

REGULAR PERTURBATION

NON-RIGOROUS DEFINITION:

We have a *regular perturbation case* if we have an equation

$$D(x, \dots, \varepsilon) = 0$$

containing a parameter ε such that the solution $x_{sol}(\dots, \varepsilon)$ approaches the solution x_0 of the simplified equation

$$D(x, \dots, 0) = 0$$

when $\varepsilon \rightarrow 0$.

METHOD:

1) Write the solution as a power series in ε :

$$x_{sol} = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \varepsilon^3 x_3 + \dots$$

2) Insert the power series in the equation and rearrange to a new power series in ε :

$$\begin{aligned} D(x_{sol}, \varepsilon) &= D(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \varepsilon^3 x_3 + \dots, \varepsilon) \\ &= D(x_0, 0) + P_1(x_0, x_1)\varepsilon + P_2(x_0, x_1, x_2)\varepsilon^2 + \dots \end{aligned}$$

3) Solve the systems recursively

$$\begin{aligned} D(x_0, 0) &= 0 \\ P_1(x_0, x_1) &= 0 \\ P_2(x_0, x_1, x_2) &= 0 \\ &\dots \end{aligned}$$

and determine $x_0, x_1, x_2, \dots$.

This applies in *many* different contexts:

- for giving approximate solutions to algebraic and transcendental equations
- for approximate expressions of definite integrals
- for ordinary and partial differential equations

Perturbation analysis is often complementary to numerical techniques:

- When ε is small, numerical methods may have problems (this actually applies to singular perturbation)
- The perturbation analysis gives asymptotic relationships that are more useful than a small number of numerical experiments.

Sometimes there are no “small” parameters. Then we have to rely on numerical solutions!
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THE PROJECTILE PROBLEM

(Lin & Segel, pp. 233)

A projectile is launched vertically from the earth with velocity V .

Equation of motion:

$$\frac{d^2 x^*}{dt^*} = -\frac{R^2 g}{(x^* + R)^2}$$
$$x^*(0) = 0$$
$$\frac{dx^*}{dt^*}(0) = V$$

Introducing **scales**:

$$x^* = \frac{V^2}{g} x,$$
$$t^* = \frac{V}{g} t,$$

Leads to the following scaled equations:

$$\frac{d^2 x}{dt^2} = \frac{1}{(1 + \epsilon x)^2},$$
$$x(0) = 0,$$
$$\frac{dx}{dt}(0) = 1.$$

Perturbation solution:

$$\begin{aligned}\ddot{x} &= \ddot{x}_0 + \varepsilon \ddot{x}_1 + \varepsilon^2 \ddot{x}_2 + \dots \\ &= -(1 + \varepsilon x)^{-2} \\ &= -\left[1 + (-2)\varepsilon x + \frac{(-2)(-3)}{2}(\varepsilon x)^2 + \dots\right] \\ &= -1 + 2\varepsilon(x_0 + \varepsilon x_1 + \dots) - 3\varepsilon^2 x_0^2 + \dots \\ &= -1 + 2\varepsilon x_0 + \varepsilon^2(2x_1 - 3x_0^2) + \dots\end{aligned}$$

We obtain a *system* of equations:

$$\begin{aligned}\ddot{x}_0 &= -1, & x_0(0) &= 0, \dot{x}_0(0) = 1, \\ \ddot{x}_1 &= 2x_0, & x_1(0) &= 0, \dot{x}_1(0) = 0, \\ \ddot{x}_2 &= 2x_1 - 3x_0^2, & x_2(0) &= 0, \dot{x}_2(0) = 0, \\ &\dots\end{aligned}$$

- Initial conditions are taken care of by the first approximation
- General solutions are the same (only particular solutions differ!)

Perturbation solution to second order:

$$x = t - \frac{1}{2}t^2 + \varepsilon\left(\frac{1}{3}t^3 - \frac{1}{12}t^4\right) + \varepsilon^2\left(-\frac{1}{4}t^4 + \frac{11}{60}t^5 + \frac{11}{360}t^6\right)$$

Perturbation analysis is best carried out using Maple!
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FLIGHT TIME TO MAXIMUM HEIGHT (to first order in ε):

$$x = t - \frac{1}{2}t^2 + \varepsilon \left(\frac{1}{3}t^3 - \frac{1}{12}t^4 \right)$$
$$\frac{dx}{dt}(t_m) = 1 - t_m + \varepsilon(t_m^2 - \frac{t_m^3}{3}) = 0$$

Thus,

$$t_m = 1 + O(\varepsilon) = 1 + a\varepsilon + O(\varepsilon^2),$$

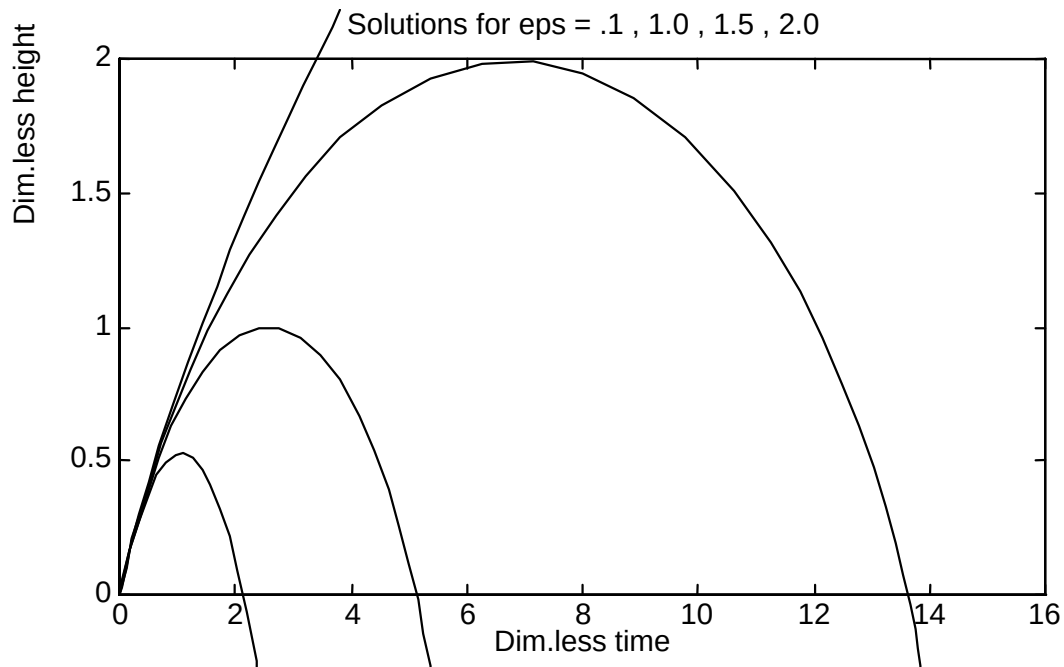
If this is inserted, we obtain

$$\begin{aligned} 0 &= 1 - (1 + a\varepsilon) + \varepsilon \left((1 + a\varepsilon)^2 - \frac{1}{3}(1 + a\varepsilon)^3 \right) \\ &= \left(-a + \frac{2}{3}\right)\varepsilon + O(\varepsilon^2) \end{aligned}$$

Finally,

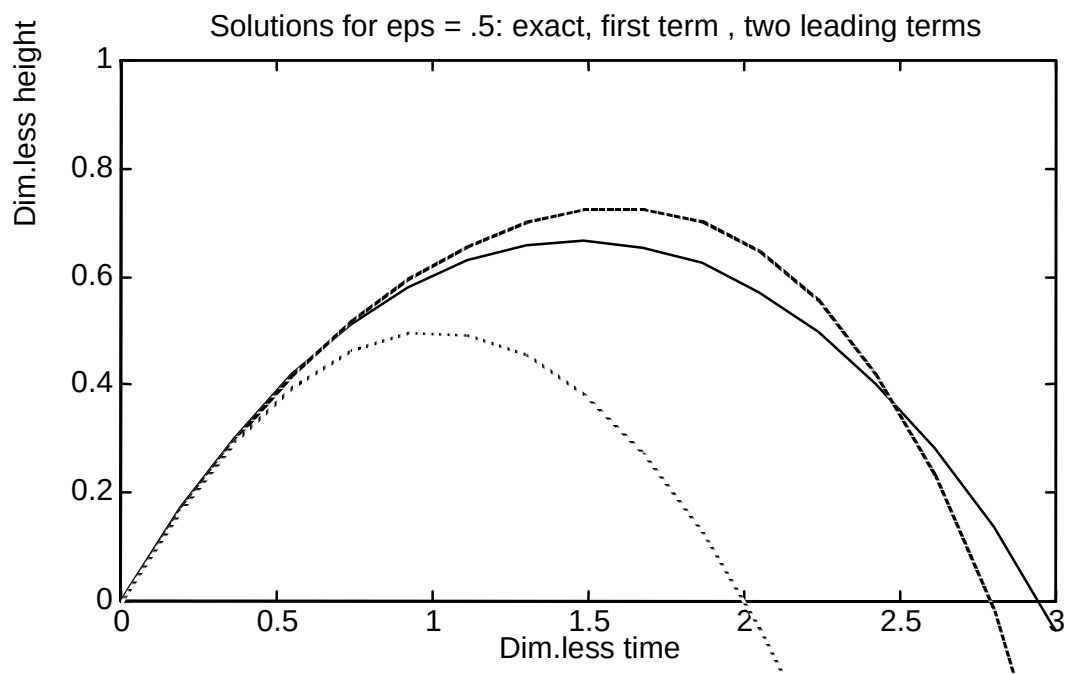
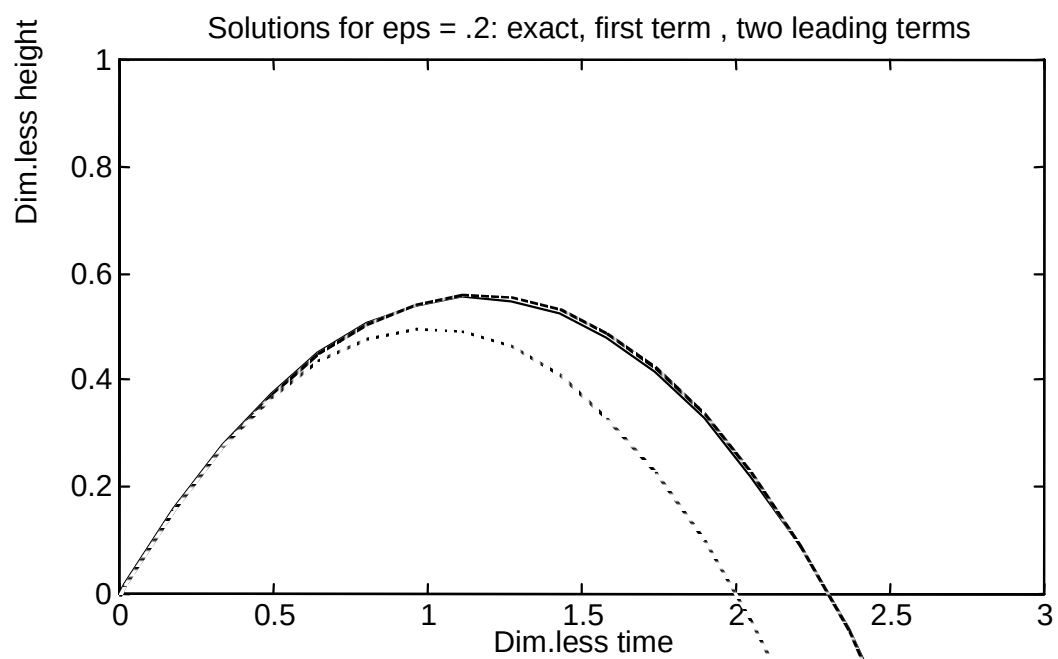
$$t_m = 1 + \frac{2}{3}\varepsilon + O(\varepsilon^2)$$

EXACT (NUMERICAL) SOLUTION



Perturbation expansion to second order:

$$x = t - \frac{1}{2}t^2 + \epsilon \left(\frac{1}{3}t^3 - \frac{1}{12}t^4 \right) + \epsilon^2 \left(-\frac{1}{4}t^4 + \frac{11}{60}t^5 + \frac{11}{360}t^6 \right)$$



THE ANALYTICAL SOLUTION

$$\ddot{x} = -\frac{1}{(1 + \varepsilon x)^2}, \quad x(0) = 0, \quad \dot{x}(0) = 1,$$

It is not possible to write the solution in closed form in terms of elementary functions.

However, multiplying the equation by $\dot{x}$ we obtain

$$\frac{d}{dt} \left(\dot{x}^2 / 2 \right) = \frac{d}{dt} \left(\frac{1}{\varepsilon} \frac{1}{1 + \varepsilon x} \right)$$

or

$$\frac{\dot{x}^2}{2} - \frac{1}{\varepsilon} \frac{1}{1 + \varepsilon x} = \text{Constant}$$

This merely says that the motion is *conservative*, and the sum of kinetic and potential energies is constant. By inserting the initial conditions, we see that the constant is $1/2 - 1/\varepsilon$. This leads to the first order equation

$$\dot{x}^2 = \frac{1 + (\varepsilon - 2)x}{1 + \varepsilon x}, \quad x(0) = 0.$$

If $\varepsilon < 2$, $\dot{x} = 0$ for

$$x_{\max} = \frac{1}{2 - \varepsilon}.$$

This is the (exact) *maximum height* for the projectile. If $\varepsilon > 2$, the velocity will always stay strictly greater than 0 and the projectile continues to the limits of the universe.

The solution (as long as $\dot{x} > 0$) can be written implicitly by

$$t = \int_0^x \sqrt{\frac{1 + \varepsilon s}{1 + (\varepsilon - 2)s}} ds$$

(NB! Different expressions for $\varepsilon < 2$ and $\varepsilon > 2$!)

The integral may be derived by using the general formula

$$\int \sqrt{\frac{a+s}{b-s}} ds =$$

$$\frac{a+b}{2} \arcsin\left(\frac{2s+a-b}{a+b}\right) - \sqrt{(a+s)(b-s)} + C = F(s, a, b) + C.$$

Thus,

$$\begin{aligned} t &= \int_0^x \sqrt{\frac{1 + \varepsilon s}{1 - (2 - \varepsilon)s}} ds = \\ &= \sqrt{\frac{\varepsilon}{2 - \varepsilon}} \int_0^x \sqrt{\frac{1/\varepsilon + s}{1/(2 - \varepsilon) - s}} ds \\ &= \sqrt{\frac{\varepsilon}{2 - \varepsilon}} \left(F\left(x, \frac{1}{\varepsilon}, \frac{1}{2 - \varepsilon}\right) - F\left(0, \frac{1}{\varepsilon}, \frac{1}{2 - \varepsilon}\right) \right) \end{aligned}$$

We already know that

$$x_{\max} = \frac{1}{2 - \varepsilon}$$

such that

$$\begin{aligned}
t_M &= \int_{s=0}^{1/(2-\varepsilon)} \sqrt{\frac{1+\varepsilon s}{1-(2-\varepsilon)s}} ds \\
&= \frac{\frac{\pi}{2} - \arcsin(1-\varepsilon) + \sqrt{(2-\varepsilon)\varepsilon}}{(2-\varepsilon)^{3/2} \varepsilon^{1/2}} = 1 + \frac{2}{3} \varepsilon + \dots
\end{aligned}$$

... which is in accordance with what we had from the perturbation analysis.