REGULAR PERTURBATION

NON-RIGOROUS DEFINITION:

We have a *regular perturbation case* if we have an equation

$$D(x,\ldots,\varepsilon)=0$$

containing a parameter ε such that the solution $x_{sol}(\ldots,\varepsilon)$ approaches the solution x_0 of the simplified equation

$$D(x,\ldots,0)=0$$

when $\varepsilon \to 0$.

METHOD:

1) Write the solution as a power series in ε :

$$x_{sol} = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \varepsilon^3 x_3 + \ldots$$

2) Insert the power series in the equation and rearrange to a *new* power series in ε :

$$\begin{aligned} D(x_{0_{sol}},\varepsilon) &= D(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \varepsilon^3 x_3 + \ldots,\varepsilon) \\ &= D(x_0,0) + P_1(x_0,x_1)\varepsilon + P_2(x_0,x_1,x_2)\varepsilon^2 + \ldots \end{aligned}$$

3) Solve the systems recursively

$$\begin{aligned} D(x_0,0) &= 0 \\ P_1(x_0,x_1) &= 0 \\ P_2(x_0,x_1,x_2) &= 0 \\ &\ldots\ldots \end{aligned}$$

and determine $x_0, x_1, x_2, \ldots$.

This applies in *many* different contexts:

- **for finding approximate solutions to algebraic and transcendental equations**
- **for approximate expressions of definite integrals**
- **for approximate solution to ordinary and partial differential equations**

Perturbation analysis is often complementary to numerical techniques:

- **When ε is small, numerical methods may have problems (this actually applies to *singular perturbation*)**
- **The perturbation analysis gives asymptotic relationships that are more useful than a small number of numerical experiments.**

Sometimes there are no "small" parameters.
Then we have to rely on numerical solutions!

THE PROJECTILE PROBLEM

(From Lin & Segel, pp. 233)

A projectile is launched vertically from the earth with velocity *V*.

Equation of motion:

$$\frac{d^2x^*}{dt^*} = -\frac{R^2 g}{(x^* + R)^2}$$

$$x^*(0) = 0$$

$$\frac{dx^*}{dt^*}(0) = V$$

Introducing scales:

$$x^* = \frac{V^2}{g} x,$$

$$t^* = \frac{V}{g} t,$$

Leads to the following scaled equations:

$$\frac{d^2x}{dt^2} = \frac{1}{(1+\varepsilon x)^2},$$

$$x(0) = 0,$$

$$\frac{dx}{dt}(0) = 1.$$

Perturbation solution:

$$\begin{aligned}
\ddot{x} &= \ddot{x}_0 + \varepsilon\ddot{x}_1 + \varepsilon^2\ddot{x}_2 + \ldots \\
&= -(1+\varepsilon x)^{-2} \\
&= -\left[1 + (-2)\varepsilon x + \frac{(-2)(-3)}{2}(\varepsilon x)^2 + \ldots\right] \\
&= -1 + 2\varepsilon(x_0 + \varepsilon x_1 + \ldots) - 3\varepsilon^2 x_0^2 + \ldots \\
&= -1 + 2\varepsilon x_0 + \varepsilon^2(2x_1 - 3x_0^2) + \ldots
\end{aligned}$$

We obtain a *system* of equations:

$$\begin{aligned}
&\ddot{x}_0 = -1, && x_0(0) = 0,\ \dot{x}_0(0) = 1, \\
&\ddot{x}_1 = 2x_0, && x_1(0) = 0,\ \dot{x}_1(0) = 0, \\
&\ddot{x}_2 = 2x_1 - 3x_0^2, && x_2(0) = 0,\ \dot{x}_2(0) = 0, \\
&\ldots
\end{aligned}$$

- **Initial conditions are taken care of by the first approximation**
- **General solutions are the same (only particular solutions differ!)**

Perturbation solution to second order:

$$x = t - \frac{1}{2}t^2 + \varepsilon\left(\frac{1}{3}t^3 - \frac{1}{12}t^4\right) + \varepsilon^2\left(-\frac{1}{4}t^4 + \frac{11}{60}t^5 + \frac{11}{360}t^6\right)$$

Perturbation analysis is best carried out using Maple!

FLIGHT TIME TO MAXIMUM HEIGHT (to first order in ε):

$$x = t - \frac{1}{2}t^2 + \varepsilon\left(\frac{1}{3}t^3 - \frac{1}{12}t^4\right)$$

$$\frac{dx}{dt}(t_m) = 1 - t_m + \varepsilon(t_m^2 - \frac{t_m^3}{3}) = 0$$

Thus,

$$t_m = 1 + O(\varepsilon) = 1 + a\varepsilon + O(\varepsilon^2),$$

If this is inserted, we obtain

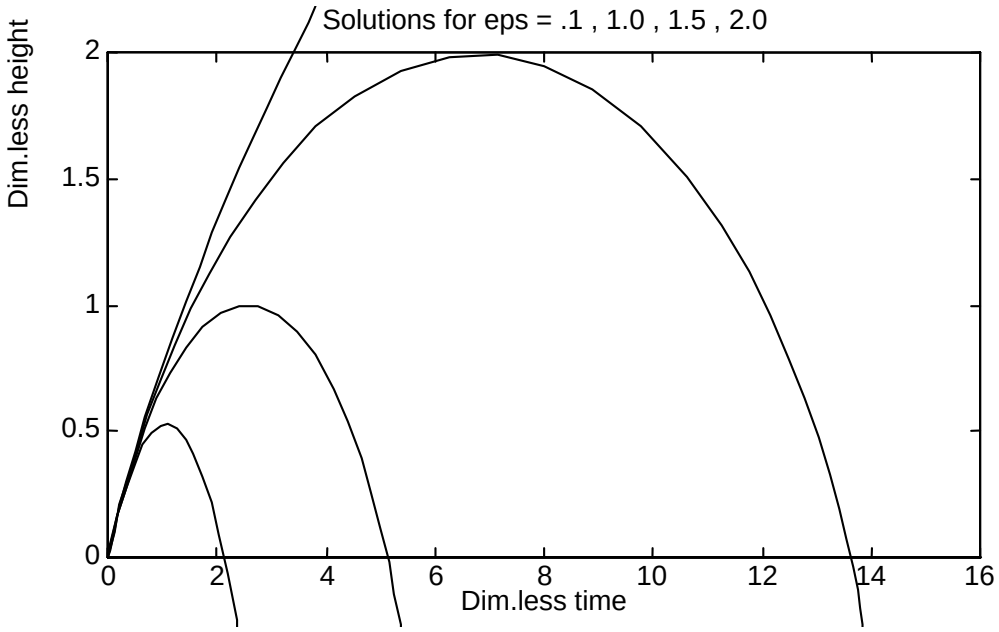
$$0 = 1 - (1 + a\varepsilon) + \varepsilon\left((1 + a\varepsilon)^2 - \frac{1}{3}(1 + a\varepsilon)^3\right)$$

$$= (-a + \frac{2}{3})\varepsilon + O(\varepsilon^2)$$

Finally,

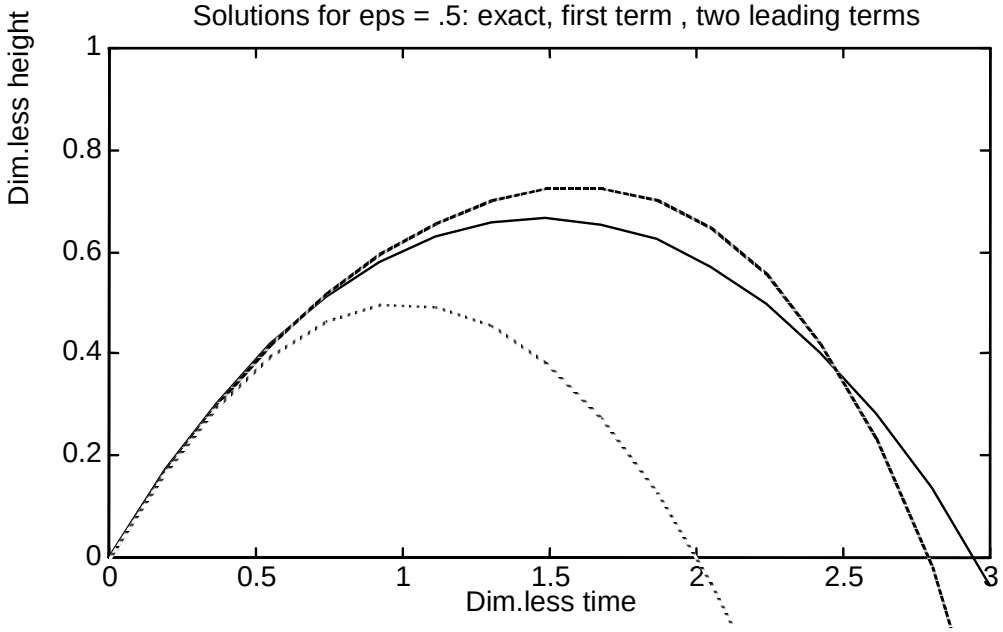
$$\boxed{t_m = 1 + \frac{2}{3}\varepsilon + O(\varepsilon^2)}$$

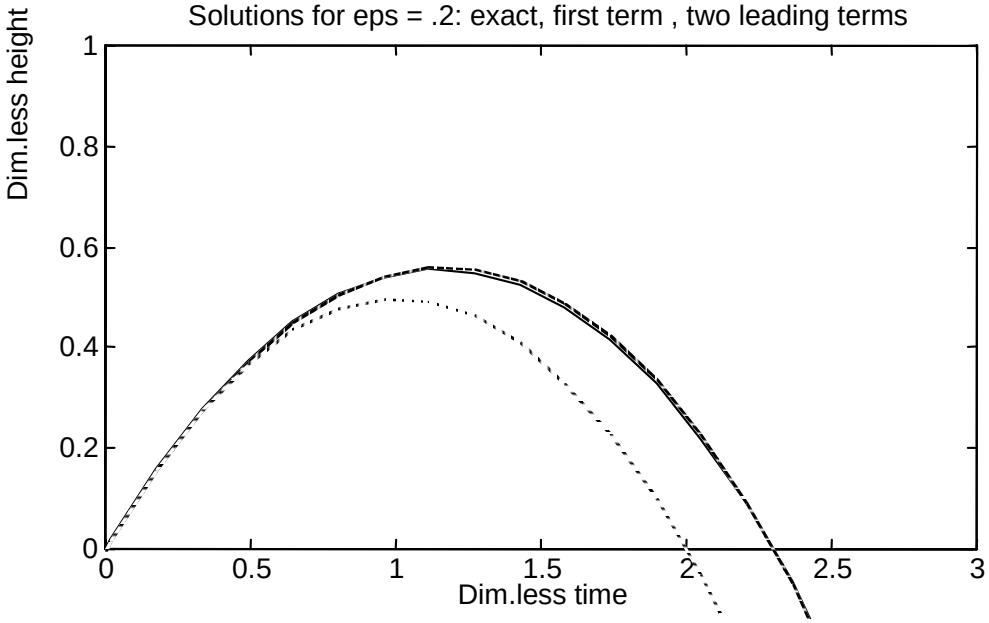
EXACT (NUMERICAL) SOLUTION



Perturbation expansion to second order:

$$x = t - \frac{1}{2}t^2 + \varepsilon\left(\frac{1}{3}t^3 - \frac{1}{12}t^4\right) + \varepsilon^2\left(-\frac{1}{4}t^4 + \frac{11}{60}t^5 + \frac{11}{360}t^6\right)$$


Solutions for eps = .5: exact, first term , two leading terms
Dim.less height
Dim.less time
0
0.2
0.4
0.6
0.8
1
0.5
1.5
2
2.5
3


Solutions for eps = .2: exact, first term , two leading terms
Dim.less height
Dim.less time
0
0.2
0.4
0.6
0.8
1
0.5
1.5
2
2.5
3

THE ANALYTICAL SOLUTION

$$\ddot{x} = -\frac{1}{(1+\varepsilon x)^2}, \quad x(0) = 0, \quad \dot{x}(0) = 1,$$

It is not possible to write the solution in closed form in terms of elementary functions.

However, multiplying the equation by $\dot{x}$ we obtain

$$\frac{d}{dt}\left(\dot{x}^2/2\right) = \frac{d}{dt}\left(\frac{1}{\varepsilon}\frac{1}{1+\varepsilon x}\right)$$

or

$$\frac{\dot{x}^2}{2} - \frac{1}{\varepsilon}\frac{1}{1+\varepsilon x} = \text{Constant}$$

This merely says that the motion is *conservative,* and the sum of kinetic and potential energies is constant. By inserting the initial conditions, we see that the constant is $1/2 - 1/\varepsilon$. This leads to the first order equation

$$\dot{x}^2 = \frac{1+(\varepsilon-2)x}{1+\varepsilon x}, x(0) = 0.$$

If $\varepsilon < 2$, $\dot{x} = 0$ for

$$x_{\max} = \frac{1}{2-\varepsilon}.$$

This is the (exact) *maximum height* for the projectile. If $\varepsilon > 2$, the velocity will always stay strictly greater than 0 and the projectile continues to the limits of the universe.

The solution (as long as $\dot{x} > 0$) can be written implicitly by

$$t = \int_0^x \sqrt{\frac{1+\varepsilon s}{1+(\varepsilon-2)s}}\,ds$$

(NB! Different expressions for $\varepsilon < 2$ and $\varepsilon > 2$!)

The integral may be derived by using the general formula

$$\int \sqrt{\frac{a+s}{b-s}}\,ds =$$

$$\frac{a+b}{2}\arcsin\left(\frac{2s+a-b}{a+b}\right) - \sqrt{(a+s)(b-s)} + C = F(s,a,b) + C.$$

Thus,

$$t = \int_0^x \sqrt{\frac{1+\varepsilon s}{1-(2-\varepsilon)s}}\,ds =$$

$$\sqrt{\frac{\varepsilon}{2-\varepsilon}}\int_0^x \sqrt{\frac{1/\varepsilon + s}{1/(2-\varepsilon) - s}}\,ds$$

$$= \sqrt{\frac{\varepsilon}{2-\varepsilon}}\left(F(x,\frac{1}{\varepsilon},\frac{1}{2-\varepsilon}) - F(0,\frac{1}{\varepsilon},\frac{1}{2-\varepsilon})\right)$$

We already know that

$$x_{\max} = \frac{1}{2-\varepsilon}$$

such that

$$t_M = \int_{s=0}^{1/(2-\varepsilon)} \sqrt{\frac{1+\varepsilon s}{1-(2-\varepsilon)s}}\, ds$$

$$= \frac{\frac{\pi}{2} - \arcsin(1-\varepsilon) + \sqrt{(2-\varepsilon)\varepsilon}}{(2-\varepsilon)^{3/2}\,\varepsilon^{1/2}} = 1 + \frac{2}{3}\,\varepsilon + \ldots$$

... which is in accordance with what we had from the perturbation analysis.