SIF 5036 Matematisk modellering

Regular Perturbation and the Projectile Problem

(Elaboration of an example in Lin and Segel)

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1 Regular Perturbation

A *regular perturbation* case is an equation

$$D(x, \cdots, \varepsilon) = 0 \tag{1}$$

containing a parameter ε such that the full solution $x_{sol}(\cdots, \varepsilon)$ approaches the solution x_0 of the simplified equation

$$D(x, \cdots, 0) = 0 \tag{2}$$

when ε tends to 0.

The basic regular perturbation methodology is simple:

1. Write the solution as a power series in ε:

$$x_{sol} = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \varepsilon^3 x_3 + \cdots \tag{3}$$

2. Insert the power series into the equation and rearrange to a *new* power series in ε:

$$\begin{aligned} D(x_{0sol}, \varepsilon) &= D(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \varepsilon^2 x_3 + \cdots, \varepsilon), \\ &= P(x_0, 0) + P_1(x_0, x_1)\varepsilon + P_2(x_0, x_1, x_2)\varepsilon^2 + \cdots \end{aligned} \tag{4}$$

3. Set each coefficient in the power series equal to 0 and solve the resulting systems

$$\begin{aligned} P_0\,(x_0, 0) &= D(x_0, 0) = 0, \\ P_1(x_0, x_1) &= 0, \\ P_2(x_0, x_1, x_2) &= 0, \\ &\cdots \end{aligned} \tag{5}$$

*recursively (*first equation first, then the next, and so on*).*

This determines x_0, x_1, $x_2, \cdots$. The idea applies in *many* different contexts:

1. For giving approximate solutions to algebraic and transcendental equations
2. For approximate expressions to definite integrals
3. For ordinary and partial differential equations

The perturbation analysis is often *complementary* to numerical techniques. When ε is small, numerical methods may have problems (this applies in particular to *singular* perturbation). Also, the perturbation analysis sometimes gives asymptotic relationships that are more useful than a small number of numerical experiments. However, in other cases there are really no "small" parameters, and we have to rely on numerical solutions.

2 The Projectile Problem

The projectile problem discussed in Lin & Segel, pp. 233, is a simple and instructive example on how regular perturbation works.

2.1 Derivation of the equation

A rocket is launched vertically from the earth with velocity V (after all the fuel is burnt in a short period of time). Assume for simplicity that the x^*-axis is vertical and that the motion of the rocket is described by $x^*(t^*)$, where

$$\begin{aligned} x^*(0) &= 0, \\ \frac{dx^*}{dt^*}(0) &= V. \end{aligned} \tag{6}$$

The rocket experiences a force given by Newton's law of gravitation,

$$F(x^*) = -G\frac{Mm}{(R+x^*)^2}, \tag{7}$$

where G is the constant of gravitation, M is the mass of the earth, R is the radius of the earth, and m the mass of the rocket. Since the force at the surface of the earth is given by $F(0) = -mg$, we have $g = GM/R^2$, and the equation of motion now follows from Newton's second law:

$$-\frac{R^2gm}{(R+x^*)^2} = m\frac{d^2x^*}{dt^{*2}}. \tag{8}$$

The equation for the full problem is therefore:

$$\frac{d^2x^*}{dt^{*2}} = -\frac{R^2g}{(R+x^*)^2},\ x^*(0) = 0, \frac{dx^*}{dt^*}(0) = V. \tag{9}$$

2.2 Scaling

We shall consider the problem for the case where V is rather small compared to the escape velocity from the earth (This velocity, which is about 11.2km/s, may actually be derived from the same equation, see below).

The problem we consider is only one of several special cases. Since the initial velocity is not very big, we do not expect the rocket to go very high. Therefore, as long as $x^* << R$,

$$\frac{d^2x^*}{dt^{*2}} \approx -g. \tag{10}$$

The simplified problem is easy to solve:

$$x^*(t^*) \approx -\frac{1}{2}gt^{*2} + Vt^*. \tag{11}$$

In particular, the approximate maximum height follows from Eqn. 11 by observing that the time to reach the maximum height is given by

$$-gt^* + V = 0, \tag{12}$$

or

$$t_{\max} = \frac{V}{g}. \tag{13}$$

Thus,

$$x_{\max} \approx -\frac{1}{2}g\left(\frac{V}{g}\right)^2 + V\left(\frac{V}{g}\right) = \frac{1}{2}\frac{V^2}{g}. \tag{14}$$

Suitable scales, where we no not care about factors of 2, are therefore

$$X = \frac{V^2}{g},\ T = \frac{V}{g}. \tag{15}$$

This is inserted into the equation:

$$\frac{d^2\left(\frac{V^2}{g}x\right)}{d\left(\frac{V}{g}t\right)^2} = -\frac{R^2 g}{\left(R+\frac{V^2}{g}x\right)^2},\ \frac{V^2}{g}x\left(0\right) = 0,\ \frac{d\left(\frac{V^2}{g}x\right)}{d\left(\frac{V}{g}t\right)} = V, \tag{16}$$

which, after simplification, gives

$$\ddot{x} = -\frac{1}{\left(1+\varepsilon x\right)^2},\ x\left(0\right) = 0,\ \dot{x}\left(0\right) = 1,\ \varepsilon = \frac{V^2}{Rg}. \tag{17}$$

It is not possible to express the solution in the form $x = x\left(t, \varepsilon\right)$ in terms of elementary functions (this is not obvious!). Observe also that ε is approximately equal to $x_{\max}/2R$.

2.3 Solution by regular perturbation

In order to solve the Eqn. 17 by regular perturbation, we introduce the expression

$$x\left(t\right) = x_0\left(t\right) + \varepsilon x_1\left(t\right) + \varepsilon^2 x_2\left(t\right) + \cdots \tag{18}$$

into the equation:

$$\begin{aligned}
\ddot{x} &= \ddot{x}_0 + \varepsilon\ddot{x}_1 + \varepsilon^2\ddot{x}_2 + \cdots = -\left(1+\varepsilon x\right)^{-2} \\
&= -\left[1 + (-2)\varepsilon x + \frac{(-2)(-3)}{2}(\varepsilon x)^2 + \cdots\right] \\
&= -1 + 2\varepsilon(x_0 + \varepsilon x_1 + ...) - 3\varepsilon^2 x_0^2 + \cdots \\
&= -1 + \varepsilon 2x_0 + \varepsilon^2(2x_1 - 3x_0^2) + \cdots.
\end{aligned} \tag{19}$$

Equating the coefficients in front of each power of ε then leads to

$$\begin{aligned}
\ddot{x}_0 &= -1, \\
\ddot{x}_1 &= 2x_0, \\
\ddot{x}_2 &= 2x_1 - 3x_0^2 \\
\ddot{x}_3 &= 2x_2 + 2x_0 x_1 - 2x_0\left(2x_1 + x_0^2\right) - 2\left(2x_1 - 3x_0^2\right)x_0
\end{aligned} \tag{20}$$

In order to derive the last equation, the expansions in Eqn. 19 have to be carried out one step further. For the initial conditions, it is sufficient to use

$$\begin{aligned}
x_0\left(0\right) &= 0,\ \dot{x}_0\left(0\right) = 1, \\
x_1\left(0\right) &= 0,\ \dot{x}_1\left(0\right) = 0, \\
x_2\left(0\right) &= 0,\ \dot{x}_2\left(0\right) = 0, \\
&\cdots
\end{aligned} \tag{21}$$

Thus, already x_0 takes care of initial conditions, which are exactly fulfilled whenever we choose to stop.

The leading order solution is simply

$$x_0(t) = t - \frac{1}{2}t^2, \tag{22}$$

and by inserting this into the next equation in 20, we obtain

$$\ddot{x}_1 = 2x_0 = 2\left(t - \frac{1}{2}t^2\right),$$

or (remembering the initial conditions),

$$x_1(t) = \frac{1}{3}t^3 - \frac{1}{12}t^4.$$

Note that only the particular solutions differ on each step, and in this case, the contribution from the homogeneous solution vanishes for x_i when $i \geq 1$.

It is common that the algebra quickly gets quite complicated, and today perturbation analysis is best carried out using some symbolic manipulation program like *Maple*, *Mathematica* or *Derive* (*Maple* was used below):

$$x(t) = t - \frac{1}{2}t^2 + \varepsilon\left(\frac{1}{3}t^3 - \frac{1}{12}t^4\right) + \varepsilon^2\left(-\frac{1}{4}t^4 + \frac{11}{60}t^5 - \frac{11}{360}t^6\right) + \mathcal{O}\left(\varepsilon^3\right). \tag{23}$$

Using Eqn. 23, we also obtain the approximate flight time to the maximum height from

$$\begin{aligned} & \frac{d}{dt}\left(t - \frac{1}{2}t^2 + \varepsilon\left(\frac{1}{3}t^3 - \frac{1}{12}t^4\right) + \varepsilon^2\left(-\frac{1}{4}t^4 + \frac{11}{60}t^5 - \frac{11}{360}t^6\right)\right) \\ = \;& 1 - t + \varepsilon t^2 - \frac{1}{3}\varepsilon t^3 - \varepsilon^2 t^3 + \frac{11}{12}\varepsilon^2 t^4 - \frac{11}{60}\varepsilon^2 t^5 = 0 \end{aligned} \tag{24}$$

This is a quintic equation, but since we expect that the solution t_m is near to 1, we try to find a perturbation solution of the form

$$t_m = 1 + a\varepsilon + b\varepsilon^2 + O(\varepsilon^3).$$

By inserting this into Eqn. 24, we obtain

$$\begin{aligned} 0 \;=\; & 1 - \left(1 + a\varepsilon + b\varepsilon^2\right) + \varepsilon\left(1 + a\varepsilon + b\varepsilon^2\right)^2 - \frac{1}{3}\varepsilon\left(1 + a\varepsilon + b\varepsilon^2\right)^3 - \varepsilon^2\left(1 + a\varepsilon + b\varepsilon^2\right)^3 + \\ & + \frac{11}{12}\varepsilon^2\left(1 + a\varepsilon + b\varepsilon^2\right)^4 - \frac{11}{60}\varepsilon^2\left(1 + a\varepsilon + b\varepsilon^2\right)^5 \\ = \;& \left(-a + \frac{2}{3}\right)\varepsilon + \left(a - \frac{4}{15} - b\right)\varepsilon^2 + O\left(\varepsilon^3\right). \end{aligned}$$

The solution to $O\left(\varepsilon^3\right)$ is therefore

$$a = \frac{2}{3},\; b = a - \frac{4}{15} = \frac{2}{5}.$$

and

$$t_m = 1 + \frac{2}{3}\varepsilon + \frac{2}{5}\varepsilon^2 + O(\varepsilon^2).$$

It is reasonable that the time to the maximum height is increased compared to the value 1, since the gravitational pull get gradually weaker as the rocket rises.

Numerically obtained solutions (using Matlab) are shown in Fig. 1.

The perturbation expansions to zeroth, first and second order are compared to the numerical solutions in Fig. 2 and 3.

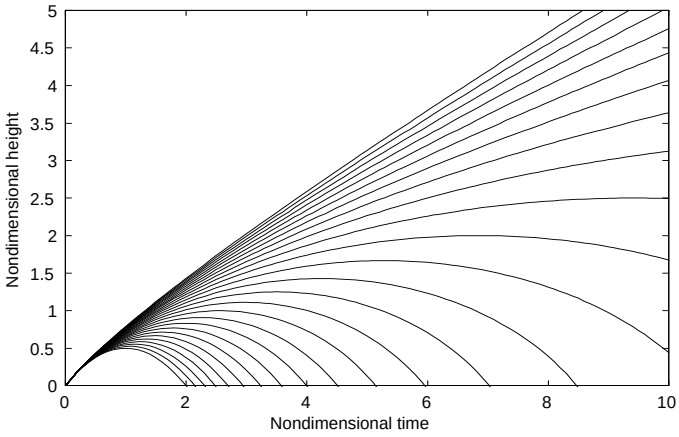

Figure 1: Numerical solutions shown for $\varepsilon = 0\,(0.1)\,3$. For $\varepsilon \geq 2$, the path never returns to $x = 0$ (see below).

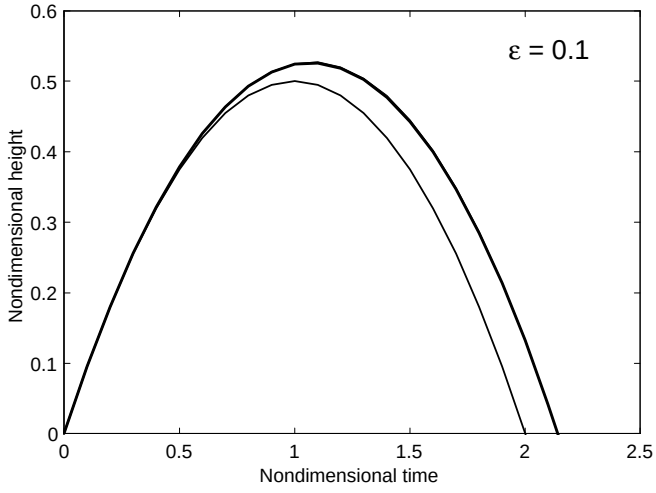

Figure 2: Numerical and perturbation solutions for $\varepsilon = 0.1$. The numerical solution (heavy line) and the perturbation expressions to first and second order co-incide on the graph. Thin curve: $x_0\,(t)$.

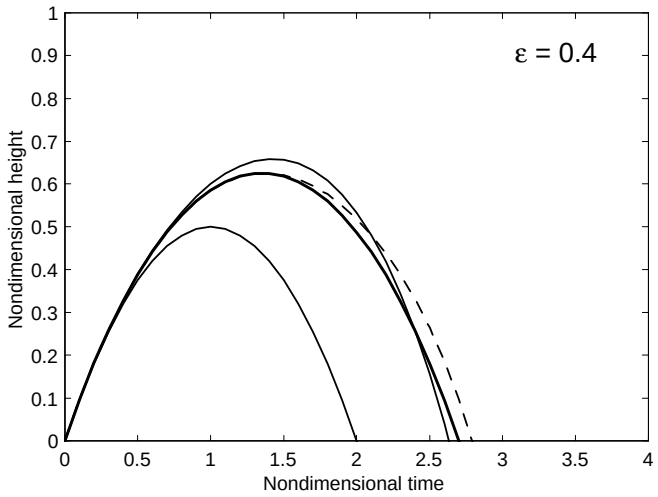

Figure 3: Similar to previous figure for $\varepsilon = 0.4$. Numerical solution: heavy curve; x_0 and $x_0 + \varepsilon x_1$: thin lines; $x_0 + \varepsilon x_1 + \varepsilon^2 x_2$: dashed line.

3 The Analytic Solution

As noted above, it is not possible to write the solution of

$$\ddot{x} = -\frac{1}{(1+\varepsilon x)^2}, \;\; x(0) = 0, \;\; \dot{x}(0) = 1 \tag{25}$$

in closed form in terms of elementary functions. However, after multiplying the equation by $\dot{x}$ we obtain

$$\frac{d}{dt}\left(\dot{x}^2/2\right) = \frac{d}{dt}\left(\frac{1}{\varepsilon}\frac{1}{1+\varepsilon x}\right), \tag{26}$$

or

$$\frac{\dot{x}^2}{2} - \frac{1}{\varepsilon}\frac{1}{1+\varepsilon x} = \text{Constant}. \tag{27}$$

This merely says that the motion is *conservative* with the sum of kinetic and potential energies equal to a constant. By inserting the initial conditions, $x\,(0) = 0$ and $\dot{x}\,(0) = 1$, we see that the constant is $1/2 - 1/\varepsilon$. This leads to the first order equation

$$\dot{x}^2 = \frac{1 + (\varepsilon - 2)x}{1 + \varepsilon x}, \;\; x(0) = 0. \tag{28}$$

If $\varepsilon < 2$, $\dot{x}$ will be equal to 0 for

$$x_{\max} = \frac{1}{2 - \varepsilon}. \tag{29}$$

This is the (exact) *maximum height* for the projectile when $\varepsilon < 2$. If $\varepsilon > 2$, the velocity will always stay strictly greater than 0 and the projectile continues to the limits of the universe. Note that $\varepsilon = 2$ corresponds to

$$V = \sqrt{2Rg} = \sqrt{2 \times \frac{40000000}{2\pi} \times 9.81 \frac{\text{m}}{\text{s}}} \approx 11.17\text{km/s}. \tag{30}$$

This gives an alternative interpretation of ε,

$$\varepsilon = \frac{V^2}{Rg} = 2\left(\frac{V}{V_0}\right)^2, \tag{31}$$

where V_0 is the escape velocity defined in Eqn. 30.

By separation, the solution of Eqn. 28, as long as stays $\dot{x} > 0$, can be written implicitly by

$$t = \int_0^x \sqrt{\frac{1+\varepsilon s}{1+(\varepsilon-2)s}} ds \tag{32}$$

Note that there are different expressions for $\varepsilon < 2$ and $\varepsilon > 2$. The integral may actually be derived by using the general formula

$$\int \sqrt{\frac{a+s}{b-s}} ds = \frac{a+b}{2} \arcsin\left(\frac{2s+a-b}{a+b}\right) - \sqrt{(a+s)(b-s)} + C = F(s,a,b) + C \tag{33}$$

Thus,

$$\begin{aligned} t &= \int_0^x \sqrt{\frac{1+\varepsilon s}{1-(2-\varepsilon)s}} ds = \sqrt{\frac{\varepsilon}{2-\varepsilon}} \int_0^x \sqrt{\frac{1/\varepsilon+\varepsilon s}{1/(2-\varepsilon)-s}} ds \\ &= \sqrt{\frac{\varepsilon}{2-\varepsilon}} \left(F(x, \frac{1}{\varepsilon}, \frac{1}{2-\varepsilon}) - F(0, \frac{1}{\varepsilon}, \frac{1}{2-\varepsilon}) \right) \end{aligned} \tag{34}$$

Although this gives an elementary expression for $t = t(x,\varepsilon)$, it is not possible to turn it around.

We already know that

$$x_{\max} = \frac{1}{2-\varepsilon}, \tag{35}$$

such that

$$\begin{aligned} t_m &= \int_{s=0}^{1/(2-\varepsilon)} \sqrt{\frac{1+\varepsilon s}{1-(2-\varepsilon)s}} ds = \frac{\frac{\pi}{2} - \arcsin(1-\varepsilon) + \sqrt{(2-\varepsilon)\varepsilon}}{(2-\varepsilon)^{3/2}\varepsilon^{1/2}} \\ &= 1 + \frac{2}{3}\varepsilon + \frac{2}{5}\varepsilon^2 + \frac{8}{35}\varepsilon^3 + O\left(\varepsilon^{7/2}\right). \end{aligned} \tag{36}$$

This is in accordance with what we had from the perturbation analysis.