phi \wedge \Psi\ ) = d\Phi \wedge \Psi + (-1)^k \Phi \wedge d\Psi\ , \quad k = \deg(\Phi)$$

There is the very important property that two consecutive exterior differentiations gives zero

$$d \circ d = d^2 = 0 : \Omega^k(M) \to \Omega^{k+2}(M)$$

Check this, for example, starting with a 0-form or 1-form in $\mathbb{R}^3$.

Example In $\mathbb{R}^3$ we calculate (using that $d^2x = d^2y = d^2z = 0$)

$$\begin{aligned} d((x^2y)dx) &= d(x^2y) \wedge dx + (x^2y)d^2x = (2xydx + x^2dy) \wedge dx + 0 \\ &= x^2dy \wedge dx = -x^2dx \wedge dy \\ d(df) &= d(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz) \\ &= (-\frac{\partial^2 f}{\partial y \partial x} + \frac{\partial^2 f}{\partial x \partial y})dx \wedge dy + (...)dy \wedge dz. + (...)dz \wedge dx = 0 \end{aligned}$$

In the second example the coefficients vanish because of a well known fact in calculus about mized derivatives. Check this !!

A k-form Φ is called **closed** if $d\Phi = 0$ and it is called **exact** if $\Phi = d\Psi$ for some (k-1)-form Ψ. Observe that the space of closed (resp. exact) k-forms are linear subspaces of $\Omega^k(M)$, and the exact ones are always closed since $d \circ d = 0$. In differential topology there is a nice way of measuring, for a given manifold, the failure of having closed forms which are not exact. Namely, let (ker d_k) be the space of closed k-forms and (Im d_{k-1}) the subspace of (ker d_k) consisting of exact k-forms. Then we can construct the quotient vector space

$$H^k(M) = \frac{(\ker\ d_k)}{(\mathrm{Im}\, d_{k-1})}$$

called the k-th (de Rham) cohomology group of M. They are most often finite dimensional vector spaces.

Example Assume M is connected, and let us calculate $H^0(M)$. Then there is no exact 0-form (since d_{-1} does not exist), so $H^0(M) = (\ker d_0)$ consists of all functions f with vanishing differential $df = 0$. Then f must be a constant, so $H^0(M) \simeq \mathbb{R}^1$ is the vector space consiting of the constant functions and hence can be identified with the real numbers.

For $k > 0$ life is more delicate, and the nonvanishing of any $H^k(M)$ expresses certain global topological properties of M. However, for $M = \mathbb{R}^n$ or an open

convex or contracible subset, H^k also vanishes for $k > 0$. As an interesting, nontrivial case, we mention the case of the n-sphere, where

$$H^k(S^n) = \begin{cases} \mathbb{R}^1 \text{ when k = 0 or n} \\ 0 \text{ otherwise} \end{cases}$$

Problem Assume the above fact that $H^1(S^1) = \mathbb{R}^1$ for the unit circle $S^1 : x^2 + y^2 = 1$ in $\mathbb{R}^2$. Find a non-exact 1-form on S^1!

Note that all 1-form are closed (since there are no nonzero 2-form on S^1), so a 1-form ω which is not the differential of a function will give us a vector $[\omega]$ spanning the 1-dimensional space $H^1(S^1)$. We will find such a 1-form later, using integration and Stokes theorem to show that our candidate is actually non-exact.

2 Integration on a manifold

2.1 How differential forms are transformed by functions between the underlying spaces

First of all, we mention that differentiable functions between manifolds induce linear maps between their tensors or tensor fields. Here we will only explain how a smooth map between manifolds

$$\alpha : M \to N$$

induces a linear map

$$\alpha^* : \Omega^k(N) \to \Omega^k(M) \tag{14}$$

between their spaces of k-forms (for any k), but observe now that the induced map is in the opposite direction. Therefore, α^* is also called the induced **pull-back** map.

Let us define the pull-back construction. Recall that α induces a linear map between tangent spaces

$$\alpha_* : T_pM \to T_{\alpha(p)}N$$

and now we claim that this actually determines α^*, as follows. Namely, let Ψ be a k-form on N, that is, a real valued k-multilinear map Ψ_q at each tangent space T_qN. Then $\alpha^*(\Psi)$ at the point $p \in M$ is the k-multilinear map on T_pM defined by

$$\alpha^*(\Psi)_p : (v_1, v_2, .., v_k) \to \Psi_{\alpha(p)}(\alpha_*(v_1), \alpha_*(v_2), .., \alpha_*(v_k))$$

The important property of the pullback map is that it commutes with exterior differentiation operator d, that is, one can show

$$d(\alpha^*(\Psi)) = \alpha^*(d(\Psi))$$

Therefore α^* maps closed forms to closed forms and exact forms to exact forms. In particular, there is thus induced a linear map

$$\alpha^* : H^k(N) \to H^k(M) \tag{15}$$

between the corresponding cohomology groups. The induced maps in (14) and (15) behaves "functorially" in the sense that the identity map $Id : M \to M$ induces the identity map Id^*, and when we have three manifolds and maps between them, then there is the "chain rule"

$$(\beta \circ \alpha)^* = \alpha^* \circ \beta^* : \Omega^k(P) \to \Omega^k(N) \to \Omega^k(M)$$

where $\beta : N \to P$, and similarly in the case of cohomology (15). In any case, α^* will be a linear isomorphism if α is a diffeomorphism.

Question Is there any differentiable map $\mathbb{R}^2 \to^{\psi} S^1$ such that the composition

$$S^1 \to^{i} \mathbb{R}^2 \to^{\psi} S^1$$

is the identity? If the answer is yes, then there are the induced linear transformations whose composition would be the identity

$$H^1(S^1) \longleftarrow^{i^*} H^1(\mathbb{R}^2) \longleftarrow^{\psi^*} H^1(S^1)$$

But the middle vector space is zero, whereas the outer two are 1-dimensional, so this is impossible. We conclude that ψ does not exist.

Remark 1 *Note that the study of differentiable maps between manifolds also includes the case of change of variable transformations we have considered earlier. This is a confusing topic, since the interpretation of the map depends on our viewpoint. Are we considering a map between different spaces, or are we considering a coordinate change for the same underlying space ?*

In the case of a coordinate transformation

$$c : (x_1, .., x_n) \to (y_1, .., y_n) \tag{16}$$

we assume φ is a diffeomorphism, and hence we can "pull back" differential forms (and more general tensor fields) in both directions using either φ or φ^{-1}. However, more general maps such as the above $\alpha : M \to N$ may not be invertible, and of course the manifolds may have different dimensions. But introducing local coordinates the map becomes

$$\alpha : (x_1, .., x_n) \to (y_1, .., y_m)$$

so formally it looks like a coordinate change (at least if $n = m$). Although α may not be invertible we have seen that any k-form on N (with y-coordinates) can be pulled back (or "transformed") to a k-form on M (with x-coordinates). But you cannot go in the other direction unless α is invertible.

In the case of (16), regarding the invertible φ as a transition between different coordinate systems on the same manifold, a k-form ω in the y-coordinates will be pulled back to a k-form $\alpha^*(\omega)$ expressed in the x-coordinates. But it is actually the same k-form on your underlying manifold and the identity $\alpha^*(\omega) = \omega$ makes sense since the two sides of the identity merely expresses a certain tensor field in two different coordinate systems. Are you confused ? Then you probably understand why the coordinate-free descriptions in modern manifold analysis have certain advantages.

2.2 Integration on a (abstract) manifold M and Stokes theorem

What do we integrate? The answer is differential forms, not functions. More precisely, if dim $M = n$ and Φ is an n-form, then we can define the integral

$$\int_M \Phi$$

and if η is a k-form and $N \subset M$ is a submanifold of dimension k, then the integral

$$\int_N \eta$$

can be defined. The basic idea is to pull back the differential form to a euclidean space using local coordinate functions and then calculate Riemann integrals in the usual way. Then we must first break up Φ into small pieces, each "lying" in a coordinate neighborhood.

So, let us first define the integral of an n-form on $\mathbb{R}^n$ (or an open set V). Recall that all n-forms on $\mathbb{R}^n$ are of type

$$\Theta = f(x_1, .., x_n)dx_1 \wedge dx_2 \wedge ... \wedge dx_n$$

and we will call $dx_1 \wedge dx_2 \wedge ... \wedge dx_n$ t**he volume form**. The choice of ordering of the dx_i in the wedge product is related to our choice of positive orientation of $\mathbb{R}^n$, namely the ordered standard basis $(\mathbf{e}_1, ..., \mathbf{e}_n)$ defines the positive orientation of $\mathbb{R}^n$. The orientation of $(\mathbf{e}_1, ..., \mathbf{e}_n)$ will stay the same if we permute the vectors by a permutation σ with signum $+1$, and the corresponding wedge product of the dx_i will be unchanged since

$$dx_{\sigma 1} \wedge dx_{\sigma 2} \wedge ... \wedge dx_{\sigma n} = sgn(\sigma)dx_1 \wedge dx_2 \wedge ... \wedge dx_n$$

So, the volume form depends on the orientation, and the two choices differ only by a sign change. Now, in an integral such as

$$\int_U f dx_1 \wedge dx_2 \wedge ... \wedge dx_n$$

we simply replace the volume form by the Riemannian volume element dV by formally removing the symbols $\wedge$

$$dx_1 \wedge dx_2 \wedge ... \wedge dx_n \rightarrow dx_1 dx_2 ... dx_n = dV$$

and so we define

$$\int_U \Theta = \int_U f \; dx_1 \wedge dx_2 \wedge ... \wedge dx_n = \int_U f \; dV \tag{17}$$

to be the Riemann integral of f on $U \subset \mathbb{R}^n$. For example, in $\mathbb{R}^2$ with the usual orientation,

$$\int_U (x^2 - xy) dy \wedge dx = \int_U -(x^2 - xy) dx \wedge dy = \int_U (-x^2 + xy) dx dy$$

Now, let us turn to the integration of an n-form Φ on an n-dimensional manifold M. We must either assume Φ has compact support (so that it vanishes outside a compact set) or that M is compact. (Otherwise the integral may not exist, similar to the case the Riemann integral). We must also assume M is oriented, so we choose an atlas $A = \{(x_\alpha, V_\alpha)\}$ consisting of orientation preserving charts $x_\alpha : V_\alpha \rightarrow U_\alpha \subset \mathbb{R}^n$.

Finally, choose a partition of unity $\{\varphi_\alpha\}$ on M subordinate to the above covering of chart neighborhoods. We can also assume each function φ_α has compact support and that there is only a finite number of terms in the sum $\Phi = \sum_\alpha \varphi_\alpha \Phi$ (otherwise we risk ending up with the conclusion that Φ is not "integrable"). Then we write the formal sum

$$\int_M \Theta = \int_M \sum_\alpha \varphi_\alpha \Theta = \sum_\alpha \int_M \varphi_\alpha \Theta = \sum_\alpha \int_{V_\alpha} \varphi_\alpha \Theta$$

where we have broken up the integral into a finite sum of integrals (still undefined), but now each integrand is an n-form with support within a chart neighborhood V_α, and that seems more promising.

Next, we must define the integral of an n-form (such as $\varphi_\alpha \Theta$) within a coordinate neighborhood V_α, and now we simply pull pack the n-form using the chart function x_α (or rather its inverse) to become an n-form on $U_\alpha \subset \mathbb{R}^n$.

Finally we refer to (17) and calculate a sum of Riemann integrals in the usual way. Hence, we are led to define

$$\int_M \Theta = \sum_\alpha \int_{V_\alpha} \varphi_\alpha \Theta = \sum_\alpha \int_{U_\alpha} (x_\alpha^{-1})^* (\varphi_\alpha \Theta) \tag{18}$$

where in the last sum each integrand is a differential form on $U_\alpha \subset \mathbb{R}^n$ and therefore of type

$$(x_\alpha^{-1})^{*)} (\varphi_\alpha \Theta) = f_\alpha dx_1 \wedge .. \wedge dx_n \tag{19}$$

The point is that the final sum is actually

- independent of the choice of the orientation preserving coordinate charts (x_α, V_α), and
- independent of the choice of partition of unity.

We omit this verification here. However, it is important that the coordinate functions are orientation preserving, since we have already seen that the transformation rule for n-forms

$$dx_1 \wedge ... \wedge dx_n = \frac{\partial(x_1, ..., x_n)}{\partial(y_1, ..., y_n)} dy_1 \wedge ... \wedge dy_n \tag{20}$$

involves a "magnification factor" which may be negative or positive, depending on whether the coordinate transformation

$$\varphi : (x_1, ..., x_n) \to (y_1, .., y_n)$$

reverses orientation or preserves orientation. Contrary to this, the transformation rule for the Riemann integral involves volume elements dV (not n-forms) and it says

$$dx_1 dx_2 .. dx_n = \left| \frac{\partial(x_1, ..., x_n)}{\partial(y_1, ..., y_n)} \right| dy_1 dy_2 .. dy_n$$

That is, the magnification factor is the abolute value of the Jacobi determinant. Hence, signs will only come out correctly, and we get the same result in (18) for any choice of charts, as long as their pairwise transition functions are orientation preserving.

Finally, let us formulate the famous Stokes Theorem for compact oriented manifolds M with boundary ∂M (possibly empty). It is assumed that ∂M has the **induced orientation** from M, briefly speaking, if you take n-1 tangent vectors $v_1, ...v_{n-1}$ at $p \in \partial M$ which is a basis for $T_p \partial M$ and choose v_n to be an "outward" pointing tangent vector at p, then $[v_1, ...v_{n-1}]_p$ gives the positive orientation of $T_p \partial M$ if and only if $[v_1, ...v_{n-1}, v_n]_p$ gives the positive orientation of $T_p M$.

Stokes Theorem Let Ψ be an (n-1)-form on the n-dimensional compact oriented manifold M. Then

$$\int_M d\Psi = \int_{\partial M} \Psi$$

For example, if $\partial M = \emptyset$ the we immediately conclude that the integral of an exact n-form on M must vanish.

Now, consider the 1-form $\omega = -ydx + xdy$ on $\mathbb{R}^2$ and use the same notation for its pullback to the unit circle $S^1 : x^2 + y^2 = 1$. We claim that it is not exact. To show this we use Stokes theorem. Then we need to calculate the integral of the 1-form on the circle, which is a 1-dimensional manifold. Note that the procedure we have used above to define the integral more generally on manifolds

will boil down to the usual calculation of line integrals in vector calculus, in this case over a parametrized curve such as S^1. So, by parametrizing S^1 by the angle θ as usual, we find

$$\int_{S^1}(-ydx+xdy)=\int_0^{2\pi}d\theta=2\pi\neq 0$$

On the other hand, if the above 1-form was really exact, that is, equal to the differential of a smooth function on S^1, then by Stokes theorem the integral would be zero since the circle has no boundary. This proves that ω is not exact, and we have solved the Problem stated at the end of Section 1.5.

3 Vector calculus in 3-space

This is an old story, but here we will relate the exterior differential operator on forms to the classical constructions of gradient, curl and divergence. Then you will see that these constructions are really exterior differentiation in disguise, and in a unifying manner.

Let us be working in some open subset $U\subset\mathbb{R}^3$ with coordinates x,y,z. The set of vector fields on U is denoted $Vect(U)$, but for convenience we introduce two identical copies of this set and distinguish them by writing $Vect(U)_1$ and $Vect(U)_2$. The classical operators are linear maps

$$C^\infty(U)\rightarrow^{grad}Vect(U)_1\rightarrow^{curl}Vect(U)_2\rightarrow^{div}C^\infty(U)$$

between vector spaces, namely

$$\begin{aligned} grad(f) &= \nabla f=(\frac{\partial f}{\partial x})\frac{\partial}{\partial x}+(\frac{\partial f}{\partial y})\frac{\partial}{\partial y}+(\frac{\partial f}{\partial z})\frac{\partial}{\partial z}\\ &= (\frac{\partial f}{\partial x})\mathbf{e}_1+(\frac{\partial f}{\partial y})\mathbf{e}_2+(\frac{\partial f}{\partial z})\mathbf{e}_3\end{aligned}$$

$$curl(\textstyle\sum g_i\mathbf{e}_i)=(\frac{\partial g_3}{\partial y}-\frac{\partial g_2}{\partial z})\mathbf{e}_1+(\frac{\partial g_1}{\partial z}-\frac{\partial g_3}{\partial x})\mathbf{e}_2+(\frac{\partial g_2}{\partial x}-\frac{\partial g_1}{\partial y})\mathbf{e}_3$$

$$div(\textstyle\sum h_i\mathbf{e}_i)=\frac{\partial h_1}{\partial x}+\frac{\partial h_2}{\partial y}+\frac{\partial h_3}{\partial z}$$

Now, consider the following diagram of linear maps

$$\begin{array}{ccc} C^\infty(U) & \rightarrow^{Id} & C^\infty(U)\\ \downarrow\nabla & & \downarrow d\\ Vect(U)_1 & \rightarrow^{\alpha_1} & \Omega^1(U)\\ \downarrow curl & & \downarrow d\\ Vect(U)_1 & \rightarrow^{\alpha_2} & \Omega^2(U)\\ \downarrow div & & \downarrow d\\ C^\infty(U) & \rightarrow^{\alpha_3} & \Omega^3(U)\end{array}\qquad(21)$$

where the horizontal maps are the following natural isomorphisms, namely

$$\begin{aligned}
\alpha_1 &: f_1\mathbf{e}_1 + f_2\mathbf{e}_2 + f_3\mathbf{e}_3 \to f_1dx + f_2dy + f_3dz \\
\alpha_2 &: g_1\mathbf{e}_1 + g_2\mathbf{e}_2 + g_3\mathbf{e}_3 \to g_1(dy \wedge dz) + g_2(dz \wedge dx) + g_3(dx \wedge dy) \\
\alpha_3 &: h \to h(dx \wedge dy \wedge dz)
\end{aligned}$$

Now, the surprising observation is that the diagram (21) is commutative. This is a simple exercise for you to verify!

So, the right column expresses exactly the same information as the classical left column. It is also easier to calculate exterior derivatives in the right column than calculate the vertical maps in the left column. How do you remember the formula for the curl? It is more straighforward to calculate the differential of a 1-form, namely the first step is (using that $d(dx) = 0$ etc.)

$$d(f_1dx + f_2dy + f_3dz) = df_1 \wedge dx + df_2 \wedge dy + df_3 \wedge dz$$

and then we substitute in the expression for df_i, for example

$$df_1 \wedge dx = (\frac{\partial f_1}{\partial x}dx + \frac{\partial f_1}{\partial y}dy + \frac{\partial f_1}{\partial z}dz) \wedge dx = \frac{\partial f_1}{\partial y}dy \wedge dx + \frac{\partial f_1}{\partial z}dz \wedge dx, \quad \text{etc.}$$

In the final step, dont forget that $dx \wedge dy = -dy \wedge dx$ and $dx \wedge dx = 0$ etc.

Recall from vector calculus that we have the identities

$$curl(\nabla f) = 0, \quad div(curl(F)) = 0$$

saying that two consecutive operators in the left column of (21) is the zero operator. But this is just the statement that $d \circ d = 0$ in the right column. Finally, we recall some well known problems in vector calculus, depending in fact on the topology of the open set U :

- **Question 1**: for a given vector field F on U, does F have a scalar potential f, that is, is F the gradient of a functiion f?
- **Qestion 2** : for a given vector field G, does G have a vector potential F, that is, is G the curl of a vector field F?

By transforming the above vector field $F \in Vect(U)_1$ and $G \in Vect(U)_2$ to a 1-form or 2-form respectively, using the maps α_1 and α_2, the question is whether the corresponding differential form is exact. Then the first thing to do is to check whether the differential form is closed, or equivalently, in Question 1 you first check whether curl$(F) = 0$, and in Question 2 you first check whether $div(G) = 0$. If the answer is yes, then there is a possibility that the vector field has a potential (of the appropriate type, scalar or vector).

Now, suppose the preliminary answer was yes. How do you finally decide whether there is a potential, that is, that the corresponding differential form

is exact ? Here is a nice result which may help clarifying the situation. We consider the cohomology groups of U

$$H^k(U) = \frac{\{\text{closed k-forms}\}}{\{\text{exact k-forms}\}}, \quad k = 1, 2$$

From this it follows in particular that all closed k-forms are exact if and only if $H^k(U) = 0$. It is an important theorem from algebraic topology that $H^k(U)$ only depends on the homotopy type of U. In many case U is convex, contractible, or even homeomorphic to $\mathbb{R}^3$. In these cases all cohomology groups are zero for all $k > 0$. We also mention that $H^1(U) = 0$ if U is simply connected.

Thus, we have barely touched upon some deeper mathematical ideas concerning the extent to which the answer is yes, no or "we must check further " for the two questions stated above.