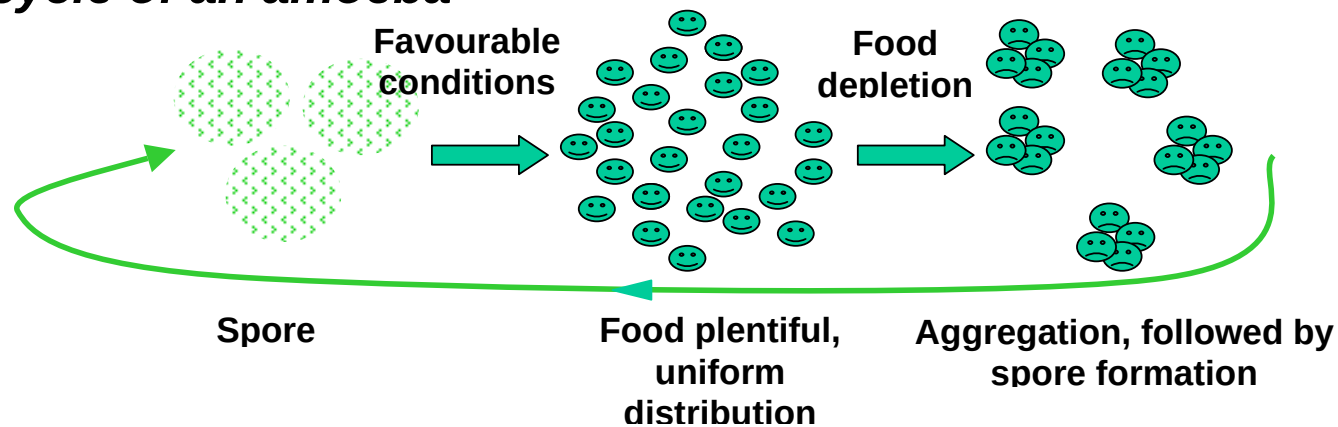


SLIME MOLD AMOEBAE AGGREGATION AS AN INSTABILITY

**A simplified model based on
E.F. Keller and L.A. Segel, *J. Theor. Biol.* (1970).**

- Amoebae spread out uniformly when sufficient food (bacteria) is available.
- Shortage of food makes the amoebae start to secrete a certain chemical (an *attractant*) that attracts other amoebae.
- Further depletion of the food supply leads to an aggregation followed by formation of spores.
- Spores develop into new amoebae when conditions are favorable.

The life cycle of an amoeba

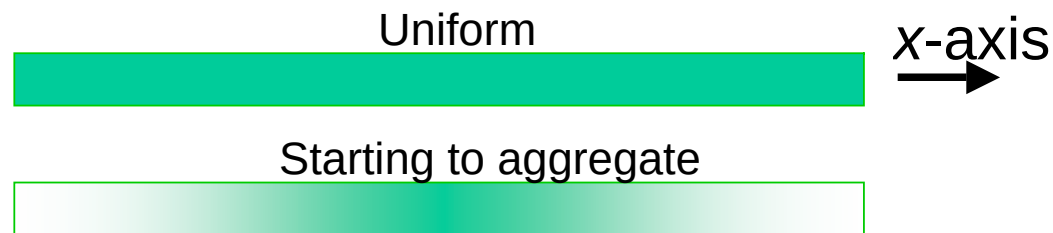


Is the onset of the aggregation due to an instability?

Biological facts:

- Amoebae diffuse due to concentration gradients
- Amoebae move towards higher concentration of attractant
- Amoebae secrete more attractant when there is shortage of food
- The attractant breaks down by time due to a chemical reaction
- The attractant diffuses in the solution by itself due to diffusion gradients

A *one-dimensional* model:



MODELLING PRINCIPLE

Conservation equations for the *density* of amoebae, $a(x,t)$, and the chemical attractant, $\rho(x,t)$.

The transport (*flux*), $\Phi_a(x,t)$, of amoebae consists of *diffusive flux* and *flux due to concentration gradients of the attractant*:

$$\Phi_a(x,t)_{diff} = -\mu \frac{\partial a}{\partial x}, \quad \Phi_a(x,t)_{att} = \chi a \frac{\partial \rho}{\partial x}$$
$$\Phi_a(x,t) = -\mu \frac{\partial a}{\partial x} + \chi a \frac{\partial \rho}{\partial x}$$

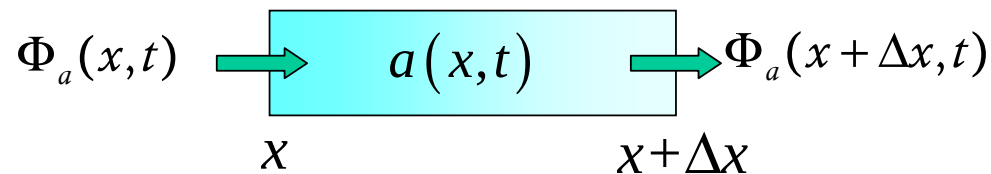
Note how $\Phi_a(x,t)_{att}$ is defined:

$$\Phi_a(x,t)_{att} \propto (\text{Amount of amoebae}) \times (\text{Gradient of attractant})$$

The *control volume* philosophy:

Change of the amount in the control volume per time unit =

What enters per time unit + What is created per time unit



The conservation law in *integral form*:

$$\frac{d}{dt} \int_x^{x+\Delta x} a(s, t) ds = \Phi_a(x, t) - \Phi_a(x + \Delta x, t)$$

Leading to

$$\frac{\partial}{\partial t} (a(\tilde{x}, t) \times \Delta x) = \Phi_a(x, t) - \Phi_a(x + \Delta x, t),$$

or the following *differential form*:

$$\boxed{\frac{\partial a}{\partial t} = \frac{\partial}{\partial x} \left(\mu \frac{\partial a}{\partial x} - \chi a \frac{\partial \rho}{\partial x} \right)}$$

A similar argument goes for the attractant:

$$\frac{d}{dt} \int_x^{x+\Delta x} \rho(s, t) ds = \Phi_{att}(x, t) - \Phi_{att}(x + \Delta x, t) + \int_x^{x+\Delta x} Q(s, t) ds ,$$

where $\Phi_{att} = -D \frac{\partial \rho}{\partial x}$.

The last term is a **source term**, where $Q(s, t) = fa - k\rho$:

- $f \times a$ is the **formation** of attractant by the amoebae (f is a constant).
- $k \times \rho$ is natural **breakdown**, proportional to the density.

This leads to the differential form $\frac{\partial \rho}{\partial t} = fa - k\rho + D \frac{\partial^2 \rho}{\partial x^2}$

We obtain a a *coupled set non-linear partial differential equations*:

$$\begin{aligned} \frac{\partial a}{\partial t} &= \frac{\partial}{\partial x} \left(\mu \frac{\partial a}{\partial x} - \chi a \frac{\partial \rho}{\partial x} \right) \\ \frac{\partial \rho}{\partial t} &= fa - k\rho + D \frac{\partial^2 \rho}{\partial x^2} \end{aligned}$$

Note that we have the uniform and stationary solution

$$a(x, t) = a_0, \rho(x, t) = \rho_0,$$

provided

$$fa_0 = k\rho_0.$$

LINEAR STABILITY ANALYSIS

$$a = a_0 + \tilde{a}$$

$$\rho = \rho_0 + \tilde{\rho}$$

Inserting and neglecting higher order terms lead to

$$\frac{\partial \tilde{a}}{\partial t} = \mu \frac{\partial^2 \tilde{a}}{\partial x^2} - \chi a_0 \frac{\partial^2 \tilde{\rho}}{\partial x^2}$$

$$\frac{\partial \tilde{\rho}}{\partial t} = f \tilde{a} - k \tilde{\rho} + D \frac{\partial^2 \tilde{\rho}}{\partial x^2}$$

Digression:

Consider the diffusion equation $u_t = \kappa u_{xx}$ with the stationary solution $u(x, t) = u_0$, and a perturbation at $t = 0$ of the form $u(x, 0) = \varepsilon \cos(kx)$. Inserting a trial solution of the form $u(x, t) = u_0 + f(t) \cos(kx)$ leads to

$$u(x, t) = u_0 + \varepsilon e^{-\kappa k^2 t} \cos(kx)$$

Thus, the perturbation dampens out for all values of $k \neq 0$.

We introduce a similar perturbation in our equations:

$$\tilde{a}(x,t) = C_1 e^{\alpha t} \cos(\beta x)$$

$$\tilde{\rho}(x,t) = C_2 e^{\alpha t} \cos(\beta x)$$

This leads to the following system for C_1 and C_2 :

$$(\alpha + \mu\beta^2)C_1 - \chi a_0 \beta^2 C_2 = 0,$$

$$-fC_1 + (\alpha + D\beta^2 + k)C_2 = 0.$$

For non-zero solutions the determinant must vanish, that is,

$$(\alpha + \mu\beta^2)(\alpha + D\beta^2 + k) - f\chi a_0 \beta^2 = 0.$$

We obtain the quadratic equation

$$\alpha^2 + (\mu\beta^2 + D\beta^2 + k)\alpha + (\mu\beta^2(D\beta^2 + k) - f\chi a_0 \beta^2) = 0.$$

The equation has real roots (check it!), and since

$$\alpha_1 + \alpha_2 = -(\mu\beta^2 + D\beta^2 + k) < 0,$$

both roots are negative if and only if

$$\alpha_1 \bullet \alpha_2 = (\mu\beta^2(D\beta^2 + k) - f\chi a_0\beta^2) > 0,$$

or

$$\mu(D\beta^2 + k) > f\chi a_0.$$

If $\mu k < f\chi a_0$, this will be violated when β is sufficiently small.

A small value of β means perturbations with a *long wavelength*, since $\beta = 2\pi / \lambda$.

**LONG ENOUGH WAVE-LIKE PERTURBATIONS
WILL GROW WITH TIME!**

$$\mu(D\beta^2 + k) > f\chi a_0$$

Instabilities are prone to occur (the inequality is violated) when

- μ is small: the diffusive mobility of the amoebae is small
- k is small: the decay of attractant is slow.
- f is large: the production of attractant is large
- χ is large: the mobility of the amoebae is large.

The analysis is quite similar in two dimensions. It is conceivable that this mechanism is qualitatively related to what we see in reality. However, the form of the aggregation patterns may be quite complicated.

APPENDIX: THE TWO-DIMENSIONAL EXTENSION

It is straightforward to derive the corresponding 2 (or 3)-dimensional model:

$$\begin{aligned}\frac{\partial a}{\partial t} &= \nabla \cdot (\mu \nabla a - \chi a \nabla \rho), \\ \frac{\partial \rho}{\partial t} &= f a - k \rho + D \nabla^2 \rho.\end{aligned}$$

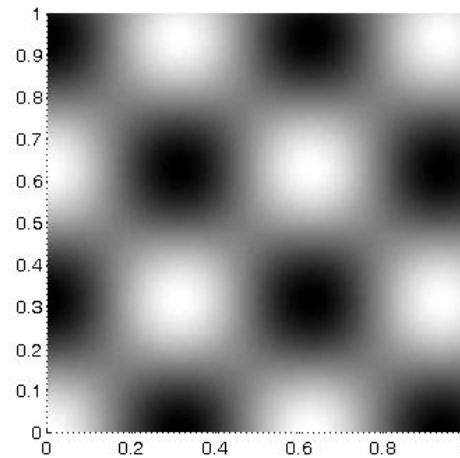
Introducing

$$\begin{aligned}a &= a_0 + \tilde{a} \\ \rho &= \rho_0 + \tilde{\rho}\end{aligned}$$

and neglecting higher order terms leads to

$$\frac{\partial \tilde{a}}{\partial t} = \mu \nabla^2 \tilde{a} - \chi a_0 \nabla^2 \tilde{\rho}, \quad \frac{\partial \tilde{\rho}}{\partial t} = f \tilde{a} - k \tilde{\rho} + D \nabla^2 \tilde{\rho}.$$

STABILITY OF A CHESS-PATTERN PERTURBATION



$$\begin{aligned}\tilde{a} &= C_1 \cos(\beta x) \cos(\beta y) e^{\alpha t}, \\ \tilde{\rho} &= C_2 \cos(\beta x) \cos(\beta y) e^{\alpha t}.\end{aligned}$$

Insert into the 2D equations:

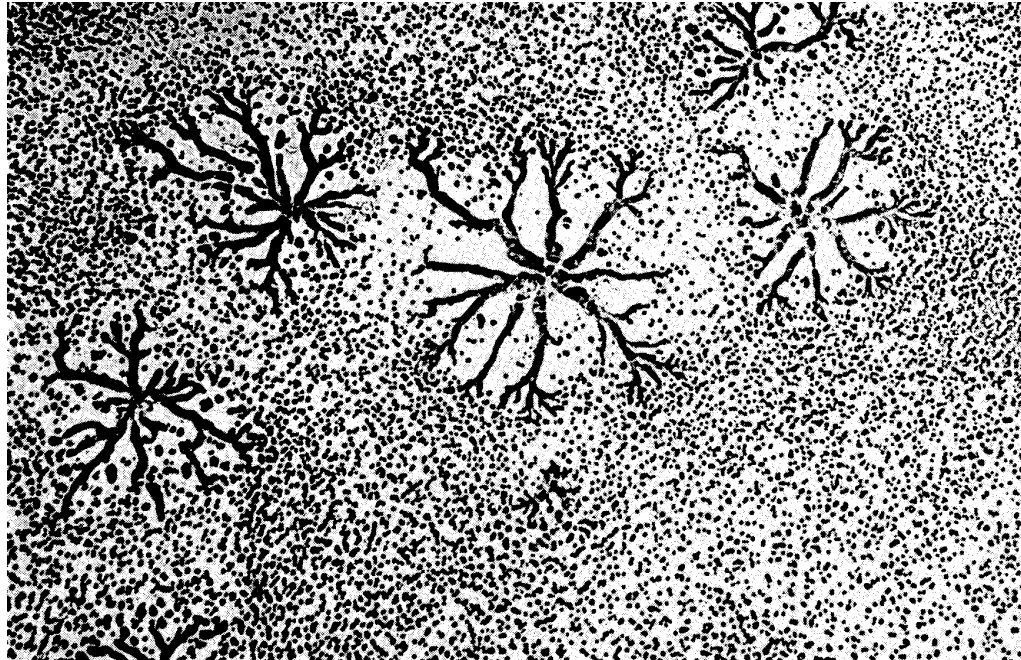
$$\begin{aligned}(\alpha + \mu(2\beta^2))C_1 - \chi a_0(2\beta^2)C_2 &= 0, \\ -fC_1 + (\alpha + D(2\beta^2) + k)C_2 &= 0.\end{aligned}$$

Same as before, with $2\beta^2$ replacing β^2 !

Almost the same criterion: Perturbations grow when

$$\mu(D2\beta^2 + k) < f\chi a_0$$

Real shapes of the aggregation:



(Copied from Lin&Segel, see references)

REFERENCES

Keller, E.F. and L.A. Segel: Initiation of Slime Mold Aggregation Viewed as an Instability, *J. Theor. Biology*, **26** (1970) pp. 399 – 415.

Lin, C.C. and L.A. Segel: *Mathematics Applied to Deterministic Problems in the Natural Sciences*, SIAM Classics in Applied Math. No. 1, SIAM (1988)

Logan, J. D.: *Applied Mathematics, A Contemporary Approach*, J. Wiley, pp. 351 – 357